BLOCK DIAGONALIZATION OF THE FERMION ACTION
IN LATTICE GAUGE THEORIES

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A B S T R A C T

A gauge invariant block diagonalization of the fermionic part of the Wilson action is proposed, which cuts down the fermionic degrees of freedom by a factor 16 in each step, keeping unaltered the infra-red behaviour of the spectrum.

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1. - INTRODUCTION

Several groups ¹⁾⁻⁴⁾ have previously attempted to calculate the hadron spectrum within the framework of lattice QCD. It turns out that numerical methods, like the Gauss-Seidel iteration meet serious convergence problems with the inversion of the so-called fermion-matrix for physical quark masses. These problems could only be circumvented at the expense of performing the inversion for unphysically high quark masses and extrapolating the results back to the physical point.

Bad convergence in iterative matrix inversion procedures signals the appearance of eigenmodes with small eigenvalues (SE modes) which cannot be avoided here : in the fermion matrix, SE modes dominate the infra-red behaviour of the propagator, i.e., the physics we want to extract from lattice calculations.

In this situation it seems to be necessary to treat the SE modes more explicitly from the very beginning. Of course, this can be achieved only by exploiting the specific structure of the fermion action, in particular the nearest neighbour interaction in a four-dimensional lattice.

In this paper we want to make a first step towards the isolation of the SE modes. Using a block spin-like method, we are able to reduce the original fermion action à la Wilson (with L^4 fermion degrees of freedom in a L^4 lattice) to an effective new action with $(L/2)^2$ degrees of freedom, but equal infra-red behaviour. We stress that this reduction can be achieved in a gauge invariant manner.

The block diagonalization method to be presented here is very much inspired by the mathematics - but not the physics! - of Susskind's staggered fermions ^{6),7)}.

The outline of the paper is as follows. In Section 2 we reformulate Wilson's action for one fermion species on a lattice with lattice spacing a in terms of 16 new fermion fields living on a sublattice with double lattice spacing $2a$. Within one block of the sublattice the fermion operator is a 16×16 matrix. In Section 3 we calculate the eigenmodes of the free fermion matrix, which provide us with the kinematical tool to perform the block diagonalization as presented in Section 4. In Section 5 we perform the integration over 15 fermion fields, retaining only one fermion field in the reduced action. By this procedure, the infra-red part of the spectrum remains unchanged.

2. - THE FERMION ACTION AS SEEN FROM A SUBLATTICE WITH DOUBLE LATTICE SPACING

Let us consider the fermion part of the Wilson action

$$S_f = S_1 + S_2 \quad (2.1)$$

where

$$S_1 = \frac{1}{2} a^{4-1} \sum_{x, \mu} \bar{\psi}(x) \gamma_\mu [U(x, x+\mu) \psi(x+\mu) - U(x, x-\mu) \psi(x-\mu)] \quad (2.2)$$

is the "naïve" fermion action and

$$S_2 = \frac{1}{2} a^{4-1} \sum_{x, \mu} \bar{\psi}(x) [U(x, x+\mu) \psi(x+\mu) + U(x, x-\mu) \psi(x-\mu) - 2 \psi(x)] \quad (2.3)$$

is the Wilson term ⁸⁾ which solves the species doubling problem. The $U(x, x+\mu)$ are $SU(N)$ matrices defined on the links of the lattice sites x and $x+\mu$, where μ is a unit vector in the positive μ direction. One could add to the action (2.1) a mass term which, however, does not play any rôle in the following and will be omitted therefore.

A four-dimensional cubic lattice with lattice spacing a can be divided into 16 cubic sublattices S_τ with lattice spacing $2a$:

$$x = a(n_0, n_1, n_2, n_3) \in S_\tau, \quad \tau = (\tau_0, \tau_1, \tau_2, \tau_3) \quad (2.4)$$

if $(-1)^{n_\mu} = \tau_\mu$ for $\mu = 0, 1, 2, 3$.

Therefore, any point x on the original lattice can be specified by a point in one preferred sublattice, say, $\tilde{x} \in S_{++++} = \tilde{S}$ and the signature vector $\tau = (\tau_0, \tau_1, \tau_2, \tau_3)$ which labels the different sublattices :

$$x = \tilde{x} + \frac{a}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 1-\tau_3) \quad (2.5)$$

In accordance, we can label the fermion fields

$$\psi(x) \rightarrow \psi_\tau(\tilde{x}) \quad \tilde{x} \in \tilde{S} \quad (2.6)$$

We think of τ in a conventional way as the four vector of relative co-ordinates within a $(2a)^4$ block $B(\tilde{x})$ with position $\tilde{x}$, rather than following Susskind ⁹⁾ who distributes internal degrees of freedom as spin and isospin over the block.

We want to rewrite the fermionic action in terms of the new co-ordinates $\tilde{x}$ and τ .

For this purpose, we first express the nearest neighbour Kronecker δ 's on the original lattice in terms of the corresponding quantities on the sublattice $\tilde{S} = S_{++++}$ with double lattice spacing :

$$\delta_{y, x+\mu} = \left(T_{\mu}^{(+)} \right)_{\tau\tau'} \frac{1}{2} \left(\delta_{\tilde{y}, \tilde{x}} + \delta_{\tilde{y}, \tilde{x}+2\mu} \right) + \left(T_{\mu}^{(-)} \right)_{\tau\tau'} \frac{1}{2} \left(\delta_{\tilde{y}, \tilde{x}} - \delta_{\tilde{y}, \tilde{x}+2\mu} \right) \quad (2.7)$$

$$\delta_{y, x-\mu} = \left(T_{\mu}^{(+)} \right)_{\tau\tau'} \frac{1}{2} \left(\delta_{\tilde{y}, \tilde{x}-2\mu} + \delta_{\tilde{y}, \tilde{x}} \right) + \left(T_{\mu}^{(-)} \right)_{\tau\tau'} \frac{1}{2} \left(\delta_{\tilde{y}, \tilde{x}-2\mu} - \delta_{\tilde{y}, \tilde{x}} \right) \quad (2.8)$$

Here, x and $\tilde{x}$ are related by Eq. (2.5) and correspondingly

$$y = \tilde{y} + \frac{a}{2} (1-\tau'_0, 1-\tau'_1, 1-\tau'_2, 1-\tau'_3) \quad (2.9)$$

