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SECOND-ORDER PERTURBATIVE THEORY
APPLIED TO DEPOLARIZING EFFECTS OF QUANTUM FLUCTUATIONS

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ABSTRACT

Depolarizing effects, which disturb the build-up of a stable level of polarization, are mainly described by the spin-orbit coupling vector $\vec{\Gamma}$. In this report $\vec{\Gamma}$ is computed for an electron storage ring in which a radial magnetic field perturbs the spin motion. A perturbative method is used, taking into account first- and second-order terms. These are more significant in the neighbourhood of synchrotron satellite spin resonances.

1. INTRODUCTION

Radiative spontaneous polarization builds up in electron beams circulating in accelerators and storage rings. Depolarizing effects, which are stronger in the vicinity of spin resonances, disturb the build-up of a stable level of polarization.

The asymptotic degree of polarization P is given by¹⁾

$$\frac{P}{P_0} = \frac{\langle \rho^{-3} \hat{z} \cdot (\hat{n} - \vec{\hat{T}}) \rangle}{\langle \rho^{-3} \left[1 - \frac{2}{9} (\hat{n} \cdot \hat{y})^2 + \frac{11}{18} |\vec{\hat{T}}|^2 \right] \rangle}, \quad (1)$$

P_0 being the maximum degree of polarization which can be reached only in ideal cases ($P_0 \approx 92\%$); ρ is the radius of curvature of the orbit; $\hat{z}$, $\hat{n}$, and $\hat{y}$ are unit vectors in the directions of the magnetic bending field, of the stable direction of polarization and of the particle velocity, respectively; $\hat{n}$ also denotes the spin closed solution; $\vec{\hat{T}} = \gamma(\partial \hat{n} / \partial \gamma)$ is the spin-orbit coupling vector²⁾; γ is the energy of the particle in units of the rest energy.

The most important cause of depolarizing effects is the term $(11/18) |\vec{\hat{T}}|^2$ appearing in the denominator of expression (1). It has been calculated elsewhere^{3,4)} with the following assumptions: synchrotron oscillations of energy modulate the spin tune ν , because of its proportionality to γ

$$\nu = \gamma G = \nu_0 + a \cos(Q_s \theta + \phi), \quad (2)$$

where $G = (g-2)/2$ is the anomalous gyromagnetic factor, Q_s is the synchrotron tune, θ the azimuth variable, and ϕ the synchrotron phase.

At one point of the ring ($\theta = \theta_0$), there is a radial magnetic field, whose intensity is much less than the intensity of the vertical bending field. The harmonic k of its Fourier analysis whose frequency is nearest to the value of the spin tune, perturbs the evolution of the spin, and gives rise to an integer spin resonance. Owing to the frequency modulation [Eq. (2)] the integer resonance is surrounded by synchrotron satellites whose strength depends on the spread of the amplitude-modulation distribution of the particles in the beam. The problem has been treated using a perturbative method, taking into account up to now only the first-order terms in the parameter ϵ proportional to the amplitude of the perturbative harmonic.

The result obtained⁴⁾ was

$$\langle |\vec{\hat{T}}|^2 \rangle = \nu_0^2 \epsilon^2 e^{-x} \sum_{p=-\infty}^{\infty} \frac{I_p(x)}{[(\delta_0 - pQ_s)^2 - Q_s^2]^2}, \quad (3)$$

where the average is taken both on the phase and over the Rayleigh distribution function of a ,

$$f(a) = \frac{2a}{\sigma_a^2} e^{-a^2/\sigma_a^2}, \quad (4)$$

[σ_a being related to the energy spread in the beam by $\sigma_a = \sqrt{2} (\sigma_\epsilon/E) v_0$]; I_p are the modified Bessel functions of integer order, $x = \sigma_a^2/2Q_s^2$ and $\delta_0 = v_0 - k$.

Equation (3) can be rewritten as

$$\langle |\vec{T}|^2 \rangle = C \langle |\vec{T}|_0^2 \rangle = C \frac{v_0^2 \epsilon^2}{\delta_0^4}, \quad (5)$$

where C is the depolarizing enhancement factor which takes into account the influence of the synchrotron motion on the spin-orbit coupling vector. The estimates of C are rather large for obtaining a good degree of polarization (see Fig. 1) for high-energy machines and for small values of Q_s ($Q_s \approx 0.1$), because the condition for having polarization higher than 50% is $11/18 \langle |\vec{T}|^2 \rangle \leq 1$.

