Ref.TH.3593-CERN

GENERALIZATIONS OF DASHEN'S FORMULA IN SUPERSYMMETRIC QCD

Graham M. Shore
CERN - Geneva

A B S T R A C T

In supersymmetric QCD there are two formulae for the masses and decay constants of pseudo-Goldstone bosons. The first one, the direct analogue of Dashen's formula, relates the "pion" mass-squared matrix to the quark-antiquark condensate, while the second one, recently obtained by Veneziano, relates the mass matrix to a chiral symmetry breaking condensate of squark fields. These formulae are established using superfield chiral Ward identities and special properties of a class of composite operators.

Ref.TH.3593-CERN
May 1983

1. - INTRODUCTION

If an approximate chiral symmetry is spontaneously broken, the resulting spectrum includes light, but non-zero mass, pseudo-Goldstone bosons (pions). The mass of these particles, which is due entirely to the presence in the Lagrangian of a small symmetry breaking term, is given by the following fundamental formula of chiral perturbation theory:

$$(m^2)_{ab} = \frac{1}{F_a F_b} \langle 0 | [Q_s^a, [Q_s^b, \mathcal{L}]] | 0 \rangle \quad (1.1)$$

This formula was originally proposed in 1969 by Dashen¹⁾. Here, Q_s^a is the generator of the chiral symmetry, F_a the pion decay constant, and $(m^2)_{ab}$ the mass-squared matrix element between pion states. In particular, in QCD the formula relates the masses and decay constants of the pions to the vacuum expectation value of the quark bilinear $\langle 0 | \bar{\psi}\psi | 0 \rangle$, an order parameter for spontaneous chiral symmetry breaking.

The purpose of this paper is to show that in the supersymmetric extension of QCD (SQCD), one can obtain a superfield formula which gives, in addition to the direct analogue of Dashen's formula, a linear pion mass formula of the same type.

Furthermore, since, if supersymmetry is unbroken, there exist degenerate fermion partners of the pseudo-Goldstone bosons, these results serve equally as mass formulae for these light composite fermions.

The basic reason for the existence of this second formula is that in SQCD there are two independent order parameters bilinear in the elementary fields. As well as the usual quark condensate $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle$ there is a further possible chiral symmetry breaking condensate $\langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle$ involving only the scalar partners of the quarks (squarks). Dashen's formula relates the mass-squared matrix elements of the pions to the quark condensate. The new formula, which holds only if supersymmetry is not spontaneously broken, relates the mass matrix element to the squark condensate. This restriction to a supersymmetric vacuum is not serious, as we already know from Witten's index theorem²⁾ that supersymmetry is not spontaneously broken in SQCD for non-zero quark mass.

An alternative explanation of the underlying physics is that the pion in SQCD differs from the QCD pion in that it contains both quark-antiquark and squark pairs. The new formula arises from looking at matrix elements describing the coupling of the pion to the squark fields.

The idea that a second Dashen-type formula might exist came originally from the observation of Taylor, Veneziano and Yankielowicz³⁾ that the product $F_{\pi\pi}^2 m_\pi^2$, unlike $F_{\pi\pi}^2 m_\pi^2$, was stable against several modifications of their effective Lagrangian. Veneziano later noticed that in all the effective Lagrangians proposed so far³⁾ such a combination was related to the expectation value of a certain chiral superfield, and then derived the actual relation directly, working in components⁴⁾. Some comments on the differences between his derivation and the one presented here are given in Appendix A.

The main technical inputs to the derivation presented here are the use of superfield functional Ward identities, and two non-trivial properties of a certain class of composite operators. These aspects of the theory of composite operators turn out to be essential in understanding how to deal with auxiliary fields when they arise in the various Green's functions of interest.

2. - SUPERFIELD WARD IDENTITIES

The model to be considered is supersymmetric QCD - the $N = 1$ supersymmetric gauge theory comprising gluons and their spin-1/2 partners, gluinos, in the adjoint representation of the gauge group $SU(N_C)$, interacting with N_f flavours of quarks and their scalar partners, squarks, in the fundamental representation. The Lagrangian, in superfield notation, is

$$\begin{aligned} \mathcal{L} = & \frac{1}{g^2} \text{Tr} \left(\int d^4x d^4\theta F^\alpha F_\alpha + \text{h.c.} \right) \\ & + \int d^4x d^4\theta d^4\bar{\theta} \left(\bar{Q}_+ e^{\tilde{\phi}} Q_+ + Q_- e^{-\tilde{\phi}} \bar{Q}_- \right) \\ & + m \left(\int d^4x d^4\theta Q_+ Q_- + \text{h.c.} \right) \end{aligned} \quad (2.1)$$

Here, F_α is the field strength, $F_\alpha = \bar{D}^2(e^{-\phi} D_\alpha e^\phi)$, and $\phi = \phi^a \tau^a$, $\tilde{\phi} = \phi^a T^a$, where $\tau^a(T^a)$ are the generators of the adjoint (fundamental) representation of $SU(N_C)$. The real superfield ϕ^a contains as physical components the gluon field A_μ^a and the gluino field λ^a , $a = 1, \dots, N_C^2 - 1$.

The left and right handed quarks $(q_+, \bar{q}_+)$ and antiquarks $(q_-, \bar{q}_-)$ and their supersymmetric partners are grouped into chiral superfields as follows

$$(Q_\pm)_{i\pm} = \left(s_\pm + \theta^\alpha q_{\pm\alpha} + \theta^2 F_\pm \right)_{i\pm} \quad (2.2)$$

$$(\bar{Q}_{\pm})_{ir} = \left(s_{\pm}^* + \bar{\theta}_{\dot{\alpha}} \bar{q}_{\pm}^{\dot{\alpha}} + \bar{\theta}^2 F_{\pm}^* \right)_{ir} \quad (2.2) \text{ cont.}$$

where $i = 1, \dots, N_f$ is the flavour index, and $r = 1, \dots, N_c$ is the colour index.

In the absence of the mass term, the Lagrangian (2.1) is invariant under the chiral symmetry $SU(N_f)_L \times SU(N_f)_R \times U_V(1) \times U_A(1) \times U_R(1)$. This is the usual invariance of QCD, with the additional $U_R(1)$ chiral symmetry belonging to the superconformal algebra. For a review of the superconformal algebra and the superfield Ward identity formalism used here, and for notation, see Ref. 5).

For simplicity of presentation, the Ward identities and proofs of the mass formulae will be given explicitly for the case of one flavour only. The generalization to N_f flavours is immediate, and the result is stated in Section 4.

First we consider in detail the $U_A(1)$ current, but neglect the axial anomaly, which clearly changes the results. This idealized example is sufficient to illustrate the key features of the derivation of the mass formulae. Then the results are generalized to a chiral symmetry $U_X(1)$, which does not commute with supersymmetry.

The results neglecting the anomaly are relevant for

- i) flavour non-singlet currents, which are not affected by the anomaly, and
- ii) an anomaly-free linear combination $U_X(1)$ of the $U_A(1)$ and $U_R(1)$ currents, if such exists.

A detailed discussion of the precise form of this anomaly-free $U_X(1)$ involves questions of the renormalization of composite operators⁶⁾ in supersymmetry which lie beyond the scope of this paper.