The 16×16 matrices $T_{\mu}^{(\pm)}$ have the following properties :

$$T_{\mu}^{(+)}_{\tau\tau'} = \begin{cases} 1 & \text{for } \tau_{\mu} \tau'_{\mu} = -1, \tau_{\nu} \tau'_{\nu} = 1 \quad \nu \neq \mu \\ 0 & \text{else} \end{cases} \quad (2.10)$$

$$T_{\mu}^{(-)}_{\tau\tau'} = \tau_{\mu} T_{\mu}^{(+)}_{\tau\tau'} \quad (2.11)$$

$$T_{\mu}^{(\pm)}_{\tau'\tau} = \pm T_{\mu}^{(\pm)}_{\tau\tau'} \quad (2.12)$$

The non-vanishing matrix elements $T_{\mu}^{(\pm)}_{\tau\tau'}$ are associated with a signature flip in μ direction $\tau_{\mu} \tau'_{\mu} = -1$ and are listed in Table 1. In Appendix A, we prove the validity of the following commutation and anticommutation relations :

$$[T_{\mu}^{(+)}, T_{\nu}^{(+)}]_{-} = 0 \quad (2.13)$$

$$[T_{\mu}^{(-)}, T_{\nu}^{(-)}]_{-} = 0 \quad (2.14)$$

$$[T_{\mu}^{(+)}, T_{\nu}^{(-)}]_{-} = 0 \quad \text{if } \mu \neq \nu \quad (2.15)$$

$$[T_{\mu}^{(+)}, T_{\mu}^{(-)}]_{+} = 0 \quad (2.16)$$

The geometrical relations (2.7) and (2.8), in conjunction with the algebraic structure (2.13)-(2.16) of the matrices $T_{\mu}^{(\pm)}$ are of fundamental importance for our block diagonalization method. To our knowledge, they have not been noticed so far.

Next, let us introduce new fermion fields ⁷⁾ :

$$\chi_{\tau}(\tilde{x}) = \tilde{U}(\tilde{x}, x) \psi(x) \quad , \quad x \in S_{\tau} \quad , \quad \tilde{x} \in \tilde{S} = S_{++++} \quad (2.17)$$

Here $\tilde{U}(\tilde{x}, x)$ is defined as the product of link variables along the path

$$\begin{aligned} \tilde{x} \quad , \quad \tilde{x} + \frac{\alpha}{2} (1-\tau_0, 0, 0, 0) \quad , \quad \tilde{x} + \frac{\alpha}{2} (1-\tau_0, 1-\tau_1, 0, 0) \\ \tilde{x} + \frac{\alpha}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 0) \quad , \quad x = \tilde{x} + \frac{\alpha}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 1-\tau_3) \end{aligned} \quad (2.18)$$

within the block $B(\tilde{x})$. The new fermion fields $\chi_{\tau}(\tilde{x})$ transform under local gauge transformations $g(x)$ on the original lattice like

$$\chi_{\tau}(\tilde{x}) \rightarrow g(\tilde{x}) \chi_{\tau}(\tilde{x}) \quad (2.19)$$

i.e., they undergo a local gauge transformation on the sublattice $\tilde{S}$.

The introduction of the new fermion fields (2.17) leads us to new gauge fields on the sublattice $\tilde{S}$:

$$\begin{aligned} U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) &= \tilde{U}(\tilde{x}, x) U(x, x+\mu) \tilde{U}(x+\mu, \tilde{x}+2\mu) \quad , \quad \text{if } \tau_{\mu} = -1 \\ U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}-2\mu) &= \tilde{U}(\tilde{x}, x) U(x, x-\mu) \tilde{U}(x-\mu, \tilde{x}-2\mu) \quad , \quad \text{if } \tau_{\mu} = 1 \end{aligned} \quad (2.20)$$

The link variables on the links in the opposite direction are defined in the usual way :

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x} \mp 2\mu) = \left(U_{\mu}^{(\tau)}(\tilde{x} \pm 2\mu, \tilde{x}) \right)^{\dagger} \quad , \quad \text{for all } \tau \quad (2.21)$$

According to Eqs. (2.7) and (2.8), one has on the sublattice $\tilde{S}$ contact terms of the gauge field in addition to the nearest neighbour interactions :

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) = \begin{cases} \tilde{U}(\tilde{x}, x) U(x, x-\mu) \tilde{U}(x-\mu, \tilde{x}) & , \text{ if } \tau_{\mu} = -1 \\ \tilde{U}(\tilde{x}, x) U(x, x+\mu) \tilde{U}(x+\mu, \tilde{x}) & , \text{ if } \tau_{\mu} = 1 \end{cases} \quad (2.22)$$

Local gauge transformations $g(x)$ on the original lattice

$$U(x, x+\mu) \rightarrow g(x) U(x, x+\mu) g(x+\mu)^{\dagger}$$

induce local gauge transformations on the sublattice $\tilde{S}$

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y}) \rightarrow g(\tilde{x}) U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y}) g(\tilde{y})^{\dagger} \quad (2.23)$$

where

$$\tilde{y} = \tilde{x}, \quad \tilde{x} \pm 2\mu$$

Here again, the index τ refers to the interior degrees of freedom within the block and labels the different gauge fields. They are ordered products of the original link variables along a path from $\tilde{x}$ via x [Eq. (2.5)] to $\tilde{y}$, the path being uniquely specified by τ . For illustration, we have listed in Tables 2a and 2b the gauge fields $U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y})$.

In terms of the new fermion fields and gauge fields, the two pieces of the fermionic action have the following form :

$$\begin{aligned} S_A = & \frac{1}{4} a^{4-1} \sum_{\tilde{x}, \mu} \sum_{\tau, \tau'} \bar{\chi}_{\tau}(\tilde{x}) \gamma_{\mu} \cdot \\ & \cdot \left\{ T_{\mu \tau \tau'}^{(+)} \left[U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) \chi_{\tau'}(\tilde{x}+2\mu) - U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}-2\mu) \chi_{\tau'}(\tilde{x}-2\mu) \right] \right. \\ & - T_{\mu \tau \tau'}^{(-)} \left[U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) \chi_{\tau'}(\tilde{x}+2\mu) + U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}-2\mu) \chi_{\tau'}(\tilde{x}-2\mu) \right. \\ & \left. \left. - 2 U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) \chi_{\tau'}(\tilde{x}) \right] \right\} \quad (2.24) \end{aligned}$$

and

$$\begin{aligned}
 S_2 = & \frac{1}{4} a^{4-1} \sum_{\tilde{x}, \mu} \sum_{\tau, \tau'} \bar{\chi}_{\tau}(\tilde{x}) \cdot \\
 & \cdot \left\{ T_{\mu \tau \tau'}^{(4)} \left[\mathcal{U}_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) \chi_{\tau'}(\tilde{x}+2\mu) + \mathcal{U}_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}-2\mu) \chi_{\tau'}(\tilde{x}-2\mu) \right. \right. \\
 & \quad \left. \left. + 2 \mathcal{U}_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) \chi_{\tau'}(\tilde{x}) \right] - 4 \delta_{\tau \tau'} \chi_{\tau'}(\tilde{x}) \right. \\
 & \quad \left. - T_{\mu \tau \tau'}^{(4)} \left[\mathcal{U}_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) \chi_{\tau'}(\tilde{x}+2\mu) - \mathcal{U}_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}-2\mu) \chi_{\tau'}(\tilde{x}-2\mu) \right] \right\} \quad (2.25)
 \end{aligned}$$

Note again that the representation is explicitly gauge invariant on the sublattice $\tilde{S}$. In the next section we will first evaluate the free case.

