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MIXED SUPERTABLEAUX OF THE SUPERUNITARY GROUPS

I - $SU(n|1)$

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ABSTRACT

Young supertableaux of the graded unitary groups $SU(n|1)$ are classified and interpreted in terms of representations of the group. A distinction between typical and atypical representations arises naturally and we propose a new type of generalized atypical supertableaux for non-fully reducible representations.

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1. - INTRODUCTION

Young supertableaux have been introduced by Balantekin and Bars^{1),2)} for a graphical description of tensorial representations of Lie superalgebras as a generalization of ordinary Young tableaux for ordinary Lie algebras. The simplest case appears to be that of superunitary groups $SU(n|m)$ defined as the set of complex linear transformations of the special graded linear group $SPL(n|m)$ leaving invariant an even regular sesquilinear form G^*

$$SU(n|m) = \{ M \in SPL(n|m) \mid M^\dagger G M = G \}$$

$$su(n|m) = \{ X \in spl(n|m) \mid X^\dagger G + G X = 0 \}$$

The relation between supertableaux and representations of basic simple supergroups has been initiated by Bars³⁾, Morel and Ruegg⁴⁾. For special superunitary groups $SU(n|m)$ the legal purely covariant and legal purely contravariant supertableaux describe one irreducible representation of $SU(n|m)$, typical or atypical but the correspondence is not one to one and the concept of equivalence between supertableaux of these types has to be introduced. Unfortunately, there exist irreducible representations of $SU(n|m)$ and also not fully reducible representations which cannot be visualized in that way, and mixed supertableaux containing both covariant and contravariant boxes must be introduced. Their discussion is considerably more complicated than for covariant and contravariant supertableaux and the aim of this paper is a classification and an interpretation of the mixed supertableaux of $SU(n|1)$. The general case for $SU(n|m)$ will be considered in a subsequent publication.

The fundamental representation of $SU(n|m)$ acts in a graded $(n+m)$ dimensional space $V = V_0 \oplus V_1$ where V_0 is bosonic and V_1 fermionic. We have two possible graduations of V and, by tensor product, two classes of representations of $SU(n|m)$

$$\begin{cases} \dim V_0 = n, & \dim V_1 = m & \text{Class I} \\ \dim V_0 = m, & \dim V_1 = n & \text{Class II} \end{cases}$$

Because of the isomorphism of the two supergroups $SU(n|m)$ and $SU(m|n)$ it is obvious that class I (II) representation of $SU(n|m)$ are also class II (I) representations of $SU(m|n)$. Equivalently a class II supertableau is simply

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Small letters indicate the superalgebra of the supergroup.

obtained from a class I supertableau by the transposition operation exchanging the roles of rows and columns. In the rest of this paper we shall limit ourselves to class I representations of $SU(n|1)$. For a measure of the number of covariant and contravariant boxes of the rows and columns of a supertableau of $SU(n|m)$ we use the notations of Ref. 5)

$$\begin{array}{l} \left| \begin{array}{c} b_j \\ \bar{b}_j \end{array} \right| \text{ counts the } \left| \begin{array}{c} \text{covariant} \\ \text{contravariant} \end{array} \right| \text{ boxes of the row } j \\ \\ \left| \begin{array}{c} c_k \\ \bar{c}_k \end{array} \right| \text{ counts the } \left| \begin{array}{c} \text{covariant} \\ \text{contravariant} \end{array} \right| \text{ boxes of the column } k \end{array}$$

with the usual constraints

$$\begin{array}{ll} b_1 \geq b_2 \geq \dots \geq b_j \geq \dots \geq 0 & \bar{b}_1 \geq \bar{b}_2 \geq \dots \geq \bar{b}_j \geq \dots \geq 0 \\ c_1 \geq c_2 \geq \dots \geq c_k \geq \dots \geq 0 & \bar{c}_1 \geq \bar{c}_2 \geq \dots \geq \bar{c}_k \geq \dots \geq 0 \end{array}$$

A class I supertableau of $SU(n|m)$ is legal if and only if the numbers of covariant and contravariant boxes of its rows satisfy one of the inequalities

$$b_{m+1-j} + \bar{b}_{j+1} \leq m \quad j = 0, 1, \dots, n \quad (1)$$

or, equivalently if the numbers of covariant and contravariant boxes of its columns satisfy one of the inequalities

$$c_{k+1} + \bar{c}_{m+1-k} \leq m \quad k = 0, 1, \dots, m \quad (2)$$

These two sets of constraints are clearly equivalent and for a legal class I supertableau of $SU(n|m)$ at least one of the inequalities (1) and one of the inequalities (2) is fulfilled.

For purely contravariant and purely covariant supertableaux the conditions given in Ref. 5)

$$\begin{array}{ll} i) \quad b_{m+1} \leq m; \quad c_{m+1} \leq m & \text{in the purely covariant case} \\ ii) \quad \bar{b}_{m+1} \leq m; \quad \bar{c}_{m+1} \leq m & \text{in the purely contravariant case} \end{array}$$

are obviously contained in Eqs. (1) or (2).

The proof of the inequalities (1) can be obtained by using two complementary techniques

- i) the reduction of a legal supertableau with respect to the subgroup $SU(n) \otimes SU(m) \otimes U(1)$ associated to the bosonic part of the superalgebra must exist; such a criterion has been partly discussed in Ref. 5);
- ii) the computation of the dimension of the supertableau with a determinant, as proposed in Ref. 1) must give, for a legal typical supertableau, a positive result.

We shall not give more details in this paper and we postpone the general proof to a subsequent publication.