The Ward identity for the $U_A(1)$ transformation

$$Q_{\pm}(x, \theta) \rightarrow e^{i\alpha} Q_{\pm}(x, \theta) \quad \bar{Q}_{\pm}(x, \theta) \rightarrow e^{-i\alpha} \bar{Q}_{\pm}(x, \bar{\theta}) \quad (2.3)$$

in the theory with $m \neq 0$ is, omitting the anomaly,

$$i \partial^{\mu} A_{\mu} = 2 \left(D^2 T - \bar{D}^2 \bar{T} \right) + \omega^A(\underline{x}) \Gamma[\underline{x}] = 0 \quad (2.4)$$

where $T = m Q_+ Q_-$, A_μ is the $U_A(1)$ current, and $\Phi = \{\phi, Q_\pm, \bar{Q}_\pm\}$ denotes generically all the elementary fields in the action Γ . ω^A is the infinitesimal symmetry operator for $U_A(1)$ transformations. Recall from Ref. 5) that for a real field, e.g., ϕ , the global symmetry operator describing the variation of the action under a variation of ϕ is,

$$\begin{aligned} Q^A(\phi) &= \int dx \omega^A(\phi) \\ &= -i \int dx d\theta d\bar{\theta} \delta^A \phi \frac{\delta}{\delta \phi} \end{aligned} \quad (2.5)$$

while for a chiral field Q ,

$$\begin{aligned} Q^A(Q, \bar{Q}) &= \int dx \omega^A(Q, \bar{Q}) \\ &= -i \int dx d\theta \delta^A Q \frac{\delta}{\delta Q} - i \int dx d\bar{\theta} \delta^A \bar{Q} \frac{\delta}{\delta \bar{Q}} \end{aligned} \quad (2.6)$$

To consider Green's functions of T , $\bar{T}$ and A_μ , introduce sources coupling to these operators and construct the generating functional,

$$\begin{aligned} &\exp i W[J_\mu^A, J_T, J_{\bar{T}}] \\ &= \int \mathcal{D}\Phi \exp i \left\{ \Gamma[\Phi] + \int dx d\theta d\bar{\theta} J_\mu^A A^\mu + \int dx d\theta J_T T + \int dx d\bar{\theta} J_{\bar{T}} \bar{T} \right\} \end{aligned} \quad (2.7)$$

The Ward identity for $W[J_\mu^A, J_T, J_{\bar{T}}]$ is,

$$\left\{ i \partial_\mu \frac{\delta}{\delta J_\mu^A} - 2 \left(D^2 \frac{\delta}{\delta J_T} - \bar{D}^2 \frac{\delta}{\delta J_{\bar{T}}} \right) + \omega^A(J_T, J_{\bar{T}}) \right\} W = 0 \quad (2.8)$$

where the symmetry operator for the sources, which follows from the variation of T , $\delta^A T = 2i\alpha T$, under a $U_A(1)$ transformation, is

$$\omega^A(\mathcal{J}_T, \mathcal{J}_{\bar{T}}) = -2 D^2 \left(\mathcal{J}_T \frac{\delta}{\delta \mathcal{J}_T} \right) + 2 \bar{D}^2 \left(\mathcal{J}_{\bar{T}} \frac{\delta}{\delta \mathcal{J}_{\bar{T}}} \right) \quad (2.9)$$

We now derive some relations amongst these Green's functions. First, differentiating Eq. (2.8) with respect to J_T gives, evaluating at the physical source values $J_\mu^A = J_T = J_{\bar{T}} = 0$

$$\begin{aligned} i \not{\partial}^\mu \frac{\delta^2 W}{\delta \mathcal{J}_\mu^A(1) \delta \mathcal{J}_T(2)} &= 2 \left(D_1^2 \frac{\delta^2 W}{\delta \mathcal{J}_T(1) \delta \mathcal{J}_T(2)} - \bar{D}_1^2 \frac{\delta^2 W}{\delta \mathcal{J}_{\bar{T}}(1) \delta \mathcal{J}_T(2)} \right) \\ &- 2 D_1^2 \left(\delta^S(1,2) \frac{\delta W}{\delta \mathcal{J}_T(1)} \right) = 0 \end{aligned} \quad (2.10)$$

where $\delta^S(1,2)$ is the delta function on the chiral subspace (x, θ) . Similarly, differentiating with respect to $J_{\bar{T}}$:

$$\begin{aligned} i \not{\partial}^\mu \frac{\delta^2 W}{\delta \mathcal{J}_\mu^A(1) \delta \mathcal{J}_{\bar{T}}(2)} &= 2 \left(D_1^2 \frac{\delta^2 W}{\delta \mathcal{J}_T(1) \delta \mathcal{J}_{\bar{T}}(2)} - \bar{D}_1^2 \frac{\delta^2 W}{\delta \mathcal{J}_{\bar{T}}(1) \delta \mathcal{J}_{\bar{T}}(2)} \right) \\ &+ 2 \bar{D}_1^2 \left(\delta^{\bar{S}}(1,2) \frac{\delta W}{\delta \mathcal{J}_{\bar{T}}(1)} \right) = 0 \end{aligned} \quad (2.11)$$

On a technical point, notice that in deriving Eqs. (2.10) and (2.11) functional derivatives with different arguments have been assumed to commute, and functional derivatives have been taken through supercovariant derivatives with different arguments. The validity of these manipulations for superfields may be checked by considering simple examples.

Now, subtracting Eq. (2.11) from (2.10) and integrating over x_1 we find the important identity,

$$\begin{aligned}
 & \int dx_1 \quad i \partial_\mu^{(1)} \left(\frac{\delta^2 W}{\delta J_\mu^A(1) \delta J_T(2)} - \frac{\delta^2 W}{\delta J_\mu^A(1) \delta J_{\bar{T}}(2)} \right) \\
 & - 2 \int dx_1 \quad \left(D_1^2 \frac{\delta^2 W}{\delta J_T(1) \delta J_T(2)} - \bar{D}_1^2 \frac{\delta^2 W}{\delta J_T(1) \delta J_T(2)} - D_1^2 \frac{\delta^2 W}{\delta J_T(1) \delta J_{\bar{T}}(2)} + \bar{D}_1^2 \frac{\delta^2 W}{\delta J_{\bar{T}}(1) \delta J_{\bar{T}}(2)} \right) \\
 & \qquad \qquad \qquad (2.12) \\
 & - 2 \left(\frac{\delta W}{\delta J_T(2)} + \frac{\delta W}{\delta J_{\bar{T}}(2)} \right) \\
 & = 0
 \end{aligned}$$

This is a superfield equation. Taking the $\theta = \bar{\theta} = 0$ component, and rewriting in canonical notation, gives

$$\begin{aligned}
 & \int dx_1 \quad i \partial_\mu^{(1)} \langle 0 | \tau \quad A^\mu(1) \quad (T_0 - \bar{T}_0)(2) \quad | 0 \rangle \\
 & - 2 \int dx_1 \quad \langle 0 | \tau \quad (T_F - \bar{T}_F)(1) \quad (T_0 - \bar{T}_0)(2) \quad | 0 \rangle \qquad (2.13) \\
 & = -2i \quad \langle 0 | \quad T_0 + \bar{T}_0 \quad | 0 \rangle
 \end{aligned}$$

where τ denotes the covariant time ordered product. T_0 and $\bar{T}_0$ denote the $\theta = 0$ and $\bar{\theta} = 0$ components of T and $\bar{T}$, respectively, and T_F and $\bar{T}_F$ their θ^2 and $\bar{\theta}^2$ components.

