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COMPOSITE OPERATORS IN THE HOPPING PARAMETER EXPANSION
IN THE FREE QUARK MODEL

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A B S T R A C T

I have calculated hopping parameter series of meson and baryon propagators up to $O(K^{32})$ in the Wilson formulation of the free quark model. The position of branch point singularities has been found with the help of Padé approximants. The values of the position of the singularities in K agreed with the exact values within 1-2% in case of mesons and 4-5% in case of baryons. It is argued that in QCD at the cross-over region the systematic errors of the method must be even smaller.

*) Part of this work has been done while the author was visiting at the Rutherford and Appleton Laboratories, UK.

Considerable effort has been devoted recently to hadron spectrum calculations in lattice QCD ¹⁾⁻⁹⁾. As experience has been accumulated it has become clear that longer series in the hopping ⁴⁾ parameter expansion, larger volumes ⁷⁾ and better statistics are needed in order to reach a quantitative understanding of the results of the Monte Carlo measurements. Fortunately, with improved computational technique, further development is expected in this direction ¹⁰⁾. If the inverse of the quark propagator is calculated by the Jacobi iterative method, we can generate very long $O(K^{30}-K^{40})$ series in the hopping parameter expansion with manageable computer time ⁷⁾.

The presence of a singularity in the hadron propagator in momentum space implies a singularity in the hadron propagator in the hopping parameter as well ⁴⁾. With long series the Padé analysis is expected to become very reliable for the determination of the position and the strength of the lowest singularity.

Therefore it is of some interest to obtain a quantitative understanding of the size of systematic errors given by the use of Padé approximants.

In Wilson's formulation of lattice QCD the hadron propagators are very simple functions of the hopping parameter in the large N_c and strong coupling limit. They are ratios of low order polynomials of the hopping parameter ^{1),4)}. Then of course the Padé analysis of even low order series gives exact result.

In the weak coupling limit, however, ($g^2 = 0$), the meson propagators have a quark-antiquark, the baryon propagators have three-quark threshold singularities at $K^{-1} = 6+2e^{Ea/n_q}$ where E is the external energy of the two-point function in the rest frame, n_q is the number of quarks and a is the lattice distance. Therefore low order hopping parameter expansion cannot show the correct singularity structure of the amplitude ¹⁾. Weaker singularities require longer series to reveal the radius of the convergence from the behaviour of the expansion coefficients.

It is expected that at finite couplings these branch point singularities are converted at higher orders into a geometric series. Therefore, it appears to be justified to assume that if the series are long enough in the free quark model to find the correct singularity structure, then the method is bound to work at finite values of g^2 in the cross-over region ($1/g^2 \approx 0.90$). With this motivation, I studied long hopping parameter series of composite operators in the free quark model.

The action of the Wilson formulation of the free quark model is given as

$$S = - \sum_n \bar{\psi}_n \psi_n + \sum_{n,\mu} K (\bar{\psi}_n (r + \gamma_\mu) \psi_{n+\mu} + \bar{\psi}_{n+\mu} (r - \gamma_\mu) \psi_n) \quad (1)$$

where n denotes the lattice sites, ψ_n 's are the quark fields and γ_μ 's are the Euclidean Dirac matrices, r may vary between 0 and 1. The positions of the branch point singularities of the hadron propagators at given K are at

$$E = \frac{1}{n_q} \ln \left\{ \frac{1-6Kr}{2K} + \frac{1+\sqrt{1+\alpha}}{1+r} \right\} \quad (2)$$

where

$$\alpha = \frac{(1-r^2)K^2}{(1-6Kr)^2}$$

E is the energy of the hadron in the rest frame, $n_q = 2$ for mesons and $n_q = 3$ for baryons. Since the threshold singularities come from the region of the loop momentum integration where

$$\vec{q}_i = 0, \quad q_0 = \ln \frac{1-6K}{2K}$$

(we use $r = 1$ for simplicity), the strength of the singularity is the same as in the continuum theory, i.e., we have square root singularities for mesons ^{*)}

$$\sim \sqrt{K_c^2 - K^2} \quad (3a)$$

and weak logarithmic singularities for baryons

$$\sim (K_c^2 - K^2)^2 \ln(K_c^2 - K^2) \quad (3b)$$

Both the position and strength of the leading (lowest singularity) are spin-independent.