2. - KAC-DYNKIN PARAMETERS OF $SU(n|1)$

1) The graded Lie algebra of the superunitary group $SU(n|1)$ has a Bose sector which is the Lie algebra of the direct product $SU(n) \otimes U(1)$. We have n Cartan operators $H_1, H_2, \dots, H_n$ the $(n-1)$ first ones belonging to $SU(n)$ and the $U(1)$ generator Q is defined by

$$Q = \frac{1}{n} \sum_{j=1}^n j H_j$$

The Kac-Dynkin parameters $\{a_1, a_2, \dots, a_{n-1}; a_n\}$ define an irreducible representation of $SU(n|1)$ ⁵⁾⁻⁷⁾. They are the eigenvalues of the Cartan generators for the highest weight of the representation chosen by Kac as corresponding to the lowest eigenvalue of Q .

The $(n-1)$ first Kac-Dynkin parameters $(a_1, a_2, \dots, a_{n-1})$ describe an irreducible representation of $SU(n)$ precisely that associated to the lowest eigenvalue of Q . As a consequence they are non-negative integers. The last parameter a_n may be any real number. In the supertableau approach it is an algebraic integer. In what follows we restrict ourselves for a_n to algebraic integer values advocating a remark on this point made in Ref. 5) even if the question does not seem to us entirely clear.

2) We have two types of irreducible representations of $SU(n|1)$, the typical representations and the atypical representations ⁵⁾⁻⁷⁾. Fixing $a_1, a_2, \dots, a_{n-1}$ and defining

$$A_j = \sum_{s=j}^{n-1} (1 + a_s) \quad j = 1, 2, \dots, n-1 \quad (3)$$

we have the following result. An irreducible representation of $SU(n|1)$ is atypical if and only if a_n takes one of the n values

$$a_n = 0, -A_1, -A_2, \dots, -A_{n-1}$$

In particular all the representations with $a_n > 0$ and $a_n < -A_{n-1}$ are typical. For typical representations of $SU(n|1)$ the dimension is even - with an equal number of bosonic and fermionic weights - and it turns out to be independent of a_n and given by⁵⁾⁻⁷⁾

$$\dim \{a_1, a_2, \dots, a_{n-1}; a_n\} = 2^n \dim(a_1, a_2, \dots, a_{n-1})$$

where the last factor is the dimension of the $SU(n)$ representation given by the well-known formula

$$\dim(a_1, a_2, \dots, a_{n-1}) = \prod_{1 \leq i \leq j \leq n-1} \left[1 + \frac{\sum_{s=i}^j a_s}{j-i+1} \right]$$

For atypical representations the dimension is always lower.

3) Finally, let us give as an illustration, three important examples

- i) the fundamental representation $\{1, 0, \dots, 0; 0\}$ has the dimension $n+1$; it is atypical and associated to the supertableau with one covariant box.
- ii) the fundamental representation $\{0, \dots, 0; -1\}$ is the contragradient of the previous one with the same dimension $n+1$; it is atypical and associated to the supertableau with one contravariant box.
- iii) the adjoint representation $\{1, 0, \dots, 0; -1\}$ has the dimension $(n+1)^2 - 1$ and is self-contragradient; it is atypical when $n \geq 3$ and associated to the supertableau with one covariant box and one contravariant box. For $SU(2|1)$ the adjoint representation is typical and it has the dimension 8.

3. - LEGAL SUPERTABLEAUX OF $SU(n|1)$

1) In $SU(n|1)$ the two sets of constraints (1) and (2) are written as

$$\left\{ \begin{array}{l} b_{n+1-j} + \bar{b}_{j+1} \leq 1 \\ c_{k+1} + \bar{c}_{2-k} \leq n \end{array} \right. \quad \begin{array}{l} j = 0, 1, 2, \dots, n \\ k = 0, 1 \end{array} \quad (4)$$

We then can distinguish between two categories of supertableaux, depending on the value of the sum of covariant and contravariant boxes of the first columns $c_1 + \bar{c}_1$

- a) when $c_1 + \bar{c}_1 < n$ the two inequalities (5) are simultaneously satisfied and we can describe these supertableaux with c_1 parameters of type b_j and $\bar{c}_1$ parameters of type $\bar{b}_j$ fixing entirely the number of boxes of each row.
- β) when $c_1 + \bar{c}_1 \geq n$ we solve the equations (4) and (5) and we easily find $2(n+1)$ solutions characterizing $2(n+1)$ types of supertableaux. In this case only $n+1$ free parameters determine the supertableau the other ones being fixed.

2) The construction and the discussion of the various types of legal supertableaux will be made by the following method. To a legal supertableau of $SU(n|1)$ we associate a point P in a $(n+1)$ dimensional space. The co-ordinates of P are chosen in such a way that they produce the same Kac-Dynkin parameters for the supertableau as those given by the $SU(n) \otimes U(1)$ analysis of the supertableau⁵⁾. We then introduce $n+1$ co-ordinates

$$x_1 \geq x_2 \geq \dots \geq x_n \quad \text{and} \quad y_1$$

The Kac-Dynkin parameters are then computed with the formulae

$$\left\{ \begin{array}{l} a_j = x_j - x_{j+1} \geq 0 \\ a_n = x_n - y_1 \end{array} \right. \quad j = 1, 2, \dots, n-1$$

Inverting these relations we are able to express the parameters b_ℓ and $\bar{b}_\ell$ as linear combinations of the Kac-Dynkin parameters a_j and of y_1 . This last quantity has a range of variation defined by the positivity constraints on the parameters $b_\ell, \bar{b}_\ell, c_1, \bar{c}_1$ and it is interpreted as describing the equivalence between supertableaux of different topology but having the same Kac-Dynkin parameters.