At $m = 0$ the second term is absent and we recover Goldstone's theorem: if the VEV $\langle 0 | T_0 + \bar{T}_0 | 0 \rangle$ is non-zero then there must exist a massless boson coupling to the current A_μ and to $(T_0 - \bar{T}_0)$. When the quark mass $m \neq 0$,

this would-be Goldstone boson is expected to acquire a non-zero mass. Assuming then that there is no massless boson in the spectrum coupling to A_μ , the first term of Eq. (2.13) vanishes, leaving the zero-momentum identity,

$$\begin{aligned} \text{F.T.} \quad & \langle 0 | \tau \left(T_F - \bar{T}_F \right) (1) \left(T_0 - \bar{T}_0 \right) (2) | 0 \rangle \Big|_{k^2=0} \\ & = i \langle 0 | T_0 + \bar{T}_0 | 0 \rangle \end{aligned} \quad (2.14)$$

To derive the second relation needed to obtain the mass formulae, take D_2^2 of Eq. (2.10), subtract $\bar{D}_2^2$ of Eq. (2.11), and again integrate over x_1 . This gives the second important identity,

$$\begin{aligned} & \int dx_1 \quad i \partial_\mu^{(1)} \left(D_2^2 \frac{\delta^2 W}{\delta \mathcal{T}_\mu^{(1)} \delta \mathcal{T}_T^{(2)}} - \bar{D}_2^2 \frac{\delta^2 W}{\delta \mathcal{T}_\mu^{(1)} \delta \mathcal{T}_{\bar{T}}^{(2)}} \right) \\ & - 2 \int dx_1 \left(D_1^2 D_2^2 \frac{\delta^2 W}{\delta \mathcal{T}_T^{(1)} \delta \mathcal{T}_T^{(2)}} - \bar{D}_1^2 D_2^2 \frac{\delta^2 W}{\delta \mathcal{T}_{\bar{T}}^{(1)} \delta \mathcal{T}_T^{(2)}} \right. \\ & \quad \left. - D_1^2 \bar{D}_2^2 \frac{\delta^2 W}{\delta \mathcal{T}_T^{(1)} \delta \mathcal{T}_{\bar{T}}^{(2)}} + \bar{D}_1^2 \bar{D}_2^2 \frac{\delta^2 W}{\delta \mathcal{T}_{\bar{T}}^{(1)} \delta \mathcal{T}_{\bar{T}}^{(2)}} \right) \\ & - 2 \left(D_2^2 \frac{\delta W}{\delta \mathcal{T}_T^{(2)}} + \bar{D}_2^2 \frac{\delta W}{\delta \mathcal{T}_{\bar{T}}^{(2)}} \right) \\ & = 0 \end{aligned} \quad (2.15)$$

Again taking the $\theta = \bar{\theta} = 0$ component and rewriting in canonical notation,

$$\int dx_1 \quad i \partial_\mu^{(1)} \langle 0 | \tau A^\mu (1) \left(T_F - \bar{T}_F \right) (2) | 0 \rangle \quad (2.16)$$

$$\begin{aligned}
 & - 2 \int dx_1 \langle 0 | \tau \left(T_F - \bar{T}_F \right) (1) \left(T_F - \bar{T}_F \right) (2) | 0 \rangle \\
 & = - 2i \langle 0 | T_F + \bar{T}_F | 0 \rangle
 \end{aligned}
 \tag{2.16}$$

cont.

Making the same assumption as above on the absence of a massless boson then gives,

$$\begin{aligned}
 & \text{F.T. } \langle 0 | \tau \left(T_F - \bar{T}_F \right) (1) \left(T_F - \bar{T}_F \right) (2) | 0 \rangle \Big|_{k^2=0} \\
 & = i \langle 0 | T_F + \bar{T}_F | 0 \rangle
 \end{aligned}
 \tag{2.17}$$

As shown in the following section, Dashen's formula for the mass of a pseudo-Goldstone boson follows from Eq. (2.17). The new formula follows in an analogous way from Eq. (2.14).

In his original derivation¹⁾, Dashen considered the two-current correlation function. To make contact with this method, we quote here the Ward identity relation for this Green's function:

$$\begin{aligned}
 & \partial_\mu^{(1)} \partial_\lambda^{(2)} \frac{\delta^2 W}{\delta J_\mu^A(1) \delta J_\lambda^A(2)} \\
 & - 4 \left(J_1^2 J_2^2 \frac{\delta^2 W}{\delta J_T(1) \delta J_T(2)} - \bar{J}_1^2 \bar{J}_2^2 \frac{\delta^2 W}{\delta J_{\bar{T}}(1) \delta J_{\bar{T}}(2)} \right. \\
 & \quad \left. - J_1^2 \bar{J}_2^2 \frac{\delta^2 W}{\delta J_T(1) \delta J_{\bar{T}}(2)} + \bar{J}_1^2 J_2^2 \frac{\delta^2 W}{\delta J_{\bar{T}}(1) \delta J_T(2)} \right) \\
 & = 4 \left(J_1^2 J_2^2 \left(\delta^S(1,2) \frac{\delta W}{\delta J_T(2)} \right) + \bar{J}_1^2 \bar{J}_2^2 \left(\delta^{\bar{S}}(1,2) \frac{\delta W}{\delta J_{\bar{T}}(2)} \right) \right) \\
 & = 0
 \end{aligned}
 \tag{2.18}$$

At zero momentum, the first term vanishes, and we recover the identity (2.17). Notice though that this approach fails to uncover Eq. (2.14), which gives us an independent mass formula in the supersymmetric theory.

In passing, we may also observe the elegance of the functional method compared to the older commutator techniques. In particular, there are no subtleties regarding covariance, and strong emphasis is placed on the Ward identity as the fundamental physical input.

Now consider a chiral symmetry $U_X(1)$ which does not commute with supersymmetry. Then, the matter fields transform as

$$Q_{\pm}(x, \theta) \rightarrow e^{i n_X \alpha} Q_{\pm}(x, \theta e^{i \alpha}) \quad \bar{Q}_{\pm}(x, \bar{\theta}) \rightarrow e^{-i n_X \alpha} \bar{Q}_{\pm}(x, \bar{\theta} e^{-i \alpha}) \quad (2.19)$$

If this were to be the $U_R(1)$ symmetry of the superconformal algebra, the chiral weight would be constrained to be $n_R = 2/3 d$, where d is the length dimension of the chiral superfield $Q_{\pm}$. More generally, we can consider a $U_X(1)$ transformation which is a linear combination of this $U_R(1)$ and the $U_A(1)$ transformation. In this case, the chiral weight n_X is determined by the particular linear combination chosen.

The Ward identity is,

$$i \partial^{\mu} X_{\mu} - (n_X(\tau) + 2) (D^2 \tau - \bar{D}^2 \bar{\tau}) + \omega^X(\tau, \bar{\tau}) \Gamma[\tau] = 0 \quad (2.20)$$

where X_{μ} is the current, and so the generating functional $W[J_{\mu}^X, J_{\tau}, J_{\bar{\tau}}]$ satisfies

$$\left\{ i \partial^{\mu} \frac{\delta}{\delta J_{\mu}^X} - (n_X(\tau) + 2) \left(D^2 \frac{\delta}{\delta J_{\tau}} - \bar{D}^2 \frac{\delta}{\delta J_{\bar{\tau}}} \right) + \omega^X(\tau, \bar{\tau}) \right\} W = 0 \quad (2.21)$$

The symmetry operator $\omega^X(J_{\tau}, J_{\bar{\tau}})$ can be written [see Ref. 5)] as

$$\omega^X(\tau, \bar{\tau}) = -(n_X(\tau) + 1) D^2 \left(\tau \frac{\delta}{\delta J_{\tau}} \right) + \left(D^2 \tau \right) \frac{\delta}{\delta J_{\tau}} - \tau \bar{D}^2 \frac{\delta}{\delta J_{\bar{\tau}}} \quad (2.22)$$

- h.c.

