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FERMION MASSES IN SU(11)

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A B S T R A C T

Introducing several different sets of Higgs fields, possible mass spectra for the three light generations of particles in the SU(11) model of Georgi are discussed. For the u, c, t quarks there are only two possibilities : either all have masses of the same order of magnitude, or two masses are of the same order and the third is about $\alpha^2 \simeq 10^{-4}$ smaller. The corresponding masses for d, s, b quarks (and e, μ , τ leptons) are : in the first case, the masses are all of the same order, but a factor α or α^2 smaller than u, c, t. In the second case, two masses are of the same order as the heavier top quarks, the third is smaller by a factor α . Neutrinos get small Majorana masses.

1. INTRODUCTION

One of the outstanding problems in particle physics is the generation puzzle : there is no satisfactory theory which explains why there is more than one family of quarks and leptons, and why their masses are so different. Several models have been proposed which embed several generations within a gauge group that contains minimal $SU(5)$ ¹⁾ grand unification. In particular, groups such as $SU(11)$, $SU(7)$, $SU(9)$ ²⁾⁻⁴⁾ have been discussed. Among them, the $SU(11)$ model of Georgi²⁾ which contains exactly three generations [e.g., three light $\bar{5}$'s and 10 's of $SU(5)$ representations of fermions] has the minimal number of irreducible fermion representations, and in addition it has the appealing property that each appears only once. So far, a detailed discussion of this model has been lacking^{*)}; in particular the question whether it could give an answer to the problem of the masses of the generations has been left open.

In this paper, we discuss possible mass spectra for the three generations in the Georgi model. We will concentrate mainly on the question whether the observed hierarchical mass pattern can arise naturally, e.g., without adjusting the various (similar) couplings as is necessary, say, in the standard model or in $SU(5)$.

Let us repeat some properties of the $SU(11)$ model. The fermions are in the following totally antisymmetric representations

$$\begin{array}{ll} \psi^{abcd} & (330) \\ \Psi_{abc} & (\overline{165}) \\ \psi_{ab} & (\overline{55}) \\ \psi_a & (\overline{11}) \end{array}$$

The indices $a, b, c, d, \dots$ go from $1, \dots, 11$, and indicate the tensor structure (ψ^{abcd} is a fourth rank antisymmetric tensor); the number in parentheses is the dimension, a bar denotes complex conjugation. This choice is dictated by requiring that there are only $1, \bar{5}, 10$ fermions [in $SU(5)$], no anomalies and no repetition of representations²⁾.

The $SU(5)$ content of the representations above is :

*) Recently, Kim⁵⁾ has begun to study the $SU(11)$ model more, in order to obtain a Peccei-Quinn-symmetry⁶⁾. His work is discussed in Section 2.

one $\psi^{\alpha\beta\gamma\delta}(\bar{5})$; six $\psi^{\alpha\beta\gamma\delta}\dots\psi^{\alpha\beta\gamma\delta}(\bar{10})$;
 fifteen $\psi^{\alpha\beta\gamma\delta}\dots\psi^{\alpha\beta\gamma\delta}(\bar{10})$; twenty $\psi^{\alpha\beta\gamma\delta}\dots\psi^{\alpha\beta\gamma\delta}(\bar{5})$;
 fifteen $\psi^{\alpha\beta\gamma\delta}\dots\psi^{\alpha\beta\gamma\delta}(\bar{1})$; one $\psi_{\alpha\beta\gamma}(\bar{10})$;
 six $\psi_{\alpha\beta\gamma}\dots\psi_{\alpha\beta\gamma}(\bar{10})$; fifteen $\psi_{\alpha\beta\gamma}\dots\psi_{\alpha\beta\gamma}(\bar{5})$ (1)
 twenty $\psi_{\alpha\beta\gamma}\dots\psi_{\alpha\beta\gamma}(\bar{1})$; one $\psi_{\alpha\beta}(\bar{10})$;
 six $\psi_{\alpha\beta}\dots\psi_{\alpha\beta}(\bar{5})$; fifteen $\psi_{\alpha\beta}\dots\psi_{\alpha\beta}(\bar{1})$;
 one $\psi_{\alpha}(\bar{5})$; six $\psi_{\alpha}\dots\psi_{\alpha}(\bar{1})$,

and Greek indices run from 1 to 5. There are thirteen $\bar{10}$'s and sixteen $\bar{10}$'s. Thus, they can form thirteen $SU(5)$ invariant (and thus very large) mass terms, and three $\bar{10}$'s remain light. This is equally so for the $\bar{5}$ and $\bar{5}$.

The Higgs sector is in principle arbitrary, except for phenomenological constraints. We have therefore chosen different sets of Higgs fields ^{*)} and determined the corresponding light fermion mass spectra. We have, however, restricted ourselves to totally antisymmetric representations. This allows for an easy transition to supersymmetric models, there are no light $SU(2)$ triplets ^{**) ,} and, furthermore, we do not believe that other representations would change our results dramatically.