4. - ATYPICAL SUPERTABLEAUX $c_1 + \bar{c}_1 < n$

1) Let us call as $N = c_1$ the number of rows with covariant boxes and $\bar{N} = \bar{c}_1$ the number of rows with contravariant boxes and let us study the case where $N + \bar{N} < n$. The corresponding supertableaux are defined by $N + \bar{N}$ positive integers $b_j, \bar{b}_j$, with the constraints

$$\left\{ \begin{array}{l} b_1 \geq b_2 \geq \dots \geq b_N \geq 1 \\ \bar{b}_1 \geq \bar{b}_2 \geq \dots \geq \bar{b}_{\bar{N}} \geq 1 \end{array} \right.$$

and they are represented in Fig. 1.

The Kac-Dynkin parameters of this supertableau are defined by the eigenvalues of the Cartan subalgebra generators for the highest weight of the representation. Such an analysis is made in Appendix A and it is equivalent to the following choice of co-ordinates for the point P of the $(n+1)$ dimensional space, associated to the supertableau

$$\left\{ \begin{array}{ll} x_s = b_s & s = 1, 2, \dots, N \\ x_s = 0 & s = N+1, \dots, n - \bar{N} \\ x_s = 1 - \bar{b}_{n+1-s} & s = n - \bar{N} + 1, \dots, n \end{array} \right. \quad (6)$$

and

$$y_1 = \bar{c}_1 = \bar{N}$$

We then compute the Kac-Dynkin parameters with the usual formulae and we get

$$\left\{ \begin{array}{ll} a_s = b_s - b_{s+1} & s = 1, 2, \dots, N-1 \\ a_N = b_N & \\ a_s = 0 & s = N+1, \dots, n - \bar{N} - 1 \\ a_{n-\bar{N}} = \bar{b}_{\bar{N}} - 1 & \\ a_s = \bar{b}_{n-s} - \bar{b}_{n-s+1} & s = n - \bar{N} + 1, \dots, n-1 \end{array} \right.$$

and

$$a_n = 1 - \bar{b}_1 - \bar{N}$$

The formulae (7) can be uniquely inverted and we immediately obtain the length of the covariant and contravariant rows in terms of the $(n-1)$ Kac-Dynkin parameters of the $SU(n)$ part

$$\left\{ \begin{array}{l} b_j = \sum_{s=1}^N a_s \\ \bar{b}_j = 1 + \sum_{s=1}^{\bar{N}} a_{n-s} \end{array} \right. \quad \begin{array}{l} j = 1, 2, \dots, N \\ j = 1, 2, \dots, \bar{N} \end{array} \quad (8)$$

The last Kac-Dynkin parameter a_n turns out to be a linear combination of the same a_j 's given by

$$a_n = -\bar{N} - \sum_{s=1}^{\bar{N}} a_{n-s} = -A_{\bar{N}}$$

It follows that a_n takes the atypical value $-A_{\bar{N}}$ previously defined in Eq. (2).

The supertableaux with $c_1 + \bar{c}_1 < n$ describe the atypical irreducible representations of $SU(n|1)$ and the correspondence between the supertableaux and the representation is one to one without equivalence. As an example the N row covariant supertableaux with $N < n$ correspond to the atypical representations $a_n = 0$.

2) The $SU(n) \otimes U(1)$ decomposition of the atypical irreducible representations is easily made by using the supertableaux. The component of the lowest eigenvalue of Q represented in Fig. 2 contains the highest weight, and by definition of the Kac-Dynkin parameters we simply have

$$q_{\min} = \frac{1}{n} \sum_{j=1}^n j a_j \quad (9)$$

The other levels in q are obtained by transferring covariant and contravariant boxes. For one box the spacing is $(1 - 1/n)$ and the total number of possible levels is $N + \bar{N} + 1$. It follows that the maximal value of q is given by

$$q_{\max} = q_{\min} + (N + \bar{N}) \left(1 - \frac{1}{n}\right)$$

When all possible $SU(n) \otimes U(1)$ components are present, simple combinatorics gives the total number of these components to be $2^{N+\bar{N}}$.

5. - TYPICAL SUPERTABLEAUX $c_1 + \bar{c}_1 \geq n$

1) We now consider the case where $c_1 + \bar{c}_1 \geq n$ and we introduce the constraints (4) and (5). We classify the supertableaux $c_1 + \bar{c}_1 \geq n$ with one constraint (4) and one constraint (5)

$$I_{\mathbb{R}}^j \Rightarrow \left\{ \begin{array}{l} b_{n+1-j} + \bar{b}_{j+1} \leq 1 \end{array} \right. \quad (10)$$

$$\left\{ \begin{array}{l} c_{k+1} + \bar{c}_{2-k} \leq n \end{array} \right. \quad (11)$$

The parameter j can take $n+1$ values $j = 0, 1, \dots, n$ and the parameter k takes two values only for $m = 1, k = 0, 1$. The total number of types of super-tableaux $c_1 + \bar{c}_1 \geq n$ is then $2(n+1)$.

The I_k^j type supertableaux are first defined by a set of n positive integers describing the non-trivial rows

$$\begin{aligned} b_1 &\geq b_2 \geq \dots \geq b_{n-j} \geq 1 \\ \bar{b}_1 &\geq \bar{b}_2 \geq \dots \geq \bar{b}_j \geq 1 \end{aligned} \quad (12)$$

The inequality (10) means that one of the two quantities b_{n-j+1} or $\bar{b}_{j+1}$ must vanish, the other one being bounded by one. The choice between these two possibilities is made by the inequality (11) and we get

$$\left\{ \begin{array}{lll} \bar{b}_{j+1} = 0 & \bar{c}_1 = j & c_1 \geq n-j \quad \text{for } I_1^j \\ b_{n-j+1} = 0 & c_1 = n-j & \bar{c}_1 \geq j \quad \text{for } I_0^j \end{array} \right.$$