The distance between two satellites is Q_s . In practical cases it is not easy to operate far from satellite resonances; so the first-order perturbative theory is not satisfactory for our problem. Second-order terms should be included to describe more exactly the behaviour near the resonances and will be the subject of this report.

2. SPIN CLOSED SOLUTION $\hat{n}$ IN SECOND-ORDER PERTURBATION THEORY

To compute $\vec{T}$ we need firstly the components of the spin closed solution $\hat{n}$. They are given by the well-known formulae³⁾:

$$n_K = - \frac{\text{Tr} (\sigma_K \Delta)}{2i \sin \pi v_p} \quad (6)$$

$$\cos \pi v_p = \frac{1}{2} \text{Tr} (\Delta),$$

σ_K being Pauli matrices and $\Delta(\theta_0, \theta)$ the rotation matrix which describes the spin evolution between the two azimuths θ_0 and θ . The Δ matrix satisfies the differential equation³⁾

$$\frac{d\Delta}{d\theta} = - \frac{i}{2} [\delta \sigma_3 + \epsilon \sigma_1] \Delta, \quad (7)$$

where $\delta = v - k$. The coordinate system has been chosen with its third axis parallel to the magnetic bending field and its first axis parallel to the perturbation³⁾.

We make a transformation to another reference system where Eq. (7) is easier to solve, defining a new rotation matrix $\mathcal{D}$:

$$\mathcal{D}(\theta_0, \theta) = \exp \left[\frac{i}{2} \chi(\theta) \sigma_3 \right] \Delta(\theta_0, \theta) \quad (8)$$

with

$$\chi(\theta) = \int_{\theta_0}^{\theta} \delta \, d\theta . \quad (9)$$

For one synchrotron period:

$$\chi\left(\theta_0 + \frac{2\pi}{Q_s}\right) = \frac{\delta_0}{Q_s} \pi \sigma_3 = \kappa \sigma_3 . \quad (10)$$

The differential equation satisfied by $\mathcal{D}$ is obtained easily:

$$\frac{d\mathcal{D}}{d\theta} = - \frac{i}{2} V(\theta) \mathcal{D} , \quad (11)$$

where $V(\theta)$ is the matrix

$$V(\theta) = \begin{pmatrix} 0 & g(\theta) \\ g^*(\theta) & 0 \end{pmatrix} \quad (12)$$

and

$$g(\theta) = \varepsilon \exp [i \chi(\theta)] .$$

The solution of Eq. (12) is an infinite series in which we will take the terms up to second order in ε

$$\mathcal{D}(\theta_0, \theta) = \mathbf{1} - \frac{i}{2} \int_{\theta_0}^{\theta} V(\theta_1) \, d\theta_1 + \left(-\frac{i}{2}\right)^2 \int_{\theta_0}^{\theta} d\theta_1 \int_{\theta_0}^{\theta_1} d\theta_2 V(\theta_1) V(\theta_2) + O(\varepsilon^3) . \quad (13)$$

As in ref. 5 we define

$$I_n(\theta) = \int_{\theta_0}^{\theta} d\theta_1 \int_{\theta_0}^{\theta_1} d\theta_2 \dots \int_{\theta_0}^{\theta_{n-1}} d\theta_n g(\theta_1) g^*(\theta_2) \dots g(\theta_{n-1}) g^*(\theta_n) \quad (14)$$

and we can write

$$\mathcal{D}(\theta_0, \theta) = \begin{bmatrix} A(\theta) & B(\theta) \\ -B^*(\theta) & A^*(\theta) \end{bmatrix} \quad (15)$$

where A and B are defined as

$$\begin{aligned} A(\theta) &= 1 - \frac{1}{4} I_2^*(\theta) \\ B(\theta) &= -\frac{i}{2} I_1(\theta) . \end{aligned} \quad (16)$$

The $\mathbb{D}$ matrix belongs to the SU(2) group. The unitarity condition ($|A|^2 + |B|^2 = 1$), expressed in terms of the integrals I_n and for second-order terms, is

$$I_2 + I_2^* = I_1 I_1^* = |I_1|^2 . \quad (17)$$

Transforming back to the Δ matrix, for one synchrotron period we have

$$\Delta\left(\theta_0, \theta_0 + \frac{2\pi}{Q_s}\right) = \begin{bmatrix} A e^{-i\kappa} & B e^{-i\kappa} \\ -B^* e^{i\kappa} & A^* e^{i\kappa} \end{bmatrix} \quad (18)$$

Of course Δ also satisfies the unitarity condition.