Proceeding as before, we find the analogue of Eq. (2.12),

$$\begin{aligned}
 & i \partial_{\mu}^{(1)} \left(\frac{\delta^2 W}{\delta \mathcal{T}_{\mu}^{(1)} \delta \mathcal{T}_{\tau}(2)} - \frac{\delta^2 W}{\delta \mathcal{T}_{\mu}^{(1)} \delta \mathcal{T}_{\bar{\tau}}(2)} \right) \\
 & - (n_X(\tau) + 2) \left(\mathcal{D}_1^2 \frac{\delta^2 W}{\delta \mathcal{T}_{\tau}(1) \delta \mathcal{T}_{\tau}(2)} - \bar{\mathcal{D}}_1^2 \frac{\delta^2 W}{\delta \mathcal{T}_{\bar{\tau}}(1) \delta \mathcal{T}_{\tau}(2)} - \mathcal{D}_1^2 \frac{\delta^2 W}{\delta \mathcal{T}_{\tau}(1) \delta \mathcal{T}_{\bar{\tau}}(2)} + \bar{\mathcal{D}}_1^2 \frac{\delta^2 W}{\delta \mathcal{T}_{\bar{\tau}}(1) \delta \mathcal{T}_{\bar{\tau}}(2)} \right) \\
 & - (n_X(\tau) + 1) \left(\mathcal{D}_1^2 \left(\delta^S(1,2) \frac{\delta W}{\delta \mathcal{T}_{\tau}(1)} \right) + \bar{\mathcal{D}}_1^2 \left(\delta^{\bar{S}}(1,2) \frac{\delta W}{\delta \mathcal{T}_{\bar{\tau}}(1)} \right) \right) \\
 & + \mathcal{D}_1^2 \left(\delta^S(1,2) \right) \frac{\delta W}{\delta \mathcal{T}_{\tau}(1)} + \bar{\mathcal{D}}_1^2 \left(\delta^{\bar{S}}(1,2) \right) \frac{\delta W}{\delta \mathcal{T}_{\bar{\tau}}(1)} \\
 & - \delta^S(1,2) \mathcal{D}_1^2 \frac{\delta W}{\delta \mathcal{T}_{\tau}(1)} - \delta^{\bar{S}}(1,2) \bar{\mathcal{D}}_1^2 \frac{\delta W}{\delta \mathcal{T}_{\bar{\tau}}(1)} \\
 & = 0
 \end{aligned} \tag{2.23}$$

Now taking the $\theta = \bar{\theta} = 0$ component using the properties,

$$\mathcal{D}_1^2 \delta^S(1,2) \Big|_{\theta=\bar{\theta}=0} = \delta(x_1, x_2) \tag{2.24}$$

$$\delta^S(1,2) = -\frac{1}{4} (\theta_1 - \theta_2)^2 \delta(x_1, x_2)$$

the previous relation (2.13) is recovered, up to numerical factors:

$$\begin{aligned}
 & \int dx_1 i \partial_{\mu}^{(1)} \langle 0 | \tau X^{\mu}(1) (\tau_0 - \bar{\tau}_0)(2) | 0 \rangle \\
 & - (n_X(\tau) + 2) \int dx_1 \langle 0 | \tau (\tau_F - \bar{\tau}_F)(1) (\tau_0 - \bar{\tau}_0)(2) | 0 \rangle
 \end{aligned} \tag{2.25}$$

$$= -i n_x(\tau) \langle 0 | T_0 + \bar{T}_0 | 0 \rangle \quad (2.25)$$

cont.

The derivation of the second identity is straightforward, and we find

$$\begin{aligned} & \int dx_1 i \partial_\mu^{(1)} \langle 0 | \tau X^\mu(1) (T_F - \bar{T}_F)(2) | 0 \rangle \\ &= (n_x(\tau) + 2) \int dx_1 \langle 0 | \tau (T_F - \bar{T}_F)(1) (T_F - \bar{T}_F)(2) | 0 \rangle \quad (2.26) \\ &= -i (n_x(\tau) + 2) \langle 0 | T_F + \bar{T}_F | 0 \rangle \end{aligned}$$

3. - PSEUDO-GOLDSTONE BOSON MASS FORMULAE

3.1 The new supersymmetric variant of Dashen's formula

The basic idea underlying the derivation of mass formulae along the lines originally proposed by Dashen¹⁾ is to dominate the Green's functions appearing in the Ward identities for current correlation functions with the contribution from pseudo-Goldstone boson intermediate states. In this way we obtain formulae, valid to lowest order in the symmetry breaking parameter, relating the decay constants and masses of the pseudo-Goldstone bosons to the VEV of the order parameters associated with the spontaneously broken symmetry.

Now consider the finite momentum version of the Ward identity (2.13). Inserting a single pion intermediate state and comparing residues of the pole at $k^2 = m_\pi^2$, we conclude, defining the pion decay constant by

$$\langle 0 | A_\mu | \pi \rangle = i k_\mu F_\pi \quad (3.1)$$

that

$$\langle 0 | T_F - \bar{T}_F | \pi \rangle = \frac{i}{2} F_\pi m_\pi^2 \quad (3.2)$$

This relation is generally valid. To derive the new variant of Dashen's formula, however, we need a further relation, whose derivation requires the assumption that supersymmetry is not spontaneously broken. This is,

$$\langle 0 | T_0 - \bar{T}_0 | \pi \rangle = \frac{i}{2} F_\pi m_\pi \quad (3.3)$$

To see this, we need the following results for the action of the supersymmetry generator Q_α on the components (ϕ, ψ, F) of a chiral superfield [see Ref. 5), Appendix A]:

$$\begin{aligned} [Q_\alpha, \phi] &= -i \psi_\alpha & [Q_\alpha, \phi^*] &= 0 \\ \{Q_\alpha, \psi_\beta\} &= -2i \epsilon_{\alpha\beta} F & \{Q_\alpha, \bar{\psi}_{\dot{\alpha}}\} &= -4 \partial_{\alpha\dot{\alpha}} \phi^* \\ [Q_\alpha, F] &= 0 & [Q_\alpha, F^*] &= 2 \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \end{aligned} \quad (3.4)$$

Assuming $Q_\alpha |0\rangle = 0$ then allows the following sequence of identities to be proved

$$\begin{aligned} \langle 0 | T F(x) F^*(y) | 0 \rangle &= -\frac{i}{2} \langle 0 | T \psi_\alpha(x) \partial^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}(y) | 0 \rangle \\ &= -\langle 0 | T \phi(x) \partial^2 \phi^*(y) | 0 \rangle \end{aligned} \quad (3.5)$$

Also,

$$\langle 0 | T F(x) F(y) | 0 \rangle = 0 \quad (3.6)$$

$$\langle 0 | T \phi(x) \partial^2 \phi(y) | 0 \rangle = 0 \quad (3.7)$$

In the case of interest here, these identities imply that

$$\langle 0 | T (T_F - \bar{T}_F)^{(1)} (T_F - \bar{T}_F)^{(2)} | 0 \rangle = -\langle 0 | T (T_0 - \bar{T}_0)^{(1)} \partial_{\alpha\dot{\alpha}}^2 (T_0 - \bar{T}_0)^{(2)} | 0 \rangle \quad (3.8)$$

Inserting the pion intermediate state and comparing residues gives the required relation,

$$|\langle 0 | T_F - \bar{T}_F | \pi \rangle|^2 = m_\pi^2 |\langle 0 | T_0 - \bar{T}_0 | \pi \rangle|^2 \quad (3.9)$$

A similar relation holds for the coupling of the degenerate fermionic superpartner of the pion to the spinor components of $T, \bar{T}$.