3. - BLOCK DIAGONALIZATION OF THE FREE FERMION ACTION

In the absence of a gluon field [$U(x, x+\mu)=1$], the free fermion propagator Δ - viewed from the sublattice $\tilde{S}$ with double lattice spacing $2a$ - can be diagonalized by Fourier transformation

$$\begin{aligned}
 \Delta^{-1}(\tilde{p} \tau, \tilde{p}' \tau') = & \delta_{\tilde{p}, \tilde{p}'} \frac{1}{2a} \cdot \\
 & \cdot \left\{ \sum_{\mu} \gamma_{\mu} \left[T_{\mu \tau \tau'}^{(4)} i \sin(\tilde{p}_{\mu} 2a) - T_{\mu \tau \tau'}^{(4)} (\cos(\tilde{p}_{\mu} 2a) - 1) \right] \right. \\
 & \left. + \sum_{\mu} \left[T_{\mu \tau \tau'}^{(4)} (\cos(\tilde{p}_{\mu} 2a) + 1) - 2 \delta_{\tau \tau'} - i T_{\mu \tau \tau'}^{(4)} \sin(\tilde{p}_{\mu} 2a) \right] \right\} \quad (3.1)
 \end{aligned}$$

The momenta range within the first Brillouin zone of $\tilde{S}$:

$$-\pi < \tilde{p}_{\mu} 2a \leq \pi$$

This has to be compared with the Fourier transform of the quark propagator on the original lattice with Wilson action :

$$\begin{aligned}
 \Delta^{-1}(p, p') = & \delta_{pp'} \frac{1}{a} \sum_{\mu} \left\{ \gamma_{\mu} i \sin(p_{\mu} a) + (\cos(p_{\mu} a) - 1) \right\} \quad (3.2) \\
 & -\pi < p_{\mu} a \leq \pi
 \end{aligned}$$

We remind the reader that, in the massless case ($m=0$) considered here, the inverse propagator (3.2) develops exactly one zero at $p_\mu a = 0$, $\mu = 0, 1, 2, 3$. In the continuum limit $p_\mu a \rightarrow 0$ the propagator Δ describes exactly one Dirac particle :

$$\Delta^{-1}(p, p') = \delta_{pp'} + i \sum_{\mu} \gamma_{\mu} p_{\mu} \quad (3.3)$$

Let us consider the continuum limit $\tilde{p}_\mu \cdot 2a \rightarrow 0$ on the sublattice $\tilde{S}$:

$$\begin{aligned} \Delta^{-1}(\tilde{p}\tau, \tilde{p}'\tau') &= \delta_{\tilde{p}, \tilde{p}'} \frac{1}{2a} \left\{ i \sum_{\mu} \gamma_{\mu} T_{\mu\tau\tau'}^{(+)}(\tilde{p}_\mu \cdot 2a) \right. \\ &\quad + 2 \sum_{\mu} (T_{\mu\tau\tau'}^{(+)} - \delta_{\tau\tau'}) \\ &\quad \left. - i \sum_{\mu} T_{\mu\tau\tau'}^{(-)}(\tilde{p}_\mu \cdot 2a) + O(a^2) \right\} \quad (3.4) \end{aligned}$$

At this stage, it is important to notice that the four 16×16 matrices $T_{\mu}^{(+)}$ $\mu = 0, 1, 2, 3$ can be diagonalized simultaneously, since they commute with each other

$$T_{\mu}^{(+)} |\chi_n\rangle = \lambda_{\mu n} |\chi_n\rangle, \quad \mu = 0, \dots, 3, \quad n = 0, \dots, 15 \quad (3.5)$$

The eigenvalues $\lambda_{\mu n}$ can be read from Table 3. Obviously, Eq. (3.4) describes 16 Dirac particles with masses

$$m_n = -\frac{1}{a} \sigma_n \quad (3.6)$$

which are given by the eigenvalues

$$\sigma_n = \sum_{\mu=0}^3 (\lambda_{\mu n} - 1) \quad (3.7)$$

of $\sum_{\mu} (T_{\mu}^{(+)} - 1)$. As can be seen from Table 3, there is exactly one zero mode :

$$T_{\mu}^{(+)} |\chi_0\rangle = |\chi_0\rangle, \quad \mu = 0, \dots, 3, \quad \sum_{\mu} (T_{\mu}^{(+)} - 1) |\chi_0\rangle = 0 \quad (3.8)$$

The fifteen remaining eigenmodes $|\chi_n\rangle$ $n = 1, \dots, 15$ correspond to fermions, the masses of which go to infinity in the continuum limit. On the original lattice, such particles appear as poles in the propagator if some components p_μ of the momentum go to π/a .

Due to the terms proportional to $T_{\mu}^{(-)}$ in Eq. (3.4), however, the fermion matrix is not yet completely diagonalized in the basis χ_n . In Appendix A [Eqs. (A.14) and (A.15)] we have calculated the explicit form of the $T_{\mu}^{(-)}$ in the basis χ_n . Thus, there is so to speak still an "interaction" between the 16 fermion species. The eigenvectors of the fermion matrix (3.4) can be calculated by a perturbation expansion in a . To first order, we get

$$|\xi_n\rangle = \frac{1}{N_n} \left\{ |\chi_n\rangle - i \sum_{m \neq n} \frac{1}{2} (\sigma_n - \sigma_m)^{-1} |\chi_m\rangle \sum_{\mu} \sin(\tilde{p}_{\mu} 2a) T_{\mu}^{(-)}{}_{nm} \right\} \quad (3.9)$$

In the new basis (3.9) (properly normalized $\langle \xi_n | \xi_n \rangle = 1$), the fermion matrix is diagonal up to terms of order a :

$$\Delta^{-1}(\tilde{p}, \tilde{p}) = \sum_n \langle \xi_n | \Delta^{-1}(\tilde{p}, \tilde{p}) | \xi_n \rangle | \xi_n \rangle \langle \xi_n | \quad (3.10)$$

The low eigenvalue spectrum is contained in the zero mode sector:

$$\begin{aligned} \langle \xi_0 | \Delta^{-1}(\tilde{p}, \tilde{p}) | \xi_0 \rangle &= \frac{1}{2a} \left\{ i \sum_{\mu} \gamma_{\mu} \sin(\tilde{p}_{\mu} 2a) + \right. \\ &\left. + \sum_{\mu} (\cos(\tilde{p}_{\mu} 2a) - 1) + \frac{1}{4} \sum_{\mu} \sin^2(\tilde{p}_{\mu} 2a) \right\} + O(a^2) \end{aligned} \quad (3.11)$$