2) Let us first discuss the n parameters b and $\bar{b}$. The x co-ordinates of the associated point P of the $(n+1)$ dimensional space S are given, as previously, in Eq. (6)

$$\left\{ \begin{array}{ll} x_s = b_s & s = 1, 2, \dots, n-j \\ x_s = 1 - \bar{b}_{n+1-s} & s = n-j+1, \dots, n \end{array} \right. \quad (13)$$

and we compute the $(n-1)$ Kac-Dynkin parameters of the $SU(n)$ part

$$\left\{ \begin{array}{ll} a_s = b_s - b_{s+1} & s = 1, 2, \dots, n-j-1 \end{array} \right. \quad (14)$$

$$\left\{ \begin{array}{ll} a_{n-j} = b_{n-j} + \bar{b}_j - 1 \end{array} \right. \quad (15)$$

$$\left\{ \begin{array}{ll} a_s = \bar{b}_{n-s} - \bar{b}_{n-s+1} & s = n-j+1, \dots, n-1 \end{array} \right. \quad (16)$$

3) Let us first study the case $j = 0$. Equations (14) only are present and they are inverted as follows

$$b_\ell = \sum_{s=\ell}^{n-1} a_s + b_n \quad \ell = 1, 2, \dots, n-1 \quad (17)$$

The positivity constraints on the b_ℓ parameters imply a trivial ordering of the x_ℓ co-ordinates

$$x_1 \geq x_2 \geq \dots \geq x_n \geq 1$$

The set of supertableaux of type I^0 is divided in two subsets

$$\left\{ \begin{array}{lll} I_1^0 & \bar{c}_1 = 0 & c_1 \geq n \\ I_0^0 & c_1 = n & \bar{c}_1 \geq 1 \end{array} \right. \quad \begin{array}{l} \text{purely covariant supertableaux} \\ \text{mixed supertableaux} \end{array}$$

The purely covariant supertableaux, I_1^0 , are represented in Fig. 3. The y_1 co-ordinate of the associated point P is chosen so that a_n agrees with the value obtained using the $SU(n) \otimes U(1)$ reduction (see Appendix B). With

$$y_1 = n - c_1$$

we compute the last Kac-Dynkin parameter a_n

$$a_n = \alpha_n - y_1 = b_n + c_1 - n \quad (18)$$

The positivity constraints $b_n \geq 1$ and $c_1 \geq n$ give a double inequality for y_1

$$1 - a_n \leq y_1 \leq 0 \quad (19)$$

The compatibility of these inequalities restricts a_n to positive values only

$$a_n \geq 1$$

As a consequence purely covariant supertableaux of type I_1^0 are associated to typical irreducible representations of $SU(n|1)$ with $a_n > 0$. Giving now a typical irreducible representation of $SU(n|1)$ by its Kac-Dynkin parameters with $a_n > 0$ the purely covariant supertableaux associated to this representation are constructed as follows

$$\left\{ \begin{array}{l} b_e = \sum_{e=1}^{n-1} a_s + a_n + y_1 \\ c_1 = n - y_1 \end{array} \right. \quad (20)$$

with the parameter y_1 varying in the range (19). We then obtain a_n equivalent typical covariant supertableaux I_1^0 .

The mixed supertableaux of type I_0^0 are represented in Fig. 5. The y_1 co-ordinate is chosen as previously (see Appendix B) and with

$$y_1 = \bar{c}_1$$

we determine the last Kac-Dynkin parameter a_n

$$a_n = x_n - y_1 = b_n - \bar{c}_1 \quad (21)$$

The positivity constraint $\bar{c}_1 \geq 1$ implies trivially

$$y_1 \geq 1$$

and we have no restriction on a_n in this case. When a_n has a typical value we obtain an infinite set of mixed supertableaux of type I_0^0 associated to a typical irreducible representation of $SU(n|1)$. These supertableaux are parametrized with the co-ordinate $y_1 \geq 1$

$$\left\{ \begin{array}{l} b_\ell = \sum_{s=1}^{n-1} a_s + a_n + y_1 \\ \bar{c}_1 = y_1 \end{array} \right. \quad (22)$$

The case where a_n has an atypical value will be discussed in Section 7.

4) We are now interested in the case $j = n$ which is the contragradient^{*)} of the previous one. Only Eqs. (16) are present and they are inverted as follows

$$\bar{b}_\ell = - \sum_{s=1-\ell}^{n-1} a_s + \bar{b}_1 \quad \ell = 2, 3, \dots, n \quad (23)$$

The positivity constraints on the $\bar{b}_\ell$ parameters imply a trivial ordering of the x_ℓ co-ordinates

$$0 \geq x_1 \geq x_2 \geq \dots \geq x_n$$

The set of supertableaux of type I^n is divided in two subsets

$$\left\{ \begin{array}{ll} I_1^n & \bar{c}_1 = n \quad c_1 \geq 1 \quad \text{mixed supertableaux} \\ I_0^n & \bar{c}_1 \geq n \quad c_1 = 0 \quad \text{purely contravariant supertableaux} \end{array} \right.$$

The purely contravariant supertableaux, I_0^n , are represented in Fig. 7. The y_1 co-ordinate is chosen on the $SU(n) \otimes U(1)$ basis as previously (see Appendix B) and we have

$$y_1 = \bar{c}_1$$

^{*)} Two supertableaux are contragradient supertableaux if one is deduced from the second by replacing covariant (contravariant) boxes by contravariant (covariant) boxes.

The last Kac-Dynkin parameter a_n has the value

$$a_n = x_n - y_1 = 1 - \bar{b}_1 - \bar{c}_1 \quad (24)$$

The positivity constraints $\bar{b}_n \geq 1, \bar{c}_1 \geq n$ give a double inequality for y_1

$$n \leq y_1 \leq -a_n - \sum_{s=1}^{n-1} a_s \quad (25)$$

The compatibility of these inequalities restricts a_n to typical values

$$a_n \leq -n - \sum_{s=1}^{n-1} a_s = -1 - A_{n-1} \quad (26)$$

As a consequence purely contravariant supertableaux I_0^n are associated to typical irreducible representations of $SU(n|1)$ with $a_n < -A_{n-1}$.