With this relation, the new mass formula follows directly from the zero-momentum limit of the Ward identity, Eq. (2.14). The leading contribution to the Green's function on the left-hand side for small quark mass m comes from the pole term from the pion intermediate state. Neglecting higher order corrections in m , we find immediately

$$\frac{1}{m_\pi^2} \langle 0 | T_F - \bar{T}_F | \pi \rangle \langle \pi | T_0 - \bar{T}_0 | 0 \rangle = - \langle 0 | T_0 + \bar{T}_0 | 0 \rangle \quad (3.10)$$

that is,

$$\frac{1}{4} F_\pi^2 m_\pi = \langle 0 | T_0 + \bar{T}_0 | 0 \rangle \quad (3.11)$$

in agreement with Ref. 4).

3.2 Counting operators and Dashen's formula in SQCD

If we now look at the second identity, Eq. (2.16), and insert the pion intermediate state as before, we are tempted to the naive conclusion that $\frac{1}{4} F_\pi^2 m_\pi^2$ is equal to the VEV $\langle 0 | T_F + \bar{T}_F | 0 \rangle$, which, being an F term, must vanish unless supersymmetry is spontaneously broken. This conclusion is false. The correct argument is rather subtle, involves non-trivial properties of composite operators, and sheds a lot of light on how auxiliary fields must be treated in complicated Green's functions.

First consider the assertion that $\langle 0 | T_F - \bar{T}_F | \pi \rangle = \frac{i}{2} F_\pi m_\pi^2$ if $\langle 0 | A_\mu | \pi \rangle = i k_\mu F_\pi$. A correct derivation of this by comparing residues of the pion pole in the Ward identity was given in Section 3.1. However, this type of relation would normally be explained as a direct consequence of the softly broken current conservation equation $i \partial^\mu A_\mu = 2(T_F - \bar{T}_F)$. But this only holds using the equations of motion. The correct equation is, in superfield notation,

$$i \partial^\mu A_\mu - 2 \left(D^2 T - \bar{D}^2 \bar{T} \right) + \omega^A(\mathbb{E}) \Gamma(\mathbb{E}) = 0 \quad (2.4)$$

The additional terms involve operators of the form $\Phi \delta\Gamma/\delta\Phi$. Such operators form a subset of the class of gauge invariant composite operators which vanish by the equations of motion ($\delta\Gamma/\delta\Phi = 0$), i.e., Class IIa, in the language of the renormalization theory of composite operators⁶⁾. A brief discussion of such operators is given in Appendix B. Here we need just two basic theorems.

Theorem 1 The on-shell matrix elements of all renormalized composite operators of Class IIa at non-zero momentum vanish.

This is a remarkable theorem. It says that if we insert an operator which vanishes using the equations of motion for the elementary fields into a matrix element involving a bound state, then the matrix element with the insertion vanishes when the bound state particle is taken to be on-shell.

Referring back to Eq. (2.4) above, we see now that the final term vanishes in on-shell matrix elements, and so in particular gives no contribution to the decay constant relation. We have, on the pion mass shell,

$$\langle 0 | i \partial^\mu A_\mu | \pi \rangle = 2 \langle 0 | T_F - \bar{T}_F | \pi \rangle \quad (3.12)$$

The second theorem applies only the subset of Class IIa operators of the form $\Phi \delta\Gamma/\delta\Phi$.

Theorem 2 A zero-momentum insertion of the renormalized composite operator $-i\Phi \delta\Gamma/\delta\Phi$ into a Green's function involving Φ counts the number of external Φ legs.

We shall therefore refer to such operators from now on as "counting operators". Note that this property is true only for insertions at zero momentum.

We now apply these general ideas to SQCD. The discussion from now on will be in terms of component fields. From Eq. (2.1) the terms in the Lagrangian involving the auxiliary fields of $Q_\pm, \bar{Q}_\pm$ are,

$$\begin{aligned} \mathcal{L}_{aux} = \int dx \left(F_+ F_+^* + F_- F_-^* \right. \\ \left. + m(s_+ F_- + F_+ s_- + q_+ q_-) + m(s_+^* F_-^* + F_+^* s_-^* + \bar{q}_+ \bar{q}_-) \right) \end{aligned} \quad (3.13)$$

The equations of motion are $\delta\Gamma/\delta F_\pm = 0, \delta\Gamma/\delta F_\pm^* = 0$, where

$$\frac{\delta \Gamma}{\delta F_{\pm}} = F_{\pm}^* + m s_{\pm} \quad \frac{\delta \Gamma}{\delta F_{\pm}^*} = F_{\pm} + m s_{\pm}^* \quad (3.14)$$

Recalling that $T = m Q_+ Q_-$, its components are

$$T_F = m (s_+ F_- + F_+ s_- + q_+ q_-) \quad (3.15)$$

$$T_0 = m s_+ s_-$$

and so,

$$\begin{aligned} T_F - \bar{T}_F &= m (q_+ q_- - \bar{q}_+ \bar{q}_-) \\ &+ F_+ \frac{\delta \Gamma}{\delta F_+} + F_- \frac{\delta \Gamma}{\delta F_-} - F_+^* \frac{\delta \Gamma}{\delta F_+^*} - F_-^* \frac{\delta \Gamma}{\delta F_-^*} \end{aligned} \quad (3.16)$$

The crucial point here is that the bosonic terms in $(T_F - \bar{T}_F)$ can be entirely written in terms of auxiliary field counting operators.

It follows immediately from Theorem 1 that these terms do not contribute to the decay constant matrix element. Thus,

$$\langle 0 | T_F - \bar{T}_F | \pi \rangle = m \langle 0 | q_+ q_- - \bar{q}_+ \bar{q}_- | \pi \rangle \quad (3.17)$$

and combining with Eq. (3.2) gives

$$m \langle 0 | q_+ q_- - \bar{q}_+ \bar{q}_- | \pi \rangle = \frac{i}{2} F_{\pi} m_{\pi}^2 \quad (3.18)$$

Now, Dashen's formula arises from the zero-momentum Ward identity (2.17). The property that the bosonic fields enter this Green's function only as counting operators enables us to simplify this identity considerably. In general, from Theorem 2,

$$-i \langle 0 | \tau \left(F_+ \frac{\delta \Gamma}{\delta F_+} + F_- \frac{\delta \Gamma}{\delta F_-} - F_+^* \frac{\delta \Gamma}{\delta F_+^*} - F_-^* \frac{\delta \Gamma}{\delta F_-^*} \right) \Big|_{k^2=0} \odot (F_{\pm}, F_{\pm}^*, \dots) | 0 \rangle \quad (3.19)$$

$$= \left(n_{F_+}(\mathcal{O}) + n_{F_-}(\mathcal{O}) - n_{F_+^*}(\mathcal{O}) - n_{F_-^*}(\mathcal{O}) \right) \langle 0 | \tau \mathcal{O}(F_+, F_+^*, \dots) | 0 \rangle$$

(3.19)
cont.

where $\langle 0 | \tau \mathcal{O} | 0 \rangle$ is a Green's function involving amongst others the auxiliary fields, and $n_{F_+}(\mathcal{O})$ denotes the number of times the operator F_+ appears in $\mathcal{O}$.