Here, we have used the explicit form [Eqs. (A.14) and (A.15)] of the matrices $T_{\mu\nu}^{(-)}$ for the calculation of

$$\sum_{m \neq n} \frac{1}{2} (\sigma_n - \sigma_m)^{-1} T_{\mu}^{(-)}{}_{nm} T_{\nu}^{(-)}{}_{mn} = -\frac{1}{4} \delta_{\mu\nu} \lambda_{\mu n} \quad (3.12)$$

The 15 non-zero mode sectors only contain eigenvalues, which go to infinity in the continuum limit $a \rightarrow 0$

$$\langle \xi_n | \Delta^{-1}(\tilde{p}, \tilde{p}) | \xi_n \rangle = \frac{1}{a} \sigma_n + O(a^0) \quad (3.13)$$

4. - BLOCK DIAGONALIZATION OF THE FERMION ACTION WITHIN ORDER a^2 :
THE INTERACTING CASE

In the free case, we observed that Wilson's fermion action describes 16 fermion species. The corresponding fields $\chi_n(\tilde{x})$ are obtained from the fields $\chi_\tau(\tilde{x})$ defined in Eq. (2.17) by a rotation R which diagonalizes the $T_\mu^{(+)}$:

$$\chi_n(\tilde{x}) = \sum_\tau R_{n\tau} \chi_\tau(\tilde{x}) \quad (4.1)$$

Therefore, let us first express the fermion action in the presence of a gluon field [Eqs. (2.24) and (2.25)] in terms of the fields $\chi_n(\tilde{x})$:

$$\begin{aligned} S_1 = & \frac{1}{4} a^{4-1} \sum_{\tilde{x}, \mu} \sum_{n, n'} \bar{\chi}_n(\tilde{x}) \gamma_\mu \cdot \\ & \cdot \left\{ \left[W_{\mu nn'}^{(+)}(\tilde{x}, \tilde{x}+2\mu) \chi_{n'}(\tilde{x}+2\mu) - W_{\mu nn'}^{(+)}(\tilde{x}, \tilde{x}-2\mu) \chi_{n'}(\tilde{x}-2\mu) \right] \right. \\ & - \left[W_{\mu nn'}^{(-)}(\tilde{x}, \tilde{x}+2\mu) \chi_{n'}(\tilde{x}+2\mu) + W_{\mu nn'}^{(-)}(\tilde{x}, \tilde{x}-2\mu) \chi_{n'}(\tilde{x}-2\mu) \right. \\ & \left. \left. - 2 W_{\mu nn'}^{(-)}(\tilde{x}, \tilde{x}) \chi_{n'}(\tilde{x}) \right] \right\} \quad (4.2) \end{aligned}$$

and

$$\begin{aligned} S_2 = & \frac{1}{4} a^{4-1} \sum_{\tilde{x}, \mu} \sum_{n, n'} \bar{\chi}_n(\tilde{x}) \cdot \\ & \cdot \left\{ W_{\mu nn'}^{(+)}(\tilde{x}, \tilde{x}+2\mu) \chi_{n'}(\tilde{x}+2\mu) + W_{\mu nn'}^{(+)}(\tilde{x}, \tilde{x}-2\mu) \chi_{n'}(\tilde{x}-2\mu) \right. \\ & + 2 W_{\mu nn'}^{(+)}(\tilde{x}) \chi_{n'}(\tilde{x}) - 4 \delta_{nn'} \chi_{n'}(\tilde{x}) \\ & \left. - \left[W_{\mu nn'}^{(-)}(\tilde{x}, \tilde{x}+2\mu) \chi_{n'}(\tilde{x}+2\mu) - W_{\mu nn'}^{(-)}(\tilde{x}, \tilde{x}-2\mu) \chi_{n'}(\tilde{x}-2\mu) \right] \right\} \quad (4.3) \end{aligned}$$

The 16 fermion fields $\chi_n(\tilde{x})$ are coupled to each other through the gauge fields :

$$W_{\mu nn'}^{(\pm)}(\tilde{x}, \tilde{y}) \equiv \sum_{\tau, \tau'} R_{n\tau} U_\mu^{(\tau)}(\tilde{x}, \tilde{y}) T_{\mu \tau \tau'}^{(\pm)} (R^\tau)_{\tau' n'} \quad (4.4)$$

Within each sector χ_n the interaction is mediated by two such fields. The first one, $W_{\mu nn}^{(+)}(\tilde{x}, \tilde{y})$ carries both nearest neighbours ($\tilde{y}=\tilde{x}\pm 2\mu$) and contact interactions ($\tilde{y}=\tilde{x}$). The second one, $W_{\mu nn}^{(-)}(\tilde{x}, \tilde{x})$ includes only contact terms, i.e.,

$$W_{\mu nn}^{(-)}(\tilde{x}, \tilde{x} \pm 2\mu) = 0 \quad (4.5)$$

This is shown in Appendix B. For the physically most important zero mode sector, the elaboration of Eq. (4.4) by use of

$$R_{0\tau} = \frac{1}{4} \quad \text{for all } \tau \quad (4.6)$$

and

$$T_{\mu \tau \tau'}^{(+)} = \tau_{\mu} T_{\mu \tau \tau'}^{(+)} \quad (4.7)$$

finally leads to the very simple and plausible results :

$$W_{\mu 00}^{(+)}(\tilde{x}, \tilde{y}) = \frac{1}{16} \sum_{\tau} U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y}) \quad (4.8)$$

$$W_{\mu 00}^{(-)}(\tilde{x}, \tilde{x}) = \frac{1}{16} \sum_{\tau} \tau_{\mu} U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) \quad (4.9)$$

Let us now turn to the non-diagonal gauge fields $W_{\mu nn'}^{(\pm)}$, $n \neq n'$, which induce species changing transitions $\chi_n \rightarrow \chi_{n'}$. It is shown in Appendix B that in the formal continuum limit $a \rightarrow 0$:

$$W_{\mu nn}^{(+)}(\tilde{x}, \tilde{y}) = O(a^2) \quad \text{if } n \neq n' \quad (4.10)$$

$$W_{\mu nn'}^{(+)}(\tilde{x}, \tilde{y}) = U(\tilde{x}, \tilde{y}) T_{\mu nn'}^{(+)} + O(a^2) \quad (4.11)$$

where

$$U(\tilde{x}, \tilde{y}) = \begin{cases} 1 & \text{if } \tilde{y} = \tilde{x} \\ U(\tilde{x}, \tilde{x} \pm \mu) U(\tilde{x} \pm \mu, \tilde{x} \pm 2\mu) & \text{if } \tilde{y} = \tilde{x} \pm 2\mu \end{cases} \quad (4.12)$$