$SU(11)$ is broken, in a first step, down to $SU(5)$, at a scale $M \gtrsim 10^{15}$ GeV. All scalars which are singlets under $SU(5)$ are expected to get a large vacuum expectation value (VEV) $\sim M$. Three $\bar{5}$'s and three $\bar{10}$'s remain massless. In the second step, $SU(5)$ breaks to $SU(3) \times SU(2) \times U(1)$; finally, at M_W , this group breaks to $SU(3) \times U(1)$. We are interested in finding the masses of the light fermions, in particular we want to see whether the observed hierarchy ($m_e \ll m_\mu \ll m_\tau$) comes out. For this we must locate these light particles in the various $SU(11)$ representations; this gives their couplings to scalars and gauge bosons and allows us to estimate their mass.

*) We will consider only Higgs fields which couple to fermions.

**) With possibly disastrous consequences for the relation $m_W/m_Z = \cos\theta_W$.

Hierarchical masses come about because some fields obtain a mass only at one- or two-loop level, whereas the others already have a tree level mass ⁷⁾⁻¹⁴⁾. This induces mass ratios of order $\sim \alpha, \alpha^2$, with $\alpha \sim 10^{-2}$. These loops contain in general both Higgs and gauge fields. We have concentrated here only on scalars for the following reason. We assume that all couplings, including the gauge coupling, are of the same order of magnitude. Therefore, scalar and gauge loops are of similar size, and one can consider one or the other. Only, if one kind of loop only occurs at, say, two-loop level, then one must make sure that the other kind does not appear at the one-loop level. We have in fact checked that in all cases where the scalars give only two loops, there are no one-loop diagrams involving gauge bosons. On the other hand, it is easy to imagine theories where only scalars can give loop masses at all. In the models discussed so far ⁷⁾⁻¹⁴⁾, the hierarchical masses emerged as a result of ad hoc symmetries added to a given model. In SU(11), it was hoped that they would emerge naturally.

In the following we carry out this programme for various choices of the scalars. In Section 2 Kim's proposal is discussed. In Section 3, we take a representation $H^{abcde}(462)$, two $H^{abc}(165)$ and a $H^a(11)$. In Section 4 we choose $H^{abcd}(330)$, two $H^{abc}(165)$, $H^{ab}(55)$ and $H^a(11)$. The results are discussed in Section 5. The Appendix contains the order of magnitude estimates.

2. H^{abcde} AND TWO H^{abc} HIGGS FIELDS ^{5),*)}

With these Higgs fields, we have the following Yukawa couplings and Higgs potential :

$$\begin{aligned} \mathcal{L}_Y = & f_1 \tilde{\psi}^{abcd} C \psi^{efgh} H_{(1)}^{ijk} \epsilon_{abcdefghijk} + \\ & + g_1 \tilde{\psi}^{abcd} C \psi_d H_{abc}^{(1)} + h_1 \tilde{\psi}_{ab} C \psi_c H_{(1)}^{abc} + (1 \leftrightarrow 2) \\ & + k_1 \tilde{\psi}_{abc} C \psi_{de} H^{abcde} \\ \tilde{\psi} = & (\psi^c)^T \quad C = i\gamma_2 \gamma_0 \quad \text{charge conjugation matrix} \end{aligned} \quad (2)$$

$$\mathcal{L}_V = M H^{abcde} H_{(1)}^{fgh} H_{(2)}^{ijk} \epsilon_{abcdefghijk} + \dots + h.c.$$

*) With this choice, the theory has an extra global $U(1)$ invariance which can play the rôle of the Peccei-Quinn symmetry ⁶⁾ and leads to an "invisible" axion ¹⁵⁾.

We have not written terms such as $(H_{abc}H^{abc})$ or $(H_{abc}H^{abc})^2$ (which are to be included in $\mathcal{L}_V$) because they are not interesting for our purposes (see footnote on p.2).

Since it is too complicated to carry out the minimization of the potential, we assume as usual that all the $SU(5)$ singlets of the Higgs fields get a vacuum expectation value (VEV) such that $SU(11) \times U_A(1)$ is broken down to $SU(5)$. The order of magnitude of the VEVs should be between M_{GUT} ($10^{14}-10^{15}$ GeV) and M_{Planck} (10^{19} GeV). As a next step, we have to determine the three light $\underline{10}$ and $\overline{5}$'s.

a) Light $\underline{10}$'s

There are fifteen $\underline{10}$'s in $\psi^{\alpha\beta 67}, \dots, \psi^{\alpha\beta 10 11}$ and one $\underline{10} \psi_{\alpha\beta\gamma}$. Thirteen of them should get a large mass, but not all the thirteen can become massive at the tree level. The others then become massive through higher order diagrams including Yukawa couplings and Higgs potential parameters. The $\underline{10} \psi_{\alpha\beta\gamma}$ becomes massive at the tree level through :

$$\tilde{\psi}_{\alpha\beta\gamma} C \psi_{\delta\epsilon} H^{\alpha\beta\gamma\delta\epsilon} . \quad (3)$$

since $H^{\alpha\beta\gamma\delta\epsilon}$ is a $SU(5)$ singlet and therefore has a large VEV m . Also six linear combinations of the fifteen $\underline{10}$ ($\psi^{\alpha\beta 67}, \psi^{\alpha\beta 68}, \dots, \psi^{\alpha\beta 10 11}$) become massive at tree level through Yukawa couplings *)

$$\tilde{\psi}^{\alpha\beta 67} C \psi^{\gamma\eta\epsilon 8} H^{\eta\epsilon 10 11} \epsilon_{\alpha\beta 67\gamma\eta\epsilon 8\eta\epsilon 10 11} . \quad (4)$$

This leaves still nine massless $\underline{10}$ but no $\overline{10}$ in ψ^{abcd} . The $\psi^{\alpha\beta 67}, \dots, \psi^{\alpha\beta 10 11}$ form mass terms with six $\overline{10} \psi_{\alpha\beta 6}, \dots, \psi_{\alpha\beta 11}$ through two-loop Feynman diagrams, shown in Fig. 1. We note that in this case there are no one-loop diagrams.