Giving now a typical irreducible representation of $SU(n|1)$ by its Kac-Dynkin parameters with $a_n < -A_{n-1}$ the purely contravariant supertableaux associated to this representation are constructed as follows

$$\left\{ \begin{array}{l} \bar{b}_e = - \sum_{s=1-e}^{n-1} a_s + 1 - a_n - y_1 \\ \bar{c}_1 = y_1 \end{array} \right. \quad (27)$$

with the parameter y_1 varying in the range (25). We then obtain $-a_n - A_{n-1}$ equivalent typical contravariant supertableaux I_0^n .

The mixed supertableaux of type I_{1-}^n are represented in Fig. 9. The y_1 co-ordinate is chosen as previously (see Appendix B) and with

$$y_1 = n - c_1$$

we determine the Kac-Dynkin parameter a_n

$$a_n = x_n - y_1 = 1 - n - \bar{b}_1 + c_1 \quad (28)$$

The positivity constraint $c_1 \geq 1$ implies trivially

$$y_1 \leq n - 1$$

and we have no restriction on a_n . When a_n has a typical value we obtain an infinite set of mixed supertableaux of type I_1^n associated to a typical irreducible representation of $SU(n|1)$. These supertableaux are parametrized with the co-ordinate y_1

$$\left\{ \begin{array}{l} \bar{b}_\ell = - \sum_{n+1-\ell}^{n-1} a_s + 1 - a_m - y_1 \\ c_1 = n - y_1 \end{array} \right. \quad (29)$$

with now $y_1 \leq n-1$.

5) For the intermediate situations $0 < j < n$ the relation (15) is present and when combined with the positivity constraints $b_{n-j} \geq 1$ and $\bar{b}_j \geq 1$ it gives

$$a_{m-j} \geq 1$$

This inequality is a necessary and sufficient condition for the existence of mixed supertableaux of type I_k^j for $0 < j < n$.

Assuming now this condition to be fulfilled we invert the relations (14), (15) and (16) using $\bar{b}_1$ as a parameter

$$\left\{ \begin{array}{l} b_\ell = \sum_{\ell}^{n-1} a_s + 1 - \bar{b}_1 \quad \ell = 1, 2, \dots, n-j \\ \bar{b}_\ell = - \sum_{n+1-\ell}^{n-1} a_s + \bar{b}_1 \quad \ell = 1, 2, \dots, j \end{array} \right. \quad (30)$$

The positivity constraints (12) imply bounds for the x co-ordinates of the point P . In particular for $x_n = 1 - \bar{b}_1$ we get

$$1 - \sum_{n-j}^{n-1} a_s \leq x_n \leq - \sum_{n-j+1}^{n-1} a_s \quad (31)$$

and for the ordering of the x co-ordinates

$$\sum_1^{n-1} a_s \geq x_1 \geq x_2 \geq \dots \geq x_{m-j} > 0 \geq x_{n-j+1} \geq \dots \geq x_n \geq 1 - \sum_{n-j}^{n-1} a_s$$

The set of mixed supertableaux of type I^j is divided in two subsets

$$\begin{array}{lll} I_1^j & \bar{c}_1 = j & c_1 \geq n-j & \text{mixed supertableaux} \\ I_0^j & c_1 = n-j & \bar{c}_1 \geq j & \text{mixed supertableaux} \end{array}$$

The mixed supertableaux of type I_0^j are represented in Fig. 11. The y_1 coordinate of the point P is given by (see Appendix B)

$$y_1 = \bar{c}_1$$

and the Kac-Dynkin parameter a_n has the value

$$a_n = x_n - y_1 = 1 - \bar{b}_1 - \bar{c}_1 \quad (32)$$

The positivity constraints $\bar{b}_j \geq 1$, $\bar{c}_1 \geq j$ give a double inequality for y_1

$$j \leq y_1 \leq -\sum_{s=1}^{n-1} a_s - a_n \quad (33)$$

The compatibility of these inequalities gives an upper bound for the values of a_n associated to supertableaux of type I_0^j

$$a_n \leq -j - \sum_{s=j+1}^{n-1} a_s = -1 - A_{j-1} \quad (34)$$

On the other hand, the third positivity constraint $b_{n-j} \geq 1$ implies a second lower limit for y_1

$$1 - \sum_{s=j}^{n-1} a_s - a_n \leq y_1 \quad (35)$$

Giving now an irreducible typical representation of $SU(n|1)$ by its Kac-Dynkin parameters with $a_{n-j} \geq 1$ and a_n typical satisfying the inequality (34) we associate to this representation mixed supertableaux of type I_0^j described by

$$\left\{ \begin{array}{ll} b_\ell = \sum_{s=\ell}^{n-1} a_s + a_n + y_1 & \ell = 1, 2, \dots, n-j \\ \bar{b}_\ell = -\sum_{s=\ell+1}^{n-1} a_s + 1 - a_n - y_1 & \ell = 1, 2, \dots, j \\ \bar{c}_1 = y_1 \end{array} \right. \quad (36)$$

with y_1 varying in the range intersection of the ranges (33) and (35). The number of equivalent supertableaux of type I_0^j describing the same typical irreducible representation of $SU(n|1)$ depends on a_n . The result is

$$\begin{array}{ll} \text{i)} & 1 - A_j \leq a_n \leq -1 - A_{j-1} \\ & 1 - A_{j-1} - a_n \text{ supertableaux} \\ \text{ii)} & a_n < -A_j \\ & a_{n-j} \text{ supertableaux} \end{array}$$