The hopping parameter series of hadron propagators have been generated by the Jacobi iterative method ⁷⁾. Although a cut is simulated by multiple pole-zero structure in Padé approximants, the position of the lowest singularity gives the position of the branch point with good accuracy. In Table 1, the results of

*) If $E = 0$ but $m_q \neq 0$ we have no branch cut for the meson amplitudes, but we have a logarithmic singularity ¹⁾ in the pion propagator $\sim m^2 \ln m^2/\Lambda^2$ given by the ultra-violet regularization of the loop integral.

8/8 Padé approximant for mesons ^{*)} and of 16/16 Padé approximants for baryons ($r = 1.0$) are given. The Padé tables of various approximants $[L, M]$ show high stability if $m_q a > 0.2$ for mesons and $m_q a \geq 0.5$ for baryons. In Table 1 (last column), the exact values calculated with the help of Eq. (1) ($r = 1.0$) are also given. The position of the lowest singularity in K has the same value for π , ρ , p and Δ . More precisely, the approximate values agree with the exact ones within $\sim 1.5\%$ in case of mesons and $\sim 4.5\%$ in case of baryons. These error bars do not change significantly with the increase of $m_q a$.

In order to see the improvement of the result if diagonal Padé approximants are used, in Table 2 we summarized the K values obtained with 15/1 and 31/1 Padé approximants for mesons and baryons, respectively.

It is also instructive to have a closer look at the coefficients of the series

$$D_M(E) = \sum_{n=0}^{16} C_n K^{2n} \quad (4a)$$

$$D_B(E) = \sum_{n=0}^{32} C_n' K^n \quad (4b)$$

In Table 3, we give the ratios $u_n = \sqrt{c_{n-1}/c_n}$ for the pion and rho ($m_q a = 0.5$) and in Table 4 the ratios $u_n = c_{n-1}/c_n$ for the proton. As $n \rightarrow \infty$ we have (ratio test)

$$K_c = \lim_{n \rightarrow \infty} u_n$$

The change in the values of the ratios u_n with the increase of n illustrates the rate of convergence as the order of the series is increased. If we know the strength of the singularity, we may correct the value of $K_c^{(n)} = u_n$ for its asymptotic value. In case of $(1-z^2)^\gamma$ we have

$$u_n = \sqrt{\frac{c_{n-1}}{c_n}} = \sqrt{1 + \frac{1+\gamma}{n-\gamma}} \quad (5a)$$

where $\gamma = \frac{1}{2}$ for mesons and if we consider $(1-x)^2 \ln(1-x)$ we obtain

^{*)} It is amusing to remark that the positions of the series of poles and zeros are so close to each other for mesons that the calculation had to be carried out in double precision using an IBM computer, in order to avoid the occurrence of instabilities due to rounding errors in the seventh digit of the coefficients.

$$u_n = \frac{C_{n-1}}{C_n} = 1 + \frac{3}{n-3} \quad (5b)$$

which is the expected asymptotics for baryons. At $n = 16$ for mesons and $n = 32$ for baryons, we have larger corrections for baryons. The corrected values calculated by the formulae

$$K_{\text{corr.}}^{(M)} = \sqrt{1 - \frac{3}{n}} K_M^{(n)} \quad (6a)$$

$$K_{\text{corr.}}^{(B)} = \left(1 - \frac{3}{n}\right) K_B^{(n)} \quad (6b)$$

are given in Table 2 (last two columns). We see that with these correction factors we obtain better agreement with the exact results than with the use of diagonal Padé approximants. (The error is less than 0.5% for mesons and $\approx 2\%$ for baryons.)