So, evaluating the left-hand side of Eq. (2.17),

$$\begin{aligned} & -i \langle 0 | \tau (T_F - \bar{T}_F) (T_F - \bar{T}_F) | 0 \rangle \Big|_{k^2=0} \\ &= -i \langle 0 | \tau m^2 (q_+ q_- - \bar{q}_+ \bar{q}_-) (q_+ q_- - \bar{q}_+ \bar{q}_-) | 0 \rangle \Big|_{k^2=0} \\ & \quad + \left(n_{F_+}(T_F) + n_{F_-}(T_F) - n_{F_+^*}(T_F) - n_{F_-^*}(T_F) \right) \langle 0 | T_F | 0 \rangle_{bos} \\ & \quad - \left(n_{F_+}(\bar{T}_F) + n_{F_-}(\bar{T}_F) - n_{F_+^*}(\bar{T}_F) - n_{F_-^*}(\bar{T}_F) \right) \langle 0 | \bar{T}_F | 0 \rangle_{bos} \end{aligned}$$

(3.20)

$$\begin{aligned} &= -i \langle 0 | \tau m^2 (q_+ q_- + \bar{q}_+ \bar{q}_-) (q_+ q_- - \bar{q}_+ \bar{q}_-) | 0 \rangle \Big|_{k^2=0} \\ & \quad + \left(\langle 0 | T_F | 0 \rangle_{bos} + \langle 0 | \bar{T}_F | 0 \rangle_{bos} \right) \end{aligned}$$

Substituting back into Eq. (2.17) then shows that at zero momentum the bosonic terms cancel, leaving

$$\begin{aligned} & m^2 \langle 0 | \tau (q_+ q_- - \bar{q}_+ \bar{q}_-) (q_+ q_- - \bar{q}_+ \bar{q}_-) | 0 \rangle \Big|_{k^2=0} \\ &= im \langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \end{aligned}$$

(3.21)

Inserting the pion intermediate state into this Green's function and using the relation (3.18) for the decay constant, we recover the supersymmetric analogue of Dashen's formula in QCD,

$$\frac{1}{4} F_{\pi}^2 m_{\pi}^2 = m \langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \quad (3.22)$$

as a relation involving only the quark fields.

Finally, in the light of these ideas, we can look at the component form of the new formula, Eq. (3.11). Provided supersymmetry is unbroken, we have

$$\begin{aligned} \langle 0 | T_0 - \bar{T}_0 | \pi \rangle &= m \langle 0 | s_+ s_- - s_+^* s_-^* | \pi \rangle \\ &= \frac{i}{2} F_{\pi} m_{\pi} \end{aligned} \quad (3.23)$$

as shown in Section 3.1. The pion therefore couples also to the squark fields.

Now in the zero-momentum Ward identity (2.14), the bosonic terms in $(T_F - \bar{T}_F)$ do not contribute, since they are counting operators for the auxiliary fields while the expression $(T_0 - \bar{T}_0)$ involves only the squark fields. So, in components,

$$m^2 \langle 0 | \tau (q_+ q_- - \bar{q}_+ \bar{q}_-) (s_+ s_- - s_+^* s_-^*) | 0 \rangle \Big|_{k^2=0} = i m \langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle \quad (3.24)$$

and inserting the pion gives,

$$\frac{1}{4} F_{\pi}^2 m_{\pi}^2 = m \langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle \quad (3.25)$$

We therefore find two relations in SQCD involving the pion decay constant and mass, one in terms of the familiar quark condensate, Eq. (3.22), and a new one, Eq. (3.25), involving a chirality breaking condensate of squark fields.

3.3 Mass formulae with the $U_X(1)$ symmetry

Before discussing the implications of Eqs. (3.22) and (3.25), we pause to collect the corresponding results for a chiral symmetry $U_X(1)$ which does not commute with supersymmetry. These follow from the Ward identities (2.25) and (2.26), and differ from the corresponding $U_A(1)$ formulae only by numerical factors.

To extract these formulae, recall the on-shell identity,

$$\langle 0 | i \partial^\mu X_\mu | \pi \rangle = (n_X(\tau) + 2) \langle 0 | T_F - \bar{T}_F | \pi \rangle \quad (3.26)$$

so defining F_π by,

$$\langle 0 | X_\mu | \pi \rangle = i k_\mu F_\pi \quad (3.27)$$

we have

$$\langle 0 | T_F - \bar{T}_F | \pi \rangle = m \langle 0 | q_+ q_- - \bar{q}_+ \bar{q}_- | \pi \rangle = \frac{i}{n_X(\tau) + 2} F_\pi m_\pi^2 \quad (3.28)$$

and

$$\langle 0 | T_0 - \bar{T}_0 | \pi \rangle = m \langle 0 | s_+ s_- - s_+^* s_-^* | \pi \rangle = \frac{i}{n_X(\tau) + 2} F_\pi m_\pi^2 \quad (3.29)$$

assuming unbroken supersymmetry. Referring to Eqs. (2.25) and (2.26), we find in this case,

$$F_\pi^2 m_\pi^2 = (n_X(\tau) + 2)^2 m \langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \quad (3.30)$$

and

$$F_\pi^2 m_\pi^2 = n_X(\tau) (n_X(\tau) + 2) m \langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle \quad (3.31)$$

3.4 Condensates

The main results of the analysis so far have been the two simple identities (3.22) and (3.25) for the $U_A(1)$ symmetry, and their generalizations (3.30) and (3.31) for $U_X(1)$. In this section we shall discuss a little of the physics of these formulae.

First, notice that if supersymmetry is not spontaneously broken then the VEV $\langle 0 | T_F + \bar{T}_F | 0 \rangle$ must be zero. This implies the identity

$$\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle = 2m \langle 0 | s_+ s_+^* + s_- s_-^* | 0 \rangle \quad (3.32)$$

in a supersymmetric vacuum. To prove this, we can rewrite the bosonic part of $\langle 0 | T_F + \bar{T}_F | 0 \rangle$ as follows,

$$\langle 0 | m (s_+ F_- + F_+ s_- + s_+^* F_-^* + F_+^* s_-^*) | 0 \rangle \quad (3.33)$$

$$= \langle 0 | m \left(s_+ \frac{\delta \Gamma}{\delta F_-^*} + s_- \frac{\delta \Gamma}{\delta F_+^*} + s_+^* \frac{\delta \Gamma}{\delta F_-} + s_-^* \frac{\delta \Gamma}{\delta F_+} \right) | 0 \rangle - 2m^2 \langle 0 | s_+ s_+^* + s_- s_-^* | 0 \rangle$$

The first set of terms all give zero, as can be seen immediately by writing the VEVs as functional integrals, e.g.,

$$\langle 0 | s_+ \frac{\delta \Gamma}{\delta F_-^*} | 0 \rangle = -i \int d\Phi s_+ \frac{\delta}{\delta F_-^*} e^{i\Gamma[\Phi]} \quad (3.34)$$

where Φ is the set of all component fields. Integrating by parts the right-hand side is clearly zero, provided as usual that surface terms in the functional integral can be neglected.