The mixed supertableaux of type I_1^j are represented in Fig. 13. The y_1 coordinate of the point P is given by (see Appendix B)

$$y_1 = n - c_1$$

and the Kac-Dynkin parameter a_n has the value

$$a_n = x_n - y_1 = 1 - n - \bar{b}_1 + c_1 \quad (37)$$

The positivity constraints $b_{n-j} \geq 1$, $c_1 \geq n-j$ give a double inequality for y_1

$$1 - \sum_{s=j}^{n-1} a_s - a_n \leq y_1 \leq j \quad (38)$$

The compatibility of these inequalities implies a lower bound for the values of a_n associated to supertableaux of type I_1^j

$$a_n \geq 1 - j - \sum_{s=j}^{n-1} a_s = 1 - A_j \quad (39)$$

On the other hand, the third positivity constraint $\bar{b}_j \geq 1$ implies a second upper limit for y_1

$$y_1 \leq - \sum_{s=j+1}^{n-1} a_s - a_n \quad (40)$$

Giving now an irreducible representation of $SU(n|1)$ by its Kac-Dynkin parameters with $a_{n-j} \geq 1$ and a_n typical satisfying the inequality (39) we associate to this representation mixed supertableaux of type I_1^j described by

$$\left\{ \begin{array}{l} b_\ell = \sum_{s=\ell}^{n-1} a_s + a_n + y_1 \quad \ell=1,2,\dots,n-j \\ \bar{b}_\ell = - \sum_{s=n-\ell+1}^{n-1} a_s + 1 - a_n - y_1 \quad \ell=1,2,\dots,j \\ c_1 = n - y_1 \end{array} \right. \quad (41)$$

with y_1 varying in the range intersection of the ranges (38) and (40). The number of equivalent supertableaux describing the same typical irreducible representation depends on a_n . The result is

$$i) \quad 1 - A_j \leq a_m \leq -1 - A_{j-1} \\ -1 + A_j + a_m \text{ supertableaux}$$

$$ii) \quad a_m > -A_{j-1} \\ a_{m-j} \text{ supertableaux}$$

Finally, the intersection of the two sets I_0^j and I_1^j corresponds to mixed supertableaux with only $(n-j)$ covariant rows and j contravariant rows. They have the shape drawn in Fig. 1 with now $N = n-j$ and $\bar{N} = j$. The length of the covariant and contravariant rows is given by Eqs. (36) and (41) with $y_1 = j$

$$b_\ell = \sum_{s=1}^{n-1} a_s + a_m + j \quad \ell = 1, 2, \dots, n-j \\ \bar{b}_\ell = -\sum_{s=n-\ell+1}^{n-1} a_s + 1 - a_m - j \quad \ell = 1, 2, \dots, j$$

and we have $c_1 = n-j$ and $\bar{c}_1 = j$ fixed. In this case where $c_1 + \bar{c}_1 = n$ we have no equivalence.

The subset of mixed supertableaux $I_1^j \cap I_0^j$ describes the typical irreducible representations of $SU(n|1)$ with a_n in the range intersection of the ranges (34) and (39)

$$1 - A_j \leq a_m \leq -1 - A_{j-1}$$

This finite subset has the dimension a_{n-j} .

For $j = 0$ and $j = n$ the analogues of this subset of typical supertableaux are respectively the n covariant row supertableaux of I_1^0 and the n contravariant row supertableaux of I_0^n .

In conclusion the full set of supertableaux $c_1 + \bar{c}_1 = n$ gives a unique description of all typical representations of $SU(n|1)$.

6) For the typical supertableaux $c_1 + \bar{c}_1 \geq n$ the relations of contragradience are simply expressed. It is straightforward to check that for all supertableaux I_k^j , the exchanges $b_j \leftrightarrow \bar{b}_j$, $c_1 \leftrightarrow \bar{c}_1$ imply the following relations between the co-ordinates of two contragradient points P and P^{CT} in the $n+1$ dimensional space S

$$\left\{ \begin{array}{l} x_j^{CT} = 1 - x_{n+1-j} \\ y_1^{CT} = n - y_1 \end{array} \right. \quad j = 1, 2, \dots, n$$

As a consequence we obtain the relations between the Kac-Dynkin parameters of two typical irreducible representations of $SU(n|1)$

$$\left\{ \begin{array}{l} a_j^{CT} = a_{n-j} \quad j = 1, 2, \dots, n-1 \\ a_n^{CT} = 1 - n - \sum_{s=1}^{n-1} a_s - a_n \end{array} \right.$$

7) The $SU(N) \otimes U(1)$ decomposition of the typical irreducible representations considered in this section is easily made using the supertableaux. The components with the lowest eigenvalue of the $U(1)$ operator Q are represented for each type of typical supertableaux in Figs. 4, 6, 8, 10, 12 and 14. We simply have

$$q_{\min} = \frac{1}{n} \sum_{j=1}^n j a_j \quad (42)$$

and the other levels in q are obtained by transferring the covariant boxes from the $SU(n)$ part to the $U(1)$ part, and the contravariant boxes from the $U(1)$ part to the $SU(n)$ part, in all the ways compatible with the symmetries of the boxes. The spacing for one box is $(1 - 1/n)$ and the number of possible levels is $n + 1$. The maximal value of q is then given

$$q_{\max} = q_{\min} + n \left(1 - \frac{1}{n}\right) = q_{\min} + n - 1$$

The level $q_j = q_{\min} + j(1 - \frac{1}{n})$ contains, at most C_n^j components $SU(n)$ and the total number of these components is simply

$$\sum_{j=0}^n C_n^j = 2^n$$

When some of the Kac-Dynkin parameters $a_1, a_2, \dots, a_{n-1}$ of a typical representation vanish the number of $SU(n) \otimes U(1)$ representations is, by accident, less than 2^n but the number of levels in q stays at $n + 1$.