We propose to approximate the action, Eqs. (4.2) and (4.3), in such a way that all species changing terms of order a^2 are neglected. Then S_1 is diagonal in the fermion fields χ_n , but S_2 is not, because of the species changing terms inherent in $T_{\mu nn'}^{(-)}$, [cf. Eq. (4.11)]. S_2 is diagonalized - up to terms of order a^2 - by introducing new fermion fields $\xi_n(\tilde{x})$:

$$\chi_n(\tilde{x}) = \xi_n(\tilde{x}) + \sum_{m \neq n} \frac{1}{4} (\sigma_n - \sigma_m)^{-1} \sum_{\mu} T_{\mu nm}^{(-)} \cdot \left\{ U(\tilde{x}, \tilde{x} + 2\mu) \xi_m(\tilde{x} + 2\mu) - U(\tilde{x}, \tilde{x} - 2\mu) \xi_m(\tilde{x} - 2\mu) \right\} \quad (4.13)$$

In this way, we finally end up with the action for 16 uncoupled fermion species $\xi_n(\vec{x})$, $n = 0, \dots, 15$

$$S_1 = \frac{1}{4} a^{4-1} \sum_{\vec{x}, \mu} \sum_{n=0}^{15} \bar{\xi}_n(\vec{x}) \gamma_\mu \cdot \left\{ W_{\mu nn}^{(+)}(\vec{x}, \vec{x}+2\mu) \xi_n(\vec{x}+2\mu) - W_{\mu nn}^{(+)}(\vec{x}, \vec{x}-2\mu) \xi_n(\vec{x}-2\mu) + 2 W_{\mu nn}^{(+)}(\vec{x}, \vec{x}) \xi_n(\vec{x}) \right\} \quad (4.14)$$

$$S_2 = \frac{1}{4} a^{4-1} \sum_{\vec{x}, \mu} \sum_{n=0}^{15} \bar{\xi}_n(\vec{x}) \cdot \left\{ W_{\mu nn}^{(+)}(\vec{x}, \vec{x}+2\mu) \xi_n(\vec{x}+2\mu) + W_{\mu nn}^{(+)}(\vec{x}, \vec{x}-2\mu) \xi_n(\vec{x}-2\mu) + 2 W_{\mu nn}^{(+)}(\vec{x}, \vec{x}) \xi_n(\vec{x}) - 4 \xi_n(\vec{x}) + \frac{1}{8} \left(\lambda_{\mu n} + \frac{1}{2} \sigma_n \right) \left[2 \xi_n(\vec{x}) - \mathcal{U}(\vec{x}, \vec{x}+2\mu) \mathcal{U}(\vec{x}+2\mu, \vec{x}+4\mu) \xi_n(\vec{x}+4\mu) - \mathcal{U}(\vec{x}, \vec{x}-2\mu) \mathcal{U}(\vec{x}-2\mu, \vec{x}-4\mu) \xi_n(\vec{x}-4\mu) \right] \right\} \quad (4.15)$$

Here, we have again used the explicit form [Eqs. (A.14) and (A.15)] of the matrices $T_{\mu n}^{(-)}$, in order to calculate

$$\sum_{m \neq n} \frac{1}{4} (\sigma_n - \sigma_m)^{-1} \left(2 + \frac{\sigma_m}{\sigma_n - \sigma_m} \right) T_{\mu nm}^{(+)} T_{\nu mn}^{(+)} = - \frac{1}{8} \left(\lambda_{\mu n} + \frac{1}{2} \sigma_n \right) \delta_{\mu\nu} \quad (4.16)$$

Remember that $\lambda_{\mu n}$ are the eigenvalues of $T_{\mu}^{(+)}$ and $\sigma_n = \sum_{\mu} (\lambda_{\mu n} - 1)$ (cf. Table 3).

We are now prepared to perform the integration of the 15 massive modes $\xi_n(\vec{x})$ in the fermionic part of the path integral. This will be done in the next section.

5. - INTEGRATION OVER THE MASSIVE FERMION MODES

The fermionic part of the path integral for the partition function

$$\prod_{n=0}^{15} \pi_{\tilde{x}} \int d\bar{\xi}_n(\tilde{x}) d\xi_n(\tilde{x}) \det(1 + \mathcal{F}\mathcal{F}^+) \exp(S_1 + S_2) \quad (5.1)$$

contains a Jacobian $\det(1 + \mathcal{F}\mathcal{F}^+)^{-1}$ which originates from the last transformation (4.13) of the fermion fields. The matrix $\mathcal{F}$ has matrix elements

$$\begin{aligned} \mathcal{F}(\tilde{x}n, \tilde{y}m) = & \frac{1}{4} \sum_{\mu} (\sigma_n - \sigma_m)^{-1} T_{\mu}^{(-)} n m \cdot \\ & \cdot \{ U(\tilde{x}, \tilde{x}+2\mu) \delta_{\tilde{y}, \tilde{x}+2\mu} - U(\tilde{x}, \tilde{x}-2\mu) \delta_{\tilde{y}, \tilde{x}-2\mu} \} \end{aligned} \quad (5.2)$$

where the $U(\tilde{x}, \tilde{x}+2\mu)$ are the double link variables, defined in Eq. (4.12).

The integration over the massive fermion modes $\xi_n(\tilde{x})$ $n = 1, \dots, 15$ yields the product of 15 fermion determinants

$$\prod_{n=1}^{15} \det(16 - D_n - N_n) \quad (5.3)$$

For later convenience, we have split the fermion matrices into a diagonal (D_n) and a non-diagonal (N_n) part. Their matrix elements can be read of Eqs. (4.14) and (4.15) :

$$D_n(\tilde{x}, \tilde{y}) = \delta_{\tilde{y}\tilde{x}} 2 \sum_{\mu} (W_{\mu nn}^{(+)}(\tilde{x}, \tilde{x}) + \gamma_{\mu} W_{\mu nn}^{(-)}(\tilde{x}, \tilde{x})) \quad (5.4)$$

$$\begin{aligned} N_n(\tilde{x}, \tilde{y}) = & \sum_{\mu} \delta_{\tilde{y}, \tilde{x}+2\mu} (1 + \gamma_{\mu}) W_{\mu nn}^{(+)}(\tilde{x}, \tilde{x}+2\mu) \\ & + \sum_{\mu} \delta_{\tilde{y}, \tilde{x}-2\mu} (1 - \gamma_{\mu}) W_{\mu nn}^{(+)}(\tilde{x}, \tilde{x}-2\mu) \\ & + \frac{1}{8} \sum_{\mu} (\lambda_{\mu n} + \frac{1}{2} \sigma_n) \{ 2 \delta_{\tilde{y}\tilde{x}} - U(\tilde{x}, \tilde{x}+2\mu) U(\tilde{x}+2\mu, \tilde{x}+4\mu) \delta_{\tilde{y}, \tilde{x}+4\mu} \\ & - U(\tilde{x}, \tilde{x}-2\mu) U(\tilde{x}-2\mu, \tilde{x}-4\mu) \delta_{\tilde{y}, \tilde{x}-4\mu} \} \end{aligned} \quad (5.5)$$