Now we can derive the components of $\hat{n}$. The normalization constant [Eq. (6)] is

$$\sin \pi \nu_p = \sin \kappa \sqrt{1 + \frac{1}{2} \cot^2 \kappa \left(\frac{1}{2} |I_1|^2 + \text{tg } \kappa \text{ Im } \{I_2\} \right)} , \quad (19)$$

where the second-order terms are inside the parentheses.

The components of the closed solution are:

$$n_K = \frac{f_K}{D} , \quad K = 1, 2, 3 \quad (20)$$

where

$$\begin{aligned} f_1 &= i(B e^{-i\kappa} - B^* e^{i\kappa}) \\ f_2 &= -(B e^{-i\kappa} + B^* e^{i\kappa}) \\ f_3 &= i(A e^{-i\kappa} - A^* e^{i\kappa}) \\ D &= 2 \sin \pi \nu_p . \end{aligned} \quad (21)$$

From the expressions of I_1 and $\text{Im } \{I_2\}$ (see Appendix A) we can verify that D does not contain pole terms. In the first-order approximation the n_K are not determined for integer values of δ_0/Q_s , on the synchrotron satellite resonances. This indeterminism disappears when we include the second-order terms.

3. SPIN-ORBIT COUPLING VECTOR

We now calculate the $\vec{I}$ vector. From Eqs. (20) and (21) we obtain:

$$|\vec{I}|^2 = \gamma^2 \sum_{K=1}^3 \left(\frac{\partial n_K}{\partial \gamma} \right)^2 = \frac{\gamma^2}{D^4} \sum_{K=1}^3 \left[D^2 \left(\frac{\partial f_K}{\partial \gamma} \right)^2 + \left(\frac{\partial D}{\partial \gamma} \right)^2 f_K^2 - 2D \frac{\partial D}{\partial \gamma} f_K \frac{\partial f_K}{\partial \gamma} \right]. \quad (22)$$

Straightforward evaluation of Eq. (21) gives

$$\sum_{K=1}^3 f_K^2 = 4[|B|^2 + |A|^2 \sin^2 (\kappa - \alpha)] \quad (23a)$$

$$\sum_{K=1}^3 \left(\frac{\partial f_K}{\partial \gamma} \right)^2 = 4 \left[\left| \frac{\partial B}{\partial \gamma} \right|^2 + \left| \frac{\partial A}{\partial \gamma} \right|^2 \sin^2 (\kappa - \alpha') \right] \quad (23b)$$

$$\sum_{K=1}^3 f_K \frac{\partial f_K}{\partial \gamma} = 2 B^* \frac{\partial B}{\partial \gamma} + A^* \left(\frac{\partial A}{\partial \gamma} - \frac{\partial A^*}{\partial \gamma} e^{2i\kappa} \right) + \text{h.c.} \quad (23c)$$

$$D \frac{\partial D}{\partial \gamma} = 2 \cos^2 \kappa \left[|I_1| \frac{\partial |I_1|}{\partial \gamma} + \text{tg } \kappa \frac{\partial}{\partial \gamma} (\text{Im } \{I_2\}) \right], \quad (23d)$$

(α and α' are the phase factors of A and $\partial A / \partial \gamma$ respectively).

Substituting these expressions in Eq. (22) and neglecting terms of order higher than second in ϵ , we can finally write:

$$|\vec{I}|^2 = \frac{\gamma^2}{D^2} \left[\frac{\partial |I_1|}{\partial \gamma} \right]^2. \quad (24)$$

It is worth comparing Eq. (24) with the equivalent expression [Eq. (15) in Ref. 4] evaluated in the first-order approximation especially in the zones very near to the satellite resonances and in the zones between them.