6. - TENSOR PRODUCTS INVOLVING THE FUNDAMENTAL REPRESENTATIONS

1) A straightforward way to construct representations of $SU(n|1)$ is to perform the tensor product of a particular representation by one of the two fundamental representations. The reduction of the tensor product of these two representations is simply obtained by considering the tensor product within $SU(n)$ of their $SU(n) \otimes U(1)$ components and by rearranging the result into representations of $SU(n|1)$.

We now consider the tensor product of an irreducible representation of $SU(n)$ given by its $n-1$ Dynkin parameters $(a_1, a_2, \dots, a_{n-1})$ by one of the two fundamental representations of $SU(n)$, $(1, 0, \dots, 0)$ or $(0, \dots, 0, 1)$. The result is

$$(a_1, a_2, \dots, a_{n-1}) \otimes (1, 0, \dots, 0) = \\ = (a_1+1, a_2, \dots, a_{n-1}) \oplus (a_1, \dots, a_{n-2}, a_{n-1}-1) \oplus \sum_{i=1}^{n-2} (a_1, \dots, a_i-1, a_{i+1}+1, \dots, a_{n-1}) \quad (43)$$

$$(a_1, a_2, \dots, a_{n-1}) \otimes (0, \dots, 0, 1) = \\ = (a_1-1, a_2, \dots, a_{n-1}) \oplus (a_1, \dots, a_{n-2}, a_{n-1}+1) \oplus \sum_{i=1}^{n-2} (a_1, \dots, a_i+1, a_{i+1}-1, \dots, a_{n-1}) \quad (44)$$

For the first two terms of the sums (43) and (44) only one Dynkin parameter a_1 or a_{n-1} is modified; for the $(n-2)$ other terms two Dynkin parameters are modified as indicated. When all the a_j 's are non-zero, the sums (43) and (44) have n terms. When some a_j 's are vanishing the corresponding terms involving a_j-1 are not present in the sums.

2) As an immediate consequence we get the following formulae for the tensor product of an irreducible representation of $SU(n|1)$ given by its n Kac-Dynkin parameters $\{a_1, a_2, \dots, a_{n-1}; a_n\}$ by one of the two fundamental representations $\{1, 0, \dots, 0; 0\}$ or $\{0, \dots, 0; -1\}$

$$\{a_1, a_2, \dots, a_{n-1}; a_n\} \otimes \{1, 0, \dots, 0; 0\} = \\ = \{a_1+1, a_2, \dots; a_n\} \oplus \{a_1, \dots, a_{n-1}; a_n+1\} \oplus \sum_{i=1}^{n-1} \{a_1, \dots, a_i-1, a_{i+1}+1, \dots; a_n\} \quad (45)$$

$$\{a_1, a_2, \dots, a_{n-1}; a_n\} \otimes \{0, 0, \dots, 0; -1\} = \\ = \{a_1-1, a_2, \dots; a_n\} \oplus \{a_1, \dots, a_{n-1}; a_n-1\} \oplus \sum_{i=1}^{n-1} \{a_1, \dots, a_i+1, a_{i+1}-1, \dots; a_n\} \quad (46)$$

When all the a_j 's, $j = 1, 2, \dots, n-1$ are non-zero, the sums (45) and (46) have $n+1$ terms. When some of these a_j 's are vanishing, the corresponding terms involving $a_j - 1$ disappear from the sum. Of course, this last remark does not apply to a_n which can be any algebraic integer and the minimal number of terms of the sums (45) and (46) is two.

The Eqs. (45) and (46) have a translation in the language of supertableaux. A detailed discussion will be given later but it is straightforward to check that there are, at most, $(n+1)$ independent ways to add (suppress) a covariant (contravariant) box to a legal supertableau of $SU(n|1)$ to obtain a new legal supertableau.

3) Let us first consider the case where the original representation is an atypical irreducible representation as studied in Section 4. From Eqs. (7) and (9) we get the Kac-Dynkin parameters of such a representation

$$\begin{cases} a_n \geq 1, & a_s = 0 \quad \text{for } s = N+1, \dots, n-\bar{N}-1 \\ a_m = -A_{\bar{N}} \end{cases}$$

with $N + \bar{N} < n$ and we have seen that such a representation is uniquely associated to a supertableau with N covariant rows and $\bar{N}$ contravariant rows.

For $N + \bar{N} < n-1$ we obtain, at most, $N + \bar{N} + 1$ terms in Eqs. (45) and (46), each term being associated to an atypical irreducible representation. The construction of these representations is trivial in the supertableau approach.

When $N + \bar{N} = n - 1$ we obtain, at most, $N + \bar{N} = n - 1$ terms associated to an atypical irreducible representation and in all cases one term associated to a typical irreducible representation. In the language of supertableaux this term has $c_1 + \bar{c}_1 = n$ and this subset of typical supertableaux has been studied in Section 5.

4) We now suppose that the original representation is a typical irreducible representation as studied in Section 5. The maximal number of terms in Eqs. (45) and (46) is $n+1$ and it corresponds to the maximal number of independent ways we have to add (suppress) a covariant (contravariant) or a contravariant (covariant) box to a legal typical supertableau, the result being again a legal supertableau.

In Eq. (45) either all terms are typical or two and only two terms become atypical, the other ones remaining typical. This last situation occurs when the original a_n parameter takes one of the values $-A_k - 1$ assumed to be typical. It is then straightforward to check that the second term and one of the other ones are atypical. For Eq. (46) the situation is entirely analogous

and the degeneracy arises when the original a_n takes one of the values $-A_k + 1$ assumed to be typical. This property will be referred, in what follows, as the mechanism of pair production of atypical representations and atypical super-tableaux.

7. - ATYPICAL MIXED SUPERTABLEAUX $c_1 + \bar{c}_1 > n$

1) We consider the mixed supertableaux studied in Section 5 for which the Kac-Dynkin parameter a_n takes one of the n possible atypical values $-A_k$ with $k = 0, 1, \dots, n-1$. As a consequence, the two parameters c_1 and $\bar{c}_1$ are now determined when the length of the rows $b_j, \bar{b}_j$ is known.