In this way, six further linear combinations of the $\psi^{\alpha\beta 67}$ get a mass, which is of the order of $\alpha^2 m$ (see Appendix), where $\alpha \sim 10^{-2}$ and m is the tree level mass, as well as the six $\overline{10}$'s in ψ_{abc} .

Finally, three linear combinations of the $\psi^{\alpha\beta 67}, \dots, \psi^{\alpha\beta 10 11}$ remain massless. In principle, it would be possible to calculate the coefficients of the linear combinations once one knows the values of the VEV. But for our purpose, it turns out that it is not important to know them in detail.

*) We will omit the indices 1, 2 on the H^{abc} for notational convenience.

b) Light $\bar{5}$'s

From (1) we see that the twenty-three $\bar{5}$'s are located in several SU(11) representations. Some become massive at the tree level : one $\bar{5}_{\alpha\beta\gamma\delta}$ through :

$$\tilde{\Psi}^{\alpha\beta\gamma\delta} C \Psi^{\epsilon 678} H^{91011} \epsilon_{\alpha\beta\gamma\delta\epsilon 67891011} \quad (5)$$

and one $\bar{5}_{\psi_{\alpha}}$ through :

$$\tilde{\Psi}^{\alpha 678} C \psi_{\alpha} H_{678} \quad (6)$$

The six $\bar{5}_{\psi_{\alpha 6} \sim \psi_{\alpha 11}}$ become massive through a one-loop Feynman diagram (Fig. 2). The mass is then of the order of $\alpha \cdot m$, where $\alpha \sim 10^{-2}$ and m is the tree level mass $\gtrsim 10^{15}$ GeV. Twelve linear combinations of the fifteen $\bar{5}_{\psi_{\alpha 67}, \dots, \psi_{\alpha 1011}}$, can become massive through the two-loop diagram of Fig. 3. These masses are then of the order of $\alpha^2 m$. Three linear combinations of the fifteen $\bar{5}_{\psi_{\alpha 67}, \dots, \psi_{\alpha 1011}}$ remain massless. More precisely, the massless linear combinations contain in principle also contributions from the other $\bar{5}$, but they appear with coefficients that are of order α , for the $\psi_{\alpha 6}, \dots, \psi_{\alpha 11}$, and of order α^2 , for $\psi^{\alpha\beta\gamma\delta}$ and ψ_{α} , smaller than the coefficients for the $\psi_{\alpha 67}, \dots, \psi_{\alpha 1011}$. This is so because $\psi_{\alpha 6}, \dots$ (and $\psi_{\alpha 67}, \dots$) get masses only at the one- (and two-loop) level, as discussed above (similarly for the 10 's). These parts of the light $\bar{5}$'s are generally negligible. However, we will see that in some cases they must be considered.

c) Masses of the light 10 and $\bar{5}$'s

At a second stage, SU(5) breaks to SU(3)×SU(2)×U(1). The three 10 and $\bar{5}$ which remained massless become massive at the third breaking stage when SU(3)×SU(2)×U(1) is broken down to SU(3)×U(1). This is achieved when the Higgs doublets under SU(2) develop a VEV which is of the order of 10^2 GeV. The three 10 get massive through the following Yukawa term :

$$\tilde{\Psi}^{\alpha\beta\gamma\delta} C \psi^{\delta 89} H^{\eta 1011} \epsilon_{\alpha\beta\gamma\delta 89\eta 1011} \quad (7)$$

where the component with $\eta=5$ of the Higgs field develops a VEV. We obtain therefore a symmetric 3×3 mass matrix, whose entries depend on the coefficients of the linear combinations of the light 10 and of the values of the VEV of the Higgs doublets of $H^{\eta ab}$ ($a, b=6, \dots, 11$; $\eta=5$). Although we do not

know these values, it is clear that all entries of the matrix have the same order of magnitude. As a consequence of this, the three eigenvalues, which are the masses of the u, c, t quarks, are in general also of the same order of magnitude. This conclusion is quite general and depends crucially on the fact that all light 10 come from the same $SU(11)$ representation.

The three $\bar{5}$, which survive at low energy, are linear combinations of the fifteen $\psi_{\alpha 67}, \dots, \psi_{\alpha 10 11}$, and they can become massive only through a two-loop Feynman diagram of the type which is shown in Fig. 4. Thus all masses are multiplied with a factor of order α^2 . This is so because there are no Yukawa terms such as

$$\tilde{\psi}_{\alpha 67} C \psi^{\alpha \beta 67} H_{\beta}$$

(or any other scalar) since the field H_a has not been introduced.