The same argument applies at zero quark mass, where the final term in Eq. (3.33) is absent and we find that in a supersymmetric vacuum,

$$\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle = 0 \quad (3.35)$$

The question of whether a quark bilinear condensate breaks supersymmetry is clearly resolved. Only for zero quark mass does $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \neq 0$ imply broken supersymmetry. For non-zero mass, it is perfectly consistent to have $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \neq 0$ with supersymmetry unbroken, as seen from Eq. (3.32).

However, we must distinguish carefully between the behaviour of the theory at $m = 0$, and its behaviour in the limit $m \rightarrow 0$. It is quite possible for instance that $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle|_{m=0}$ and $\lim_{m \rightarrow 0} \langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle$ are different.

In particular, we cannot conclude that unbroken supersymmetry implies $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle = 0$ in the limit $m \rightarrow 0$, without assuming that the squark condensate $\langle 0 | s_+ s_+^* + s_- s_-^* | 0 \rangle$ is at least as regular as m^{-1} . It is not necessarily unreasonable that $\langle 0 | s_+ s_+^* + s_- s_-^* | 0 \rangle$ should be singular as $m \rightarrow 0$ ⁴⁾.

A model in which the behaviour at $m = 0$ and in the limit $m \rightarrow 0$ are different has been given by Peskin ⁴⁾. This model has two candidate vacua. In the first, the quark condensate $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle$ is $O(\Lambda^3)$, as in conventional QCD, and so we know immediately from Eq. (3.35) that at $m = 0$

supersymmetry is broken in this vacuum. In the second, supersymmetry is unbroken, ensuring agreement with the index theorem, but for small m , $\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \sim \nu m^{-3}$ and $\langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle \sim m^{-1}$. So, in this vacuum, both condensates are singular in the limit $m \rightarrow 0$, implying, according to our formulae, a singular behaviour of m_π . In fact $F_\pi \sim m$, $m_\pi \sim m^{-2}$ for small m . The second vacuum is apparently energetically favoured for all $m \neq 0$, so the naive picture that emerges is one of unbroken supersymmetry for all finite m , but a phase transition at $m = 0$ into a supersymmetry breaking vacuum. The details of this picture could, however, be changed for small m due to dynamical effects in the supersymmetric vacuum.

Returning to the two Dashen-type formulae (3.22) and (3.25), we see now that there are two chiral symmetry breaking condensates relevant in SQCD, viz.

$\langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle$ and $\langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle$. The values of these two independent condensates determine the physical parameters F_π and m_π . This is in contrast to ordinary QCD, where only the combination $F_\pi^2 m_\pi^2$ is fixed by a condensate.

Of course, we can combine the formulae (3.22), (3.25) and (3.32) in various ways. One expression of particular interest may be the following for the pion mass in terms of the chirality breaking condensates, not involving the quark mass parameter explicitly,

$$m_\pi = \langle 0 | q_+ q_- + \bar{q}_+ \bar{q}_- | 0 \rangle \left(\langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle \right)^{-1} \quad (3.36)$$

An alternative formula, relating the pion mass to a ratio of squark condensates, is

$$m_\pi = 2m \langle 0 | s_+ s_+^* + s_- s_-^* | 0 \rangle \left(\langle 0 | s_+ s_- + s_+^* s_-^* | 0 \rangle \right)^{-1} \quad (3.37)$$

Clearly it would be very important to our understanding of the chiral limit of SQCD^{3),7),8)} to develop means of reliably estimating the m dependence of these basic condensates. Some results for the squark condensate have already been suggested⁸⁾ and their implications for the new formula have been discussed⁴⁾.

The existence of two independent condensates in SQCD is one way to understand why we find here two formulae of the Dashen type. Alternatively, it is a reflection of the fact that the pion in SQCD is significantly different from the QCD pion, being a bound state of both quark-antiquark and squark pairs. The two formulae describe the different couplings of the pion to the quark and squark fields.

4. - GENERALIZATIONS

The basic results (3.22), (3.25) and (3.30), (3.31) have been derived here for the special case of one-flavour SQCD. They can, however, be immediately generalized to the case of a non-Abelian chiral symmetry and a multiplet of pion states.

Suppose then that there exists a non-anomalous, partially conserved current J_μ^a corresponding to a spontaneously broken symmetry transformation generated by charges $Q^a = \int d^3x J_0^a$, which commutes with supersymmetry. Define the decay constants,

$$\langle 0 | J_\mu^a | \pi_b \rangle = i k_\mu F_\pi^a \delta^{ab} \quad (4.1)$$

where $|\pi_a\rangle$ denotes the multiplet of pseudo-Goldstone boson states. For the case of chiral $SU(N_F)_L \times SU(N_F)_R$ breaking to diagonal $SU(N_F)$, the index a runs from 1 to $N_F^2 - 1$.

Recalling that in the derivation of Eqs. (3.22) and (3.25), Green's functions of the divergence of the partially conserved current were studied, and that this divergence is simply the variation of the Lagrangian under the required transformation, we see that the generalization of our formulae are,

$$F_\pi^a (m_\pi^2)_{ab} F_\pi^b = \langle 0 | [Q^a, [Q^b, \hat{\mathcal{L}}_F]] | 0 \rangle \quad (4.2)$$

and

$$F_\pi^a (m_\pi)_{ab} F_\pi^b = \langle 0 | [Q^a, [Q^b, \mathcal{L}_0]] | 0 \rangle \quad (4.3)$$

(no sum on a, b). Here $\mathcal{L}_0$ denotes the lowest ($\theta = \bar{\theta} = 0$) component of the Lagrangian, or equivalently just its symmetry breaking part. Similarly, $\hat{\mathcal{L}}_F$ denotes its F component, but modified to omit terms involving the auxiliary fields. The mechanism of cancellation of these terms from the zero-momentum Ward identity displayed in Section 3.2 generalizes straightforwardly.

To interpret these formulae, note that as the mass matrix need not in general be diagonal in the space of $|\pi_a\rangle$ states defined by Eq. (4.1), the symbols $(m_\pi^2)_{ab}$ and $(m_\pi)_{ab}$ in Eqs. (4.2) and (4.3) are to be understood as the matrix elements of the mass-squared and mass operators, respectively, in the space of pion states $|\pi_a\rangle$.

Finally, if the transformation generated by Q^a does not commute with supersymmetry, the formulae (4.2) and (4.3) are changed by chiral weight factors in analogy with the results (3.30) and (3.31) for the $U_X(1)$ symmetry.

ACKNOWLEDGEMENTS

I would like to thank G. Veneziano for initially suggesting to me that a second Dashen-type formula might exist, and for many interesting conversations. I am also grateful to P. Stevenson for useful discussions.