In order to evaluate the fermion determinants (5.3), we first factor out the diagonal parts and then expand the logarithms of the remaining determinants :

$$\det(16 - \mathcal{D}_n - N_n) = \det(16 - \mathcal{D}_n) \det(1 - (16 - \mathcal{D}_n)^{-1} N_n) \quad (5.6)$$

where

$$\det(16 - \mathcal{D}_n) = \pi \det \left[16 - 2 \sum_{\mu} \left\{ W_{\mu nn}^{(H)}(\tilde{x}, \tilde{x}) + \gamma_{\mu} W_{\mu nn}^{(E)}(\tilde{x}, \tilde{x}) \right\} \right] \quad (5.7)$$

and

$$\log \det(1 - (16 - \mathcal{D}_n)^{-1} N_n) = - \text{Tr} \left((16 - \mathcal{D}_n)^{-1} \cdot N_n \right)^2 \quad (5.8)$$

The series (5.8) is indeed a hopping parameter expansion, which becomes more apparent if we consider the free case where

$$(16 - \mathcal{D}_n)^{-1} = (8 - 2\sigma_n)^{-1} \cdot 1 \quad (5.9)$$

Here, $(8 - 2\sigma_n)^{-1}$ is the hopping parameter k which is $1/8$ for the zero mode ($\sigma_0 = 0$), but smaller ($1/12, 1/16, 1/20, 1/24$) for the massive modes (with $\sigma_n = -2, -4, -6, -8$).

Therefore, we expect a better convergence for the series (5.8) than for the usual hopping parameter expansion.

In a quite analogous way, we expand the Jacobian determinant in the path integral :

$$\log \det(1 + \mathcal{F}\mathcal{F}^*)^{-1} = -\text{Tr} \mathcal{F}\mathcal{F}^* + \text{Tr} \mathcal{F}\mathcal{F}^* \mathcal{F}\mathcal{F}^* \quad (5.10)$$

As can be verified from Eq. (5.2), $\text{Tr} \mathcal{F}\mathcal{F}^*$ is independent of the gauge fields, whereas $\text{Tr} \mathcal{F}\mathcal{F}^* \mathcal{F}\mathcal{F}^*$ contains the plaquette variables on the sublattice $\tilde{S}$, i.e., 2×2 Wilson loops on the original lattice.

We propose to neglect, in a zeroth order, approximation contributions from Eqs. (5.8) and (5.10) altogether. Then the net effect of the integration of the 15 massive fermion modes ξ_n , $n = 1, \dots, 15$, is a local effective action on the sublattice $\tilde{S}$:

$$\tilde{S}_g = \sum_{\vec{x}} \sum_{n=1}^{15} \text{tr} \log \left(16 - 2 \sum_{\mu} [W_{\mu nn}^{(+)}(\vec{x}, \vec{x}) + \gamma_{\mu} W_{\mu nn}^{(-)}(\vec{x}, \vec{x})] \right) \quad (5.11)$$

In this approximation, our final result for the partition function is

$$Z = \prod_{x, \mu} \int dU(x, x+\mu) \exp(S_g + \tilde{S}_g) \cdot \prod_{\vec{x} \in \tilde{\mathcal{S}}} \int d\bar{\xi}_0(\vec{x}) d\xi_0(\vec{x}) \exp(\tilde{S}_{\xi_0}) \quad (5.12)$$

where the reduced fermion action can be taken from Eqs. (4.14) and (4.15) :

$$\begin{aligned} \tilde{S}_{\xi_0} = & \frac{1}{4} a^{4+} \sum_{\vec{x}, \mu} \bar{\xi}_0(\vec{x}) \cdot \\ & \cdot \left\{ W_{\mu 00}^{(+)}(\vec{x}, \vec{x}+2\mu) (1+\gamma_{\mu}) \xi_0(\vec{x}+2\mu) + W_{\mu 00}^{(+)}(\vec{x}, \vec{x}-2\mu) (1-\gamma_{\mu}) \xi_0(\vec{x}-2\mu) \right. \\ & + 2 \left[W_{\mu 00}^{(+)}(\vec{x}, \vec{x}) + \gamma_{\mu} W_{\mu 00}^{(-)}(\vec{x}, \vec{x}) - 2 \right] \xi_0(\vec{x}) \\ & + \frac{1}{8} \left[2 \xi_0(\vec{x}) - U(\vec{x}, \vec{x}+2\mu) U(\vec{x}+2\mu, \vec{x}+4\mu) \xi_0(\vec{x}+4\mu) \right. \\ & \left. \left. - U(\vec{x}, \vec{x}-2\mu) U(\vec{x}-2\mu, \vec{x}-4\mu) \xi_0(\vec{x}-4\mu) \right] \right\} \quad (5.13) \end{aligned}$$

The first part of the reduced action (5.13) looks very similar to the original one, Eqs. (2.2) and (2.3). The new link variables $W_{\mu 00}^{(+)}(\vec{x}, \vec{x} \pm 2\mu)$ [Eq. (4.8)] are defined as averages over 16 open Wilson lines, as described in Table 2a. These new link variables are no longer $SU(N)$ matrices, but nevertheless they are still gauge covariant

$$W_{\mu 00}^{(+)}(\vec{x}, \vec{y}) \rightarrow g(\vec{x}) W_{\mu 00}^{(+)}(\vec{x}, \vec{y}) g(\vec{y})^{\dagger}$$

The presence of contact terms, $W_{\mu 00}^{(+)}(\vec{x}, \vec{x})$, and four link interactions [e.g., $U(\vec{x}, \vec{x}+2\mu) U(\vec{x}+2\mu, \vec{x}+4\mu)$] is a consequence of the block diagonalization.

6. - DISCUSSION AND OUTLOOK

The block diagonalization procedure for the fermion action advocated above is very much in the spirit of Wilson's real space renormalization group program for lattice gauge theories. It has the nice feature to respect gauge invariance throughout all reduction steps. The physical idea behind this block diagonalization consists in :

- grouping the lattice co-ordinates x into block sites $\tilde{x}$ and relative co-ordinates τ inside one block $B(\tilde{x})$, and
- retaining only those degrees of freedom from the block that carries the infra-red behaviour.

We think that the block diagonalization method can be of great help to compute hadron masses in lattice gauge theories, because it cuts down the rank of the fermion matrix by a factor 16 in each step. This would be highly welcome since the numerical inversion of the fermion matrix is presently still a barring obstacle to perform realistic calculations on a sufficiently large lattice. Work along this line is in progress.

ACKNOWLEDGEMENTS

One of us (K.H.M.) would like to thank P. Hasenfratz, B. Petersson and I.O. Stamatescu for discussions.