We obtain a 3×3 mass matrix whose entries are all of the same order of magnitude. As a consequence, the three eigenvalues which are the masses of the d, s, b quarks (and of e, μ, τ) are in general of the same order of magnitude, and are a factor α^2 less than the corresponding values of the u, c, t quarks. It is worth while to note that the other pieces (see discussion above) which appear in the linear combinations forming the three light $\bar{5}$ do not change this conclusion. For instance, the $\psi_{\alpha 6}, \dots, \psi_{\alpha 11}$ appear in the linear combinations with a factor α and can couple to the $\psi^{\alpha \beta 67}, \dots, \psi^{\alpha \beta 10 11}$ through the one-loop Feynman diagram of Fig. 5. Therefore their contributions to the masses are also proportional to α^2 . In conclusion, we see that this choice of the Higgs fields leads to a wrong mass spectrum for the light quarks and leptons.

3. H^{abcde} , TWO H^{abc} AND ONE H^a AS HIGGS FIELDS

If we have an additional Higgs field H^a , there remains a global $U(1)$ symmetry. The Yukawa couplings are the same as in Eq. (2) with the following addition :

$$\tilde{\psi}^{abcd} C \psi_{abc} H_d \quad (8)$$

We again assume that all $SU(5)$ singlets of the Higgs fields get a VEV of the same order or magnitude as in the previous example. Then the symmetry $SU(11) \times U_A(1)$ is broken down to $SU(5)$. In this model the three light 10 's are again linear combinations of the fifteen $\psi^{\alpha \beta 67}, \dots, \psi^{\alpha \beta 10 11}$

only. The location of the light $\bar{5}$ is not the same as before : they are now linear combinations of the six $\bar{5} \psi_{\alpha 6}, \dots, \psi_{\alpha 11}$. This is because all $\bar{5}$'s, except the six $\bar{5} \psi_{\alpha 6}, \dots, \psi_{\alpha 11}$, can become massive at the tree level; especially the fifteen $\bar{5} \psi_{\alpha 67}, \dots, \psi_{\alpha 10 11}$ through the Yukawa couplings :

$$\bar{\psi}^{\alpha 678} \subset \psi_{\alpha 67} H_8 \quad (9)$$

Three out of the six $\psi_{\alpha 6}, \dots, \psi_{\alpha 11}$ become massive only through a one-loop Feynman diagram of the type shown in Fig. 2.

The three massless linear combinations obviously contain contributions of all other $\bar{5}$, but they appear with coefficients that are of order α , and are therefore negligible.

The three light 10 's become then massive through the same Yukawa term as in Eq. (7). We have thus the same situation as in the case before, and again we can conclude that the three masses of the u, c, t quarks are of the same order of magnitude. On the other hand, the three light $\bar{5}$'s can become massive only through a one-loop Feynman diagram of the type shown in Fig. 5. The entries of the 3×3 mass matrix are, however, all of the same order of magnitude and thus, in general, also the three eigenvalues. The masses of the d, s, b quarks (and e, μ, τ leptons) are of the same order, but a factor α less than the corresponding masses of the u, c, t quarks. The other contributions that appear in the linear combinations of the three light $\bar{5}$'s do not change these conclusions. Thus also this choice of the Higgs fields leads unavoidably to a wrong mass spectrum for quarks and leptons. It is, however, interesting to note that the up and down masses are related, such that $m_d < m_u$.

From the previous analysis, we see that a main point, why the mass spectrum comes out wrong, is the fact that the light 10 's are all in the same $SU(11)$ representation, and thus all get masses of the same order of magnitude. If we want to avoid this, we must choose the Higgs fields such that also the $10 \psi_{\alpha \beta \gamma}$ remains light; actually, the $SU(11)$ model contains 10 under $SU(5)$ only in two representations : namely in ψ^{abcd} and in ψ_{abc} . Due to this fact, for the three light 10 's there are only two possibilities : either they are all linear combinations of the fifteen 10 's in ψ^{abcd} , but then we get a mass of the same order of magnitude for all up-quarks, or two are linear combinations of the fifteen 10 's in ψ^{abcd} and one is in ψ_{abc} . In this last case we will have the two in ψ^{abcd} with about the same mass; the other in ψ_{abc} will get quite a different mass, since it can acquire it only through loop diagrams, this because a Yukawa term like :

$$\tilde{\psi}_{\alpha\beta\gamma} \subset \psi_{\delta\epsilon\eta} H^{\alpha\beta\gamma\delta\epsilon\eta} \quad (\alpha, \beta, \gamma, \delta, \epsilon, \eta = 1, \dots, 5)$$

vanishes due to antisymmetry.

Analogously, for the three light $\bar{10}$'s, it would be good if they came from different $SU(11)$ representations. Then they could get different masses. However, it is difficult, if not impossible, to carry out this programme, because once the Higgs fields are chosen such that two light 10 's are in the ψ^{abcd} and one in ψ_{abc} , also the location of the $\bar{10}$ is fixed and it turns out that they are all in the same $SU(11)$ representation.

We will now try to construct a model which realizes this second possibility for the three light 10 's. For this purpose we introduce the following Higgs fields H^a, H^{ab} , two H^{abc}, H^{abcd} .