We also considered the ratio

$$W_n = \begin{cases} \sqrt{\frac{C_n^2}{C_{n-1} C_n}} & , \text{ for mesons} \\ \frac{C_n^2}{C_{n-1} C_n} & , \text{ for baryons} \end{cases} \quad (7a)$$

$$(7b)$$

which give a good measure to show both the rate of convergence and the strength of the leading singularity (see Tables 3 and 4). In case of π and ρ , the strength of the leading singularity is consistent with $\gamma = 0.6 \pm 0.15$ [see Eq. (5a)]. The somewhat slower approach to the asymptotic value in case of baryons is explained by the weaker singularity of the baryon propagators and the presence of more important subleading contributions. Even for $n > 20$ we can observe the remnants of the structure of the strong coupling limit where only the coefficients $C'_{(3 \times k)}$ ($k = 0, 1, 2, \dots$) are non-vanishing.

We expect that at intermediate values of $1/g^2$, where both the mesons and baryons have a pole as leading singularity, the method will be even more reliable. If the statistical errors can be kept under control ^{*)}, the calculation of long hopping parameter series will provide us with a powerful tool to obtain the values of the low lying hadron masses in lattice QCD in the cross-over region.

^{*)} In QCD at finite coupling $\gamma = -1$. It is expected that the size of the statistical errors in γ with ≈ 20 configuration will not be larger than $\Delta\gamma \approx 0.3-0.5$, at least at larger quark mass values.

Here I would like to emphasize that the analysis was reliable only above certain mass values ($m_q a \geq 0.25$ for mesons and $m_q a \geq 0.5$ for baryons). It is not clear what are the allowed mass values at finite g^2 for series of $O(K^3)$.

I also carried out the analysis at smaller values of r ($r = 0.5, 0.25$) and it was found that subleading effects increase with the decrease of r . This suggests that the choice $r = 1.0$ is preferred if the hopping parameter expansion is used. We listed the expansion coefficients of S_{eff} at $r = 1.0, 0.5$ and 0.25 in Table 5 up to $O(K^3)$. It is remarkable that one needs only ~ 3 min IBM 3081 CPU computer time to generate the exact coefficients of Table 5.

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$m_q a$	K_c^π	K_s^0	K_c^p	K_c^Δ	K_c^{exact}
0.25	0.119	0.120	0.138	0.135	0.1167
0.50	0.1096	0.1094	0.1106	0.1104	0.1076
1.00	0.0889	0.0888	0.0924	0.0924	0.0874
2.00	0.0489	0.0489	0.0508	0.0507	0.04813

Table 1 - The position of the branch point singularities of π , ρ , proton and delta propagators as given by the diagonal Padé approximants 8/8 and 16/16 of $O(K^{32})$ series. The last column shows the exact value [see Eq. (2) with $r = 1$].

$m_q a$	K_c^π	K_c^p	K_c^{exact}	$K_\pi^{\text{corr.}}$	$K_p^{\text{corr.}}$
0.25	0.121	u.	0.1167	0.115	u.
0.50	0.1133	u.	0.1076	0.108	u.
1.00	0.0926	0.0965	0.0874	0.0876	0.0875
2.00	0.0507	0.0544	0.0481	0.0483	0.0493

Table 2 - Position of the branch point singularities of π and proton in K given by off-diagonal Padé approximants 1/7 and 1/15. At $m_q a \leq 0.5$ there is no sign of convergence yet for proton. The last column shows the corrected values of K_c [see Eq. (6)].