APPENDIX A - THE MATRICES $T_{\mu}^{(\pm)}$

The matrices $T_{\mu}^{(\pm)}$ defined through Eqs. (2.7) and (2.8) by expressing the nearest neighbour Kronecker δ 's on the original lattice in terms of the corresponding quantities on the sublattice $\tilde{S} = S_{++++}$ with double lattice spacing $2a$, can be extracted from Table 1 and the relations (2.10)-(2.12). A first step towards the diagonalization of these matrices is achieved by the following change of the basis :

$$\begin{aligned} |1\pm\rangle &= \frac{1}{\sqrt{2}} (|++++\rangle \mp |----\rangle) & |5\pm\rangle &= \frac{1}{\sqrt{2}} (|-+++\rangle \mp |+-\rangle) \\ |2\pm\rangle &= \frac{1}{\sqrt{2}} (|---+\rangle \mp |+-\rangle) & |6\pm\rangle &= \frac{1}{\sqrt{2}} (|+-++\rangle \mp |+-\rangle) \\ |3\pm\rangle &= \frac{1}{\sqrt{2}} (|-+-+\rangle \mp |+-\rangle) & |7\pm\rangle &= \frac{1}{\sqrt{2}} (|++--\rangle \mp |+-\rangle) \\ |4\pm\rangle &= \frac{1}{\sqrt{2}} (|-++-\rangle \mp |+-\rangle) & |8\pm\rangle &= \frac{1}{\sqrt{2}} (|++--\rangle \mp |+-\rangle) \end{aligned} \quad (A.1)$$

A suitable form of the matrices $T_{\mu}^{(\pm)}$ is obtained by ordering columns and rows as follows :

$$\begin{aligned} &|1+\rangle, |2+\rangle, |3+\rangle, |4+\rangle, |1-\rangle, |2-\rangle, |3-\rangle, |4-\rangle \\ &|5+\rangle, |6+\rangle, |7+\rangle, |8+\rangle, |5-\rangle, |6-\rangle, |7-\rangle, |8-\rangle \end{aligned} \quad (A.2)$$

Then

$$T_{\mu}^{(+)} = \begin{pmatrix} 0 & 0 & \hat{T}_{\mu} & 0 \\ 0 & 0 & 0 & \check{T}_{\mu} \\ \hat{T}_{\mu} & 0 & 0 & 0 \\ 0 & \check{T}_{\mu} & 0 & 0 \end{pmatrix} \quad \mu = 0, 1, 2, 3 \quad (A.3)$$

$$T_0^{(-)} = \begin{pmatrix} 0 & 0 & 0 & \tilde{T}_0 \\ 0 & 0 & \tilde{T}_0 & 0 \\ 0 & -\tilde{T}_0 & 0 & 0 \\ -\tilde{T}_0 & 0 & 0 & 0 \end{pmatrix}, \quad T_j^{(-)} = \begin{pmatrix} 0 & 0 & 0 & \tilde{T}_j \\ 0 & 0 & -\tilde{T}_j^+ & 0 \\ 0 & \tilde{T}_j & 0 & 0 \\ -\tilde{T}_j^+ & 0 & 0 & 0 \end{pmatrix} \quad (A.4)$$

The 4x4 submatrices $\hat{T}_\mu$, $\check{T}_\mu$ and $\tilde{T}_\mu$ are given by

$$\hat{T}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{T}_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \hat{T}_2 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \hat{T}_3 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \quad (A.5)$$

$$\check{T}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \check{T}_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \check{T}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \check{T}_3 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad (A.6)$$

$$\tilde{T}_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -1 \end{pmatrix}, \tilde{T}_1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tilde{T}_2 = \begin{pmatrix} 0 & 1 \\ -\sigma_3 & 0 \end{pmatrix}, \tilde{T}_3 = \begin{pmatrix} 0 & \sigma_1 \\ i\sigma_2 & 0 \end{pmatrix} \quad (A.7)$$

where 1 is the 2x2 unit matrix and σ_j , $j = 1, 2, 3$ are the usual Pauli matrices. From this explicit form of the matrices $T_\mu^{(\pm)}$ it is straightforward to derive the commutation and anticommutation relations (2.13)-(2.16).

Since the matrices $T_\mu^{(+)}$, $\mu = 0, 1, 2, 3$ are Hermitian and commute with each other, they can be diagonalized simultaneously. This is achieved by the following orthogonal matrix R :

$$R = R_3 \cdot R_2 \cdot R_1 \quad (A.8)$$

where R_1 provides the first step (A.1) in the diagonalization and the proper ordering of states (A.2). The diagonalizing matrices for (A.3) are:

$$R_2 = \begin{pmatrix} \hat{R}_2 & 0 & 0 & 0 \\ 0 & \check{R}_2 & 0 & 0 \\ 0 & 0 & \hat{R}_2 & 0 \\ 0 & 0 & 0 & \check{R}_2 \end{pmatrix} \quad (A.9)$$

with the 4x4 submatrices

$$\hat{R}_2 = \frac{1}{2} \begin{pmatrix} 1 & -\sigma_1 \\ \sigma_1 & 1 \end{pmatrix} \begin{pmatrix} 1+i\sigma_2 & 0 \\ 0 & 1+i\sigma_2 \end{pmatrix} \quad (A.10)$$

$$\check{R}_2 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+i\sigma_2 & 0 \\ 0 & 1+i\sigma_2 \end{pmatrix} \quad (A.11)$$

and

$$R_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad (A.12)$$

Here 1 denotes the 4x4 unit matrix. The 16 eigenvalues of $T_{\mu}^{(+)}$, $\mu = 0,1,2,3$ are contained in Table 3. There we have also listed the eigenvalues of $\sum_{\mu} (T_{\mu}^{(+)} - 1)$, which has the meaning of a mass matrix (cf. Section 3) and turns out to have precisely one zero mode. For convenience, we call the corresponding eigenvector $|\chi_0\rangle$ and the non-zero modes $|\chi_n\rangle$ $n = 1, \dots, 15$.

The eigenvector $|\chi_0\rangle$ is characterized by the fact that the four matrices $T_{\mu}^{(+)}$ have the same eigenvalue 1 :

$$T_{\mu}^{(+)} |\chi_0\rangle = |\chi_0\rangle \quad \mu = 0, \dots, 3 \quad (\text{A.13})$$

Finally, we present the matrices $T_{\mu}^{(-)}$ in the basis $|\chi_n\rangle$ $n = 0,1,\dots,15$:

$$T_0^{(-)} = \begin{pmatrix} 0 & 0 & 0 & \tilde{T}_0 \\ 0 & 0 & \tilde{T}_0 & 0 \\ 0 & -\tilde{T}_0 & 0 & 0 \\ -\tilde{T}_0 & 0 & 0 & 0 \end{pmatrix} \quad T_j^{(-)} = \begin{pmatrix} 0 & \tilde{T}_j & 0 & 0 \\ -\tilde{T}_j^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{T}_j \\ 0 & 0 & \tilde{T}_j^+ & 0 \end{pmatrix} \quad (\text{A.14})$$

with the 4x4 submatrices :

$$\tilde{T}_0 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 0 & i\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} -1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \tilde{T}_3 = \begin{pmatrix} 0 & \sigma_3 \\ i\sigma_2 & 0 \end{pmatrix} \quad (\text{A.15})$$

From Eqs. (A.14) and (A.15), one determines the tensors defined in Eqs. (3.12) and (4.16).