4. H^a, H^{ab} , TWO H^{abc} AND H^{abcd} AS HIGGS FIELDS

With these Higgs fields, we have the following Yukawa couplings and Higgs potential ^{*)} :

$$\begin{aligned} \mathcal{L}_Y = & f_1 \tilde{\psi}^{abcd} \subset \psi^{efgh} H_{(1)}^{ijk} \epsilon_{abcdefghijk} \\ & + g_1 \tilde{\psi}^{abcd} \subset \psi_d H^{(1)}_{abc} \\ & + h_1 \tilde{\psi}^{abcd} \subset \psi_{cd} H_{ab} + k_1 \tilde{\psi}^{abcd} \subset \psi_{abc} H_d \\ & + \ell_1 \tilde{\psi}_{ab} \subset \psi_c H_{(1)}^{abc} + m_1 \tilde{\psi}_{abc} \subset \psi_d H^{abcd} \\ & + n_1 \tilde{\psi}_{ab} \subset \psi_{cd} H^{abcd} + (1 \longleftrightarrow 2) \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{L}_V = & r H_{(1)}^{abc} H_{(2)}^{def} H^{ghij} H^k \epsilon_{abcdefghijk} \\ & + s H^{abcd} H^{efgh} H^{ij} H^k \epsilon_{abcdefghijk} \\ & + M H^{abcd} H^{efgh} (t_1 H_{(1)}^{ijk} + t_2 H_{(2)}^{ijk}) \epsilon_{abcdefghijk} + \dots \end{aligned}$$

In the potential we have written only those terms that are of interest for our purposes. M is a mass.

*) We use the same letters as above for the coupling constants, since we are not interested in their exact value.

As in previous cases, we will assume that all the $SU(5)$ singlets become a VEV value of the same order of magnitude, except the six singlets $H_6 \sim H_{11}$, which should receive a VEV, that is, about a factor α less than the others. This choice is because we want to have one light $\underline{10}$ in ψ_{abc} . This can be seen by going through a lengthy argument which we omit here. The fields that get a mass through Yukawa terms that contain the $H_6 \sim H_{11}$ (and are therefore heavy), can also be made massive through one-loop Feynman diagrams (without a VEV of the $H_6, \dots, H_{11}$) and thus have masses that are about a factor α less. From this we conclude that even if the $H_6, \dots, H_{11}$ had a VEV much less than $\alpha \cdot m$ ($m \gtrsim 10^{15}$ GeV), we would have to perform any further calculations as if they had an effective VEV of the order of $\alpha \cdot m$.

Due to this choice, out of the three light $\underline{10}$'s, two are linear combinations of the fifteen $\underline{10} \psi^{\alpha\beta 67}, \dots, \psi^{\alpha\beta 10 11}$ and one is $\psi_{\alpha\beta\gamma}$. In particular, this last one has also contributions from the other fifteen $\underline{10}$'s, but they appear with coefficients that are of order α .

The three light $\bar{\underline{5}}$ are linear combinations of the fifteen $\bar{\underline{5}} \psi_{\alpha 67}, \dots, \psi_{\alpha 10 11}$, and contributions from all the other $\bar{\underline{5}}$'s, which appear, however, with coefficients of order α . This is a direct consequence of the fact that the $H_6, \dots, H_{11}$ get a VEV that is of order α less than the VEV of the $SU(5)$ singlets of the other Higgs fields.

a) Masses of the light $\underline{10}$'s

In the following we will write the three light $\underline{10}$'s as :

$$\begin{aligned} \psi_1^{10} &= a \psi_{\beta\gamma\delta} + \alpha [a_1 \psi^{\epsilon\eta 67} + \dots + a_{15} \psi^{\epsilon\eta 10 11}] \\ \psi_2^{10} &= \alpha b \psi_{\beta\gamma\delta} + [b_1 \psi^{\epsilon\eta 67} + \dots + b_{15} \psi^{\epsilon\eta 10 11}] \\ \psi_3^{10} &= \alpha c \psi_{\beta\gamma\delta} + [c_1 \psi^{\epsilon\eta 67} + \dots + c_{15} \psi^{\epsilon\eta 10 11}] \end{aligned} \quad (11)$$

The states ψ_i^{10} are normalized and orthogonal to each other. ψ_2^{10} and ψ_3^{10} can acquire a mass through the Yukawa term :

$$\tilde{\psi}^{\alpha\beta 67} C_1 \psi^{\gamma\delta 89} H^{\eta 10 11} \epsilon_{\alpha\beta 67\gamma\delta 89 10 11}$$

where the component with $\eta=5$ of the Higgs field develops a VEV as $SU(3) \times SU(2) \times U(1)$ is broken down to $SU(3) \times U(1)$. We will therefore take the corresponding entries of the mass matrix to be of order 1. Couplings

between ψ_1^{10} and ψ_2^{10} or ψ_3^{10} are only possible through a one-loop Feynman diagram such as shown in Fig. 6, and are therefore of order α . A mass term for ψ_1^{10} is only possible through a two-loop Feynman diagram (see Fig. 7) and is of order α^2 . The contributions of order α of the fifteen $10_{\alpha\beta 67}, \dots, 10_{\alpha\beta 1011}$ to ψ_1^{10} [see Eq. (11)] do not change these conclusions. Thus we obtain the following mass matrix, where the entries are indicated only by their order of magnitude

$$\begin{array}{c|ccc}
 & \psi_1^{10} & \psi_2^{10} & \psi_3^{10} \\
 \hline
 \psi_1^{10} & \alpha^2 & \alpha & \alpha \\
 \psi_2^{10} & \alpha & 1 & 1 \\
 \psi_3^{10} & \alpha & 1 & 1
 \end{array} \quad (12)$$