APPENDIX A - COMMENTS ON THE SUPERFIELD METHOD

A few comments on the general approach of using superfield Ward identities may be helpful. An alternative way of deriving component field results such as (3.21) or (3.24) would be to eliminate the auxiliary fields right at the start, work in components, and consider the Ward identities arising from the chiral transformation $s_{\pm} \rightarrow e^{i\alpha} s_{\pm}$, $q_{\pm} \rightarrow e^{i\alpha} q_{\pm}$. This is essentially the approach followed by Veneziano⁴⁾ in his derivation. While of course this method, used carefully, gives the same results for physical quantities, it suffers from two technical problems.

First, by eliminating auxiliary fields, one is left with a supersymmetry algebra which only closes on shell. This would create difficulties in any calculation involving the (anti-) commutators of supersymmetry charges with themselves, or other generators of the superconformal algebra [e.g., $U_R(1)$ transformations].

The second point is that after eliminating the auxiliary fields the above chiral transformation on $s_{\pm}$, $q_{\pm}$ does not commute with supersymmetry. To see this, consider the equation of motion for the auxiliary fields, $F_{\pm} = -ms_{\pm}^*$. This is not invariant under the transformation $Q_{\pm} \rightarrow e^{i\alpha} Q_{\pm}$, but respects an alternative transformation where $s_{\pm} \rightarrow e^{i\alpha} s_{\pm}$, $q_{\pm} \rightarrow e^{i\alpha} q_{\pm}$, $F_{\pm} \rightarrow e^{-i\alpha} F_{\pm}$. The method of eliminating the auxiliary fields essentially amounts to using this second assignment of the chiral transformation for $F_{\pm}$, which clearly does not respect supersymmetry. The Ward identities so derived differ in general from those in the superfield approach, although the physical consequences are the same.

The main advantage of the methods used here is that supersymmetry remains clear and manifest. The price to be paid is learning how to handle auxiliary fields properly when they occur in complicated Green's functions involving composite operators. It will be clear from the calculations in the text that only in very special circumstances is it correct to eliminate auxiliary fields from Green's functions using their equations of motion. However, once understood, the methodology involving properties of counting operators is simple and unambiguous, and manipulations involving auxiliary fields can be made without difficulty.

APPENDIX B - COMPOSITE OPERATORS

In this appendix, we discuss briefly the two theorems concerning "counting operators" of the form $\mathcal{O}(x) = -i \phi \delta\Gamma/\delta\phi$ that were used in Section 3.2. For simplicity we just consider here a theory with one field ϕ .

The essential property of such an operator is that when inserted into a Green's function it merely changes the momentum of the external legs. To be precise, defining

$$G(p_1, \dots, p_n) = \text{F.T.} \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle \quad (\text{B.1})$$

and

$$G_0(q; p_1, \dots, p_n) = \text{F.T.} \langle 0 | T \phi(x_1) \dots \phi(x_n) \mathcal{O}(x) | 0 \rangle \quad (\text{B.2})$$

then

$$G_0(q; p_1, \dots, p_n) = \sum_{i=1}^n G(p_1, \dots, p_i + q, \dots, p_n) \quad (\text{B.3})$$

That this is true may be seen diagrammatically⁹⁾ by inserting the operator $\mathcal{O}$ with momentum q into a multiloop diagram contributing to G . The contributions from insertions on internal lines and vertices cancel because of the special form of $\mathcal{O}$, $\delta\Gamma/\delta\phi$ being the field equation operator. The only surviving contribution comes from the free part of $\mathcal{O}$ acting on an external leg.

We can however give an elegant formal derivation using functionals:

$$\begin{aligned} G_0(q; p_1, \dots, p_n) &= - \prod_{i=1}^n \int dx_i e^{i p_i \cdot x_i} \int dx e^{i q \cdot x} \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) \phi(x) \frac{\delta \Gamma}{\delta \phi(x)} e^{i \Gamma} \\ &= \prod_{i=1}^n \int dx_i e^{i p_i \cdot x_i} \int dx e^{i q \cdot x} \int \mathcal{D}\phi \sum_i \delta(x - x_i) \phi(x_1) \dots \phi(x_{i-1}) \phi(x) \phi(x_{i+1}) \dots \phi(x_n) e^{i \Gamma} \end{aligned}$$

by a functional integration by parts,

$$\begin{aligned} &= \sum_i \int dx_1 \dots dx_n e^{i(p_1 x_1 + \dots + (p_i + q) x_i + \dots + p_n x_n)} \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{i \Gamma} \\ &\quad (\text{B.4}) \end{aligned}$$

$$= \sum_i G(p_1, \dots, p_i + q, \dots, p_n)$$

Theorem 2 of Section 3.2 is now obvious. A zero-momentum insertion of $\mathcal{O}$ into a Green's function merely counts the number of external ϕ lines.

Theorem 1, that on-shell matrix elements of these counting operators at non-zero momentum vanish, is much more difficult to prove, and here we must refer to the literature¹⁰⁾ on composite operators and bound states in field theory. In outline, it is possible to write a generalized reduction formula in which a matrix element between bound states is related to a Green's function G involving the elementary constituent fields, acted on by Klein-Gordon operators which vanish on the mass shell of the physical bound states. The essential point is that by changing the external momenta, the insertion of the operator $\mathcal{O}$ shifts the position of the bound state poles in G , at least for non-exceptional momenta, and subsequently the action of the Klein-Gordon operators gives zero.

In fact, as mentioned in the text, this theorem holds for a larger class of composite operators, viz. Class IIa - gauge invariant operators which vanish using the equations of motion.

REFERENCES

- 1) R. Dashen, Phys. Rev. 183 (1969) 1245 and Phys. Rev. D3 (1971) 1879.
- 2) E. Witten, Nucl. Phys. B202 (1982) 253;
S. Cecotti and L. Girardello, Phys. Lett. 110B (1982) 39;
E. Cohen and C. Gomez, Harvard preprint HUTP 83/A007 (1983).
- 3) G. Veneziano and S. Yankielowicz, Phys. Lett. 113B (1982) 321;
T.R. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. B218 (1983) 493;
M.E. Peskin, SLAC preprint PUB-3061 (1983);
A.C. Davis, M. Dine and N. Seiberg, IAS preprint (1983).
- 4) G. Veneziano, CERN preprint TH.3574 (1983).
- 5) G.M. Shore, CERN preprint TH.3474 (1982), to appear in Nucl. Phys. B.
- 6) J.A. Dixon and J.C. Taylor, Nucl. Phys. B78 (1974) 552;
H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D12 (1975) 467, 482, 3159;
S.D. Joglekar and B.W. Lee, Ann. Phys. (N.Y.) 97 (1976) 160;
W.S. Deans and J.A. Dixon, Phys. Rev. D18 (1978) 1113;
R. Tarrach, Nucl. Phys. B196 (1982) 45;
D. Espriu and R. Tarrach, Zeit. für Physik C16 (1983) 77.
- 7) E. Witten, Nucl. Phys. B188 (1981) 513;
M. Dine, W. Fischler and M. Srednicki, Nucl. Phys. B189 (1981) 575;
S. Dimopoulos and S. Raby, Nucl. Phys. B192 (1981) 353;
H.P. Nilles, Phys. Lett. 112B (1982) 455.
- 8) G. Veneziano, CERN preprint TH.3478 (1982) and addendum (1983).
- 9) C.G. Callan, in Methods in Field Theory, Les Houches 1975, North Holland (1976).
- 10) W.S. Deans and J.A. Dixon, Ref. 6);
K. Huang and H.A. Weldon, Phys. Rev. D11 (1975) 257.