4. ANALYSIS OF $|\vec{I}|^2$ BEHAVIOUR

- i) When we are just at the middle point between two satellite resonances (corresponding to half integer values of the ratio δ_0/Q_s), the second-order terms appearing in Eq. (24) vanish as they are multiplied by $\cos \kappa$, and we obtain the same expression as in the first-order approximation: the second-order treatment does not affect the minimum value of the enhancement factor C .

ii) In the vicinity of these points (δ_0/Q_s very near to a half-integer value), we can approximate

$$\begin{aligned} |\vec{I}|^2 &= \frac{\gamma^2}{4 \sin^2 \kappa} \left[\frac{\partial |I_1|}{\partial \gamma} \right]^2 \left[1 - \frac{\varepsilon^2}{2} \cot^2 \kappa F_1(\theta_0) \right] \\ &\approx \frac{\gamma^2}{4 \sin^2 \kappa} \left[\frac{\partial |I_1|}{\partial \gamma} \right]^2 + O(\varepsilon^4), \end{aligned} \quad (25)$$

since F_1 is a positive function not depending on ε . In this region the correction to the first-order perturbation theory is of the fourth order in ε , and hence negligible compared to other terms. Anyway, its value being negative, the enhancement factor C is slightly diminished by it.

iii) Just on the satellites ($\delta_0/Q_s = n = \text{integer value}$), taking the limit of expression (24) for $\delta_0/Q_s \rightarrow n$, one obtains (see Appendix A):

$$|\vec{I}|^2 \rightarrow \frac{\gamma^2}{4} \frac{Q_s^2}{\varepsilon^2 \pi^2} \left(\frac{\partial |I_1|}{\partial \gamma} \right)^2.$$

Owing to the proportionality of $|I_1|$ to ε , $|\vec{I}|^2$ does not depend on ε on the satellites.

iv) In the neighbourhood of the satellites, the following approximations can be made

$$\cos \kappa = \pm \cos \mu \pi \approx 1 - \frac{(\mu \pi)^2}{2}$$

$$\sin \kappa = \pm \sin \mu \pi \approx \pm \mu \pi$$

where $\mu = \delta_0/Q_s - n$.

Substituting these expressions in $|I_1|^2$ and $\text{Im} \{I_2\} \text{tg } \kappa$, we have

$$\begin{aligned} |I_1|^2 &= \frac{4\varepsilon^2 \pi^2}{Q_s^2} \pm \frac{8\varepsilon^2 \mu \pi^2}{Q_s^2} \sum_{p \neq q} \frac{J_p(a/Q_s) J_q(a/Q_s)}{\delta_0/Q_s + q} \cos(p-q) Q_s \theta_0 \\ \text{tg } \kappa \text{ Im } \{I_2\} &= -\mu \frac{2\varepsilon^2 \pi^2}{Q_s^2} \sum_{p \neq q} \frac{J_p(a/Q_s) J_q(a/Q_s)}{\delta_0/Q_s + q} \cos(p-q) Q_s \theta_0. \end{aligned}$$

So

$$|\vec{T}|^2 \rightarrow \frac{\gamma^2}{4} \left[\frac{\partial |I_1|}{\partial \gamma} \right]^2 \left[\frac{2\epsilon^2 \pi^2}{Q_s^2} + a_1 \left(\frac{\delta_0}{Q_s} - n \right) \sum_{p \neq q} \frac{J_p(a/Q_s) J_q(a/Q_s)}{\delta_0/Q_s + q} \cos(p-q) Q_s \theta \right]^{-1}$$

$$\left(a_1 = \begin{cases} 1 & \text{for } n \text{ even} \\ -3 & \text{for } n \text{ odd} \end{cases} \right)$$

neglecting terms in μ^2 .

5. CONCLUSIONS

An integer spin resonance is surrounded by synchrotron satellite resonances due to the energy modulation from synchrotron oscillations and to the energy spread in the beam. Each resonance causes depolarization. For electrons the depolarizing effects are mainly described by the modulus squared of the spin-orbit coupling vector, $\vec{T}$.

Up to now $|\vec{T}|^2$ had been calculated to first-order with perturbative methods^{3,4}). In order to have a good degree of polarization $|\vec{T}|^2$ should be as small as possible.

In this report we have included the second-order terms, with the following results:

The minimum value of $|\vec{T}|^2$ (in the points between two satellites) is not much affected by the second-order terms, but in general the value of $|\vec{T}|^2$ is slightly diminished. The correction becomes stronger just in the satellites and their neighbourhoods. On the whole the second-order treatment gives more encouragement for having high polarization than does the first-order one, mainly because the polarization does not disappear even if we are on a resonance: however if we operate between two resonances the two treatments are equivalent.