The three eigenvalues are the masses of the u, c, t quarks. In general, we can conclude from (12) that two eigenvalues will be of order 1, and one eigenvalue will be of order α^2 . If the Yukawa couplings are of the order of the gauge couplings, then the corresponding values are $m_u \sim \text{few MeV}$, $m_c, m_t \lesssim M_W$.

b) Masses of the light $\bar{5}$'s

We will proceed analogously as in the previous section and write the three light $\bar{5}$'s as :

$$\begin{aligned}
 \psi_1^{\bar{5}} &= \bar{a}_1 \psi_{\alpha 67} + \dots + \bar{a}_{15} \psi_{\alpha 1011} \\
 \psi_2^{\bar{5}} &= \bar{b}_1 \psi_{\alpha 67} + \dots + \bar{b}_{15} \psi_{\alpha 1011} \\
 \psi_3^{\bar{5}} &= \bar{c}_1 \psi_{\alpha 67} + \dots + \bar{c}_{15} \psi_{\alpha 1011} .
 \end{aligned} \quad (13)$$

Again we will estimate the order of magnitude of the entries of the mass matrix. The $\psi_i^{\bar{5}}$ ($i=1,2,3$) couple to the ψ_2^{10} and ψ_3^{10} through a Yukawa term like :

$$\tilde{\psi}^{\alpha\beta 67} G \psi_{\alpha 67} H_{\beta} , \quad (14)$$

where the component $\beta=5$ of the Higgs field becomes a VEV as $SU(3) \times SU(2) \times U(1)$ is broken down to $SU(3) \times U(1)$. These entries will therefore be of order 1. For the couplings of the $\psi_i^{\bar{5}}$ to ψ_1^{10} , we must be more careful, because $\psi_{\alpha\beta\gamma}$

can couple to $\psi_{\alpha 67}$ only through a two-loop Feynman diagram, but ψ_1^{10} contains also contributions of the $\psi^{\alpha\beta 67}$, which can couple to the $\psi_i^{\bar{5}}$ through a Yukawa term like that in Eq. (14). However, these contributions of the $\psi^{\alpha\beta 67}$ appear with coefficients of order α and therefore these entries of the mass matrix will be of order α .

We thus obtain the following mass matrix :

$$\begin{array}{c|ccc}
 & \psi_1^{\bar{5}} & \psi_2^{\bar{5}} & \psi_3^{\bar{5}} \\
 \hline
 \psi_1^{10} & \alpha & \alpha & \alpha \\
 \psi_2^{10} & 1 & 1 & 1 \\
 \psi_3^{10} & 1 & 1 & 1
 \end{array} \tag{15}$$

The three eigenvalues are the masses of the d,s,b quarks (or e, μ , τ leptons). In general, we can conclude that two eigenvalues are of order 1 and one of order α .

From this analysis, we conclude that also in the case where two light 10 's are linear combinations of the $\psi^{\alpha\beta 67}, \dots, \psi^{\alpha\beta 10 11}$ and one is in ψ_{abc} the mass spectrum of the quarks and leptons one obtains is not satisfactory. Also it seems difficult to get large enough a Cabibbo angle ¹⁶⁾.

We will nevertheless analyze in the following the Majorana masses of the neutrinos and discuss the possible number of light Higgs doublets in this model.

Before we go further, we would like to mention that one could also have chosen the Higgs fields such that the light $\bar{5}$'s are located in different SU(11) representations. For instance, if one introduces the Higgs fields H^a, H^{ab}, H^{abcd} and H^{abcde} , then the light $\bar{5}$'s are located : one in $\psi^{\alpha\beta\gamma\delta}$, one in ψ_α and one is a linear combination of the fifteen $\psi_{\alpha 67}, \dots, \psi_{\alpha 10 11}$ and six $\psi_{\alpha 6}, \dots, \psi_{\alpha 11}$.

However, the price one has to pay is that then all the three light 10 's are linear combinations of the fifteen $\psi^{\alpha\beta 67}, \dots, \psi^{\alpha\beta 10 11}$, and thus, in general, with masses of the same order of magnitude.

In this model, it turns out that the light 10 's become massive through a one-loop diagram. One light $\bar{5}$ becomes a mass at the tree level, and the two others through a one-loop diagram.

c) Majorana masses for the neutrinos

A Majorana mass for the neutrinos can be obtained if there is a Higgs field that transforms like a 15 under $SU(5)$. (The Yukawa coupling is then like $\bar{5} \times \bar{5} \times 15$.)

The Higgs fields that we introduced in this model do not contain any 15 under $SU(5)$. A consequence of this fact is that a Majorana mass can only be produced through radiative corrections. In fact it is possible to obtain a Majorana mass through a one-loop Feynman diagram of the type shown in Fig. 8. This diagram gives a contribution to the mass of the neutrinos of the order :

$$\delta m_\nu \approx \alpha \frac{M_3^2}{M} \quad (16)$$

where $\alpha \simeq 10^{-2}$, M_3 is a typical direct mass of the third generation, and $M \gtrsim 10^{15}$ GeV is the scale of the breaking of $SU(11)$ to $SU(5)$.