n	$\sqrt{c_{n-1}/c_n} \pi$	$\sqrt{c_{n-1}/c_n} \rho$	W_n^π	W_n^0	$W_n^{\text{sq root}}$
1	0.2346	0.2656	0.8123	0.8547	-
2	0.1906	0.2270	0.8734	0.8947	0.8367
3	0.1664	0.2031	0.8993	0.8204	0.8944
4	0.1497	0.1666	0.9195	0.7749	0.9949
5	0.1376	0.1291	0.9416	0.8815	0.9661
6	0.1296	0.1138	0.9610	0.9801	0.9770
7	0.1245	0.1116	0.9746	0.9929	0.9834
8	0.1214	0.1124	0.9828	1.0099	0.9874
9	0.1193	0.1135	0.9876	1.0052	0.9910
10	0.1178	0.1141	0.9905	1.0002	0.9921
11	0.1167	0.1143	0.9923	0.9991	0.9935
12	0.1158	0.1142	0.9935	0.9974	0.9946
13	0.1150	0.1139	0.9945	0.9974	0.9954
14	0.1144	0.1136	0.9953	0.9965	0.9960
15	0.1139	0.1132	0.9959	0.9969	0.9965
16	0.1134	0.1129			

Table 3 - Ratios of the coefficients of the meson series [see Eqs. (4a) and (7a)] at $m_0 a = 0.5$. For comparison, the W_n ratios are also given for $(1-x^2)^{1/2}$.

n	$c_{n-1}/c_n)_p$	$c_{n-1}/c_n)_\Delta$	w_n^p
17	0.1194	0.1125	0.9246
18	0.1104	0.1089	0.8567
19	0.09458	0.09456	0.8669
20	0.1091	0.10863	1.0832
21	0.10072	0.1001	0.9232
22	0.09956	0.09945	0.9885
23	0.10315	0.10269	0.9574
24	0.09876	0.09847	1.0113
25	0.09972	0.09951	0.9637
26	0.09961	0.09928	0.9842
27	0.09804	0.9783	1.0031
28	0.09834	0.09817	0.9944
29	0.09779	0.09755	0.9937
30	0.09717	0.09699	0.9997
31	0.09714	0.09694	0.9945
32	0.09660	0.09642	

Table 4 - Ratios of the coefficients of the baryon series [see Eqs. (4b) and (7b)] at $m_q a = 1$.

n	$S_{\text{eff}}(r=1.0)_n$	$S_{\text{eff}}(r=0.5)_n$	$S_{\text{eff}}(r=0.25)_n$
0			
2	0.0	0.12×10^2	0.15×10^2
4	$+0.96 \times 10^2$	0.15×10^1	-0.5166×10^2
6	$+0.2186 \times 10^4$	-0.64×10^2	$+0.2285 \times 10^3$
8	$+0.8198 \times 10^5$	-0.2576×10^3	-0.1134×10^4
10	$+0.2458 \times 10^7$	$+0.2152 \times 10^2$	$+0.6128 \times 10^4$
12	$+0.7508 \times 10^8$	$+0.6186 \times 10^4$	-0.3580×10^5
14	$+0.2302 \times 10^{10}$	$+0.4964 \times 10^5$	$+0.2257 \times 10^6$
16	$+0.6948 \times 10^{11}$	$+0.1558 \times 10^6$	-0.1537×10^7
18	$+0.2001 \times 10^{13}$	-0.2312×10^7	$+0.1131 \times 10^8$
20	$+0.5147 \times 10^{14}$	-0.4066×10^8	-0.8988×10^8
22	$+0.9456 \times 10^{15}$	-0.2287×10^9	$+0.7719 \times 10^9$
24	-0.8001×10^{16}	$+0.1106 \times 10^{10}$	-0.7132×10^{10}
26	-0.2296×10^{19}	$+0.2779 \times 10^{11}$	
28	-0.1804×10^{21}	$+0.1325 \times 10^{12}$	
30	-0.1130×10^{23}	-0.1688×10^{13}	
32	-0.6456×10^{24}	-0.3406×10^{14}	

Table 5 - The coefficients of the hopping parameter series of the effective action at three values of r , $r = 1.0, 0.5$ and 0.25 .
 $S_{\text{eff}} = \sum_n S_{\text{eff},n} k^{2n}$.

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