APPENDIX B - THE GAUGE FIELDS $W_{\mu}^{(\pm)}(\tilde{x}, \tilde{y})$

The gauge fields $U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y})$, as defined through Eqs. (2.20)-(2.22) are not completely independent ; e.g., if

$$\tau'_{\mu} = -\tau_{\mu} \quad , \quad \tau'_s = \tau_s \quad s \neq \mu \quad (B.1)$$

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) = U_{\mu}^{(\tau')}(\tilde{x}, \tilde{x}+2\mu) \quad (B.2)$$

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) = (U_{\mu}^{(\tau')}(\tilde{x}, \tilde{x}))^{+} \quad (B.3)$$

The first constraint (B.2) leads to Eq. (4.5). For the proof, let us start from the definition (4.4) together with Eq. (4.7)

$$W_{\mu n n}^{(-)}(\tilde{x}, \tilde{x}+2\mu) = \sum_{\tau, \tau'} R_{n\tau} R_{n\tau'} T_{\mu \tau \tau'}^{(+)} \tau_{\mu} U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) \quad (B.4)$$

$R_{n\tau} R_{n\tau'} T_{\mu \tau \tau'}^{(+)}$ is symmetric under the interchange $\tau \leftrightarrow \tau'$ and non-vanishing only if (B.1) is satisfied. In this case, however,

$$\tau_{\mu} U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) + \tau'_{\mu} U_{\mu}^{(\tau')}(\tilde{x}, \tilde{x}+2\mu) = 0 \quad (B.5)$$

which means that the right-hand side of Eq. (B.4) is zero.

Next, let us consider the new gauge fields $W_{\mu}^{(\pm)}(\tilde{x}, \tilde{y})$ in the continuum limit $a \rightarrow 0$. Since the contact variables $U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x})$ are closed Wilson loops, they behave as :

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) - 1 = O(a^2) \quad (B.6)$$

Inserting Eq. (B.6) into the definition (4.4) yields :

$$W_{\mu}^{(\pm)}(\tilde{x}, \tilde{x}) = T_{\mu}^{(\pm)} + O(a^2) \quad (B.7)$$

A similar argument applies to the gauge fields $W_{\mu}^{(\pm)}(\tilde{x}, \tilde{y}+2\mu)$. We express them in terms of the closed Wilson loops

$$V_{\mu}^{(\tau)}(\tilde{x}) = U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) U(\tilde{x}+2\mu, \tilde{x}) \quad (B.8)$$

where $U(\tilde{x}, \tilde{x}+2\mu)$ is the double link variable introduced in Eq. (4.12). Then one finds, in the continuum limit :

$$W_{\mu}^{(\pm)}(\tilde{x}, \tilde{x}+2\mu) = U(\tilde{x}, \tilde{x}+2\mu) T_{\mu}^{(\pm)} + O(a^2) \quad (B.9)$$

	----	+++	---+	++-	+-+	++-	+++	----
----	q_0	0	q_1	0	q_2	0	q_3	0
+++	0	$\pm q_0$	0	$\pm q_1$	0	$\pm q_2$	0	$\pm q_3$
---+	$\pm q_1$	0	$\pm q_0$	0	0	$\pm q_3$	0	$\pm q_2$
++-	0	q_1	0	q_0	$\pm q_3$	0	$\pm q_2$	0
+-+	$\pm q_2$	0	0	q_3	$\pm q_0$	0	0	q_1
++-	0	q_2	$\pm q_3$	0	0	q_0	$\pm q_1$	0
+++	$\pm q_3$	0	0	q_2	0	q_1	$\pm q_0$	0
----	0	q_3	$\pm q_2$	0	$\pm q_1$	0	0	q_0

Table 1 - The matrices $T_{\mu}^{(\pm)} q_{\mu}$

$$\begin{aligned}
U_{\circ}^{(++++)} &= U_{\circ}^{(----)} = W(\hat{0}, \hat{0}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{0}, -\hat{3}, -\hat{2}, -\hat{1}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{1}, \hat{0}, -\hat{1}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{2}, \hat{3}, \hat{0}, -\hat{3}, -\hat{2}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{2}, \hat{0}, -\hat{2}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{1}, \hat{3}, \hat{0}, -\hat{3}, -\hat{1}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{3}, \hat{0}, -\hat{3}) \\
U_{\circ}^{(----)} &= U_{\circ}^{(++++)} = W(\hat{0}, \hat{1}, \hat{2}, \hat{0}, -\hat{2}, -\hat{1})
\end{aligned}$$

Table 2a - The new gauge field variables $U_{\circ}^{(\tau)}(\tilde{x}, \tilde{x}+2\hat{0})$ are products of link variables $U(x, x+\mu)$ along a path ϕ starting at $\tilde{x}$ ending at $\tilde{x}+2\hat{0}$. The sequence of links along this path ϕ is indicated in the brackets.

$$\begin{aligned}
U_{\circ}^{(++++)} &= (U_{\circ}^{(----)})^+ = 1 \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{0}, \hat{1}, \hat{2}, \hat{3}, -\hat{0}, -\hat{3}, -\hat{2}, -\hat{1}) \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{0}, \hat{1}, -\hat{0}, -\hat{1}) \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{2}, \hat{3}, \hat{0}, -\hat{3}, -\hat{2}, -\hat{0}) \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{0}, \hat{2}, -\hat{0}, -\hat{2}) \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{1}, \hat{3}, \hat{0}, -\hat{3}, -\hat{1}, -\hat{0}) \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{0}, \hat{3}, -\hat{0}, -\hat{3}) \\
U_{\circ}^{(----)} &= (U_{\circ}^{(++++)})^+ = W(\hat{1}, \hat{2}, \hat{0}, -\hat{2}, -\hat{1}, -\hat{0})
\end{aligned}$$

Table 2b - The new gauge field variables $U_{\circ}^{(\tau)}(\tilde{x}, \tilde{x})$ are Wilson loops starting and ending at $\tilde{x}$. The sequence of links along the closed path ϕ is indicated in the brackets.

$T_0^{(+)}$	$T_1^{(+)}$	$T_2^{(+)}$	$T_3^{(+)}$	$\sum_{\mu} (T_{\mu}^{(+)} - 1)$
1	1	1	-1	-2
1	-1	1	1	-2
1	-1	-1	-1	-6
1	1	-1	1	-2
1	1	-1	-1	-4
1	-1	-1	1	-4
1	1	1	1	0
1	-1	1	-1	-4
-1	-1	-1	1	-6
-1	1	-1	-1	-6
-1	1	1	1	-2
-1	-1	1	-1	-6
-1	-1	1	1	-4
-1	1	1	-1	-4
-1	-1	-1	-1	-8
-1	1	-1	-1	-4

Table 3 - The eigenvalues of the matrices $T_{\mu}^{(+)}$

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