The neutrinos can also couple to the singlets of $SU(5)$, which get a large mass of the order M , as $SU(11)$ breaks down to $SU(5)$. Due to this coupling, the neutrinos can acquire a Dirac mass of the order of M_3 , if they couple at the tree level with the $SU(5)$ singlets, otherwise of order αM_3 if in one-loop. Thus we obtain an upper bound for the neutrino masses of the order ¹⁷⁾

$$m_\nu \lesssim \frac{M_3^2}{M} \quad (17)$$

in the first case, or of the order

$$m_\nu \lesssim \alpha \frac{M_3^2}{M} \quad (18)$$

in the second case, where then the main contribution comes from (16). From (17), we obtain for $M_3 \simeq 10^2$ GeV, and $M \gtrsim 10^{15}$ GeV :

$$m_\nu \lesssim 10^{-2} \text{ eV}.$$

d) Number of light Higgs doublets

The number of light Higgs doublets is important because it enters into the determination of the value of $\sin^2 \theta_w$ ¹⁸⁾. In general, the Higgs doublets will acquire a large mass [for instance of the order of the scale of the breaking of $SU(11)$], but then they cannot develop a VEV. Therefore it is necessary to have a fine tuning of the parameters in the potential, such that they get a light mass and develop a VEV of the order of 10^2 GeV.

In the above model only the following Higgs doublets are necessary to give a mass to the light particles (see diagrams in Figs. 6 and 7) : one in H_α ; fifteen in $H^{\alpha 67}, \dots, H^{\alpha 10 11}$ [note that it does not matter whether it is $H_{(1)}^{\alpha 67}$ or $H_{(2)}^{\alpha 67}$ or a linear combination of the two]. It is therefore necessary that they are kept light; all the others will acquire a large mass and consequently will not develop a VEV.

We note that the fifteen $H^{\alpha ab}$ with $a, b = 6, \dots, 11$ transform under $SU(6)$ (which acts on the indices a, b) like an antisymmetric second-rank tensor. With a $SU(6)$ rotation, which is to be performed before we break $SU(11)$, we can reduce the number of independent doublets from fifteen to three. The antisymmetric second-rank tensor can be written as a six-by-six antisymmetric matrix, which can be brought in the form

$$\begin{pmatrix} a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}, \quad (19)$$

after an $SU(6)$ rotation¹⁹⁾.

Thus the total number h of light Higgs doublets is 4; with this value for h we can calculate $\sin^2 \theta_w$ at M_W using the formula¹⁸⁾

$$\sin^2 \theta_w = \frac{3}{8} \left(1 - \frac{\alpha}{4\pi} \left(\frac{110-h}{9} \right) \log \left(\frac{M_x}{M_W} \right)^2 \right), \quad (20)$$

and current values for α, M_x, M_W . We obtain

$$\sin^2 \theta_w = 0.221,$$

quite consistent with the experimental value²⁰⁾.

5. CONCLUSIONS

We have considered the $SU(11)$ grand unified model of Georgi in some detail. In particular, we have investigated the masses of the three light generations of fermions, concentrating on the possibility that the hierarchy of masses, e.g., $m_e \ll m_\mu \ll m_\tau$ comes out naturally; for example, that m_e arises only at the two-loop level.

We have taken three different sets of Higgs fields. The first two lead to an extra $U(1)$ symmetry⁵⁾. In this case, all three up-quarks are of the same order of magnitude (unless one introduces vastly different couplings),

that is, they all get a mass in the same way. The three down-quark masses are also all of the same order of magnitude, but a factor α^2 smaller than the up-quarks. Here, α^2 is the fourth power of coupling constants; typically $\alpha \gtrsim 10^{-2}$ in grand unified theories, at least for the gauge couplings. A possible "natural" generation of the observed masses would be that $m_d \simeq \alpha m_s \simeq \alpha^2 m_b$, and similarly for the other fermions. This has been realized in "ad hoc" constructions ⁷⁾⁻¹⁴⁾; and the hope was that in SU(11) this could come out naturally. Our result indicates that it is not the case. However, we obtain a relation between the up and down masses, which, although not correct, is noteworthy.

The second choice of Higgs fields gives a similar situation. This time, however, the down-quarks are only a factor α smaller than the up-quarks. This is to be compared with the measured values which yield the (surprising) results $m_c \sim 10 m_s$; $m_t \sim 10 m_b$. Our result might explain why the ups (excluding the u quark) are considerably heavier than the respective down-quarks or leptons.

The third choice gives naturally the following orders of magnitude :

$$\begin{aligned} m_c \sim m_t & ; \quad m_u \sim \alpha^2 m_c ; \quad m_b \sim m_t \\ m_s \sim m_t & ; \quad m_d \sim \alpha m_b . \end{aligned}$$

None of these Higgs field choices gives a satisfactory result. One might wonder whether other sets of scalars can lead to more hopeful results. We believe that this is not the case. Due to the particular assignments of fermion fields, the light up-quarks can only appear as such linear combinations that at least two of them are in the same irreducible SU(11) representation. Then we expect these two to have generally masses of the same order of magnitude. It might be that in larger groups, this can be overcome. But, as discussed by Georgi ²⁾, they have more generations and possibly two (or more) of them will again have similar masses.