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Fig. 1 Depolarization enhancement factor C in first-order perturbative theory, for LEP

APPENDIX A

COMPUTATION OF I_1 AND I_2

The integral I_1 is defined as:

$$I_1\left(\theta_0 + \frac{2\pi}{Q_s}\right) = \varepsilon \int_{\theta_0}^{\theta_0 + 2\pi/Q_s} \exp [i \chi(\theta)] d\theta . \quad (A1)$$

It may be solved using the following property of Bessel functions:

$$\exp \left(i \frac{a}{Q_s} \sin \theta \right) = \sum_{p=-\infty}^{\infty} J_p\left(\frac{a}{Q_s}\right) \exp (i p \theta) . \quad (A2)$$

The modulus squared of I_1 is then³⁾:

$$|I_1|^2 = 4\varepsilon^2 \sin^2 \kappa \left[\sum_{p=-\infty}^{\infty} \frac{J_p^2(a/Q_s)}{(\delta_0 + pQ_s)^2} + 2 \sum_{p \neq q} \frac{J_p(a/Q_s) J_q(a/Q_s)}{(\delta_0 + pQ_s)(\delta_0 + qQ_s)} \cos (p - q) Q_s \theta_0 \right] . \quad (A3)$$

The integral I_2 is defined as

$$I_2\left(\theta_0 + \frac{2\pi}{Q_s}\right) = \varepsilon^2 \int_{\theta_0}^{\theta_0 + 2\pi/Q_s} d\theta_1 \exp [i \chi(\theta_1)] \int_{\theta_0}^{\theta_1} d\theta_2 \exp [-i \chi(\theta_2)] . \quad (A4)$$

Using the same property (A2) and performing the double integral, we obtain:

$$\begin{aligned} I_2\left(\theta_0 + \frac{2\pi}{Q_s}\right) &= 2i \varepsilon^2 \sum_{p, q=-\infty}^{\infty} \frac{J_p(a/Q_s) J_q(a/Q_s)}{\delta_0 + qQ_s} \exp [-i (p - q) Q_s \theta_0] \times \\ &\times \left\{ \frac{\sin (p - q) \pi}{(p - q) Q_s} \exp [i (p - q) \pi] - \frac{\sin (\delta_0/Q_s + p) \pi}{\delta_0 + pQ_s} \exp [i (\delta_0/Q_s + p) \pi] \right\} . \end{aligned} \quad (A5)$$

We are interested only in the imaginary part of I_2 , which turns out to be:

$$\begin{aligned} \text{Im } \{I_2\} = \epsilon^2 \sum_p \frac{J_p^2(a/Q_s)}{\delta_0 + pQ_s} \left[\frac{2\pi}{Q_s} - \frac{\sin 2\pi \delta_0/Q_s}{\delta_0 + pQ_s} \right] \\ - 2 \epsilon^2 \sum_{p \neq q} \frac{J_p(a/Q_s) J_q(a/Q_s)}{(\delta_0 + pQ_s)(\delta_0 + qQ_s)} \sin \kappa \cos \left[(p-q) Q_s \theta_0 + \frac{\delta_0}{Q_s} \pi \right]. \end{aligned} \quad (\text{A6})$$

When δ_0/Q_s is an integer value, let us say $\delta_0/Q_s = -n$, $|I_1|^2$ goes to the limit

$$|I_1|^2 \rightarrow \frac{4\epsilon^2\pi^2}{Q_s^2} \sum_p J_p^2(a/Q_s) = \frac{4\epsilon^2\pi^2}{Q_s^2} \quad (\text{A7})$$

and the term $\text{tg } \kappa \text{ Im } \{I_2\}$ is

$$\text{tg } \kappa \text{ Im } \{I_2\} \rightarrow \frac{\pi}{Q_s} \epsilon^2 \sum_p J_p^2(a/Q_s) \left(\frac{2\pi}{Q_s} - \frac{2\pi}{Q_s} \right) = 0. \quad (\text{A8})$$

Hence

$$\frac{1}{2} |I_1|^2 + \text{tg } \kappa \text{ Im } \{I_2\} \rightarrow \frac{2\epsilon^2\pi^2}{Q_s^2}.$$