Besides this, we considered the masses of neutrinos; they are all extremely light [Majorana mass $\sim m_q^2/M_{\text{GUT}}$ ¹⁷⁾], but necessarily non-zero. This is a consequence of the SU(11) structure. Furthermore, we have shown that, as far as the renormalization of $\sin^2 \theta_w$ from M_{GUT} down to M_w is concerned, no bad surprises occur.

In conclusion, our results suggest that the SU(11) model, although quite appealing at first, does not automatically lead to satisfactory fermion masses.

A P P E N D I X

A typical one-loop Feynman diagram, such as that in Fig. 2, gives a contribution to the mass of the order (up to numerical coefficients such as π) :

$$\delta m^{(1)} \cong g^3 t v^2 / M , \quad (A.1)$$

where $g \simeq 10^{-1}$ is the Yukawa coupling constant (all Yukawa couplings are of the same order of magnitude), t is the trilinear Higgs coupling constant which is of order $g \cdot M^{11}$; v is the value of the VEV, and M is a typical direct mass.

With $g \cdot v \cong M$ and $g^2 = \alpha \simeq 10^{-2}$ we can write (A.1) as :

$$\delta m^{(1)} \cong \alpha M . \quad (A.2)$$

Analogously, we obtain for a typical two-loop diagram (see Fig. 3), a mass term of the order :

$$\delta m^{(2)} \cong \alpha^2 M , \quad (A.3)$$

where the quartic Higgs coupling constant is of the order of α . The relations (A.2) and (A.3) are also valid for the diagrams which give mass contributions to the light particles, such as in Figs. 4 and 5, but M is then to be replaced by a direct mass, typical for the third family of light particles.

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R E F E R E N C E S

- 1) H. Georgi and S.L. Glashow, Phys.Rev.Letters 32 (1974) 438.
- 2) H. Georgi, Nuclear Phys. B156 (1979) 126.
- 3) P.H. Frampton, Phys.Letters B88 (1979) 2991.
- 4) P.H. Frampton and S. Nandi, Phys.Rev.Letters 43 (1979) 1460.
- 5) J. Kim, Phys.Rev. D21 (1980) 1687;
J. Kim, Phys.Rev. D24 (1981) 3007.
- 6) R.D. Peccei and H. Quinn, Phys.Rev.Letters 38 (1977) 1440.
- 7) R. Barbieri and D.V. Nanopoulos, Phys.Letters 91B (1980) 369; 95B
(1980) 43.
- 8) S. Barr, Phys.Rev. D21 (1980) 1424.
- 9) H. Georgi, Harvard preprint HUTP 80/A030 (1980).
- 10) M. Bowick and P. Ramond, Phys.Letters B103 (1981) 355.
- 11) R. Barbieri, D.V. Nanopoulos and D. Wyler, Phys.Letters B103 (1981) 433.
- 12) L. Ibañez, Nuclear Phys. B193 (1982) 317.
- 13) D.V. Nanopoulos, 4th Kyoto Summer Institute, Eds. M. Konuma and
T. Maskawa (World Science Publishing Co., Singapore 1981).
- 14) R. Barbieri and D. Wyler, Erice Workshop (1981).
- 15) J. Kim, Phys.Rev.Letters 43 (1979) 103;
M. Dine, W. Fischler and M. Srednicki, Phys.Letters 104B (1981) 402.
- 16) D. Wyler, Phys.Letters B105 (1981) 426.
- 17) M. Gell-Mann, P. Ramond and R. Slansky, in Supergravity, Eds.
P. van Nieuwenhuizen and D. Freedman, North Holland Publ.Co.
(1979), P.317;
T. Yanagida, KEK Workshop (1979). See also :
E. Witten, Phys.Letters 91B (1980) 81.
- 18) H. Georgi, H. Quinn and S. Weinberg, Phys.Rev.Letters 33 (1974) 45;
W. Marciano, Phys.Rev. D20 (1979) 274.
- 19) L.F. Li, Phys.Rev. D9 (1974) 1723.
- 20) See, e.g., L.M. Sehgal, in Proceedings 21st International Conference
on High Energy Physics, Paris, Eds. P. Petiau and M. Porneuf,
J.Phys. 43 (1982).

FIGURE CAPTIONS

- Figure 1 : A two-loop diagram which gives a heavy mass to $\underline{10}$'s and $\overline{10}$'s.
- Figure 2 : A one-loop diagram which gives a heavy mass to $\underline{5}$'s and $\overline{5}$'s.
- Figure 3 : A two-loop diagram which gives a heavy mass to $\underline{5}$'s and $\overline{5}$'s.
- Figure 4 : A two-loop diagram which gives a mass to the three light $\overline{5}$'s. $H_{\alpha\gamma\delta e11}$ transforms like a $\underline{5}$ under $SU(5)$ and its fifth component gets a VEV of the order of 10^2 GeV.
- Figure 5 : A one-loop diagram which gives a mass to the three light $\overline{5}$'s.
- Figure 6 : A one-loop diagram which gives an off-diagonal term in the mass matrix of the light $\underline{10}$'s.
- Figure 7 : A two-loop diagram which gives a mass to the up-quark located in $\psi_{\alpha\beta\gamma}$.
- Figure 8 : A one-loop diagram which gives a contribution to the Majorana mass of the neutrinos.

FIG. 2

FIG. 4

FIG. 1

FIG. 3

FIG. 6

FIG. 8

FIG. 5

FIG. 7