For the mixed supertableaux of type I_0^j with $j = 0, 1, \dots, n-1$, we have $c_1 = n-j$ and from Eqs. (21) and (32) we get

$$\bar{c}_1 = j + b_{n-j} + A_k - A_j$$

Because of the positivity constraint $c_1 + \bar{c}_1 > n$ we immediately see that k takes only $(n-j)$ values, $k = j, j+1, \dots, n-1$.

For the mixed supertableaux of type I_1^j with $j = 1, 2, \dots, n$, we have $\bar{c}_1 = j$ and from Eqs. (28) and (37) we have

$$c_1 = n-j + \bar{b}_j + A_{j-1} - A_k$$

Again the positivity constraint $c_1 + \bar{c}_1 > n$ restricts k to j values $k = 0, 1, \dots, j-1$.

2) We now come back to Eqs. (45) and (46) in the degenerate case. Two terms, $\mathcal{C}_1$ and $\mathcal{C}_2$ become simultaneously atypical and it is straightforward to study their $SU(n) \otimes U(1)$ content and to deduce, from that content, their dimension. This $SU(n) \otimes U(1)$ analysis exhibits, for $\mathcal{C}_1$ and for $\mathcal{C}_2$ two atypical components and one of these components is the same for $\mathcal{C}_1$ and $\mathcal{C}_2$

$$\mathcal{C}_1 = a_1 + a_2 \quad \mathcal{C}_2 = a_2 + a_3$$

This result is identical to that found by Marcu⁹⁾ in the $SU(2|1)$ case and his conclusion was that the four atypical components are the basis of a non-fully reducible representation of $SU(2|1)$ with the structure

We conjecture that the same result holds also for $SU(n|1)$.

3) In the language of supertableaux the appearance of two atypical terms in Eqs (45) and (46) correspond to the appearance of two atypical supertableaux T_1 and T_2 . We have no consistent way to relate individually T_1 or T_2 to the terms $\mathcal{C}_1$ or $\mathcal{C}_2$ or, equivalently to find a definite $SU(n) \otimes U(1)$ content of T_1 and T_2 separately.

However, it is formally possible to compute the dimension of T_1 and T_2 by using the determinant technique¹⁾ and we make the following observations

- i) the results $\dim T_1$ and $\dim T_2$ are not always positive definite for a legal atypical mixed supertableau $c_1 + \bar{c}_1 > n$;
- ii) the sum $\dim T_1 + \dim T_2$ is positive definite;
- iii) the simple relation holds

$$\dim T_1 + \dim T_2 = \dim \mathcal{C}_1 + \dim \mathcal{C}_2$$

These features are clearly in favour of a non-fully reducible representation with four atypical components and we introduce the notation $\llbracket a_2 \rrbracket$ for such a representation. In fact, they express the impossibility in this tensor representation to separate the components T_1 and T_2 of the full tensor, one of them being associated to a trace.

An atypical supertableau is only a part of a non-fully reducible representation of $SU(n|1)$ and only the consideration of pairs of neighbouring atypical supertableaux in the space S makes sense. We then introduce new generalized atypical mixed supertableaux incorporating the two atypical components whose sum is associated to a non-fully reducible representation of $SU(n|1)$. In a generalized mixed supertableau, double bars will indicate which ordinary atypical supertableaux are contained in the generalized one.

4) Let us be a little more precise for the $SU(2|1)$ case for which the non-full reducibility has been proved. We only have two Kac-Dynkin parameters, $a_1 \geq 0$ and a_2 and the atypical values of a_2 are 0 and $-(1+a_1)$. We then have non-fully reducible representations $\llbracket a_1; 0 \rrbracket$ and $\llbracket a_1; -1-a_1 \rrbracket$. The four atypical components of these representations are

$$\llbracket a_1, 0 \rrbracket = \{a_1; 0\} \begin{cases} \nearrow \{a_1+1; 0\} \\ \searrow \{a_1-1; 0\} \end{cases} \rightarrow \{a_1, 0\} \quad \text{for } a_1 \geq 1$$

$$[(0;0)] = [0;0] \begin{cases} \nearrow \{1;0\} \\ \searrow \{0;-1\} \end{cases} \rightarrow [0;0] \quad \begin{array}{l} \text{self-contragradient} \\ \text{dimension 8} \end{array}$$

$$[a_1; -1-a_1] = [a_1; -1-a_1] \begin{cases} \nearrow \{a_1+1; -2-a_1\} \\ \searrow \{a_1-1; -a_1\} \end{cases} \rightarrow [a_1; -1-a_1] \quad \text{for } a_1 > 1$$

$$[(0;-1)] = [0;-1] \begin{cases} \nearrow \{1;-2\} \\ \searrow \{0;0\} \end{cases} \rightarrow [0;-1] \quad \text{dimension 12}$$

We have illustrated in Figs. 15 - 19 the five particular cases of generalized atypical supertableaux $[(2;0)]$, $[(1;0)]$, $[(0;0)]$, $[(0;-1)]$ and $[(1;-2)]$ giving the full set of equivalent generalized supertableaux.

5) For the general case of $SU(n|1)$ things are analogous but more complicated. The non-fully reducible representation is defined by its four atypical components

Giving now an atypical component a_2 by its Kac-Dynkin parameters $a_2 = \{a_1, a_2, \dots, a_{n-1}; -A_k\}$ with $k = 0, 1, \dots, n-1$ we determine its two partners a_1 and a_3 . The general result is the following

$$a_1 = \{a_1, \dots, a_{n-1-k} + 1, a_{n-k} - 1, \dots, a_{n-1}; -A_k + 1\}$$

$$a_3 = \{a_1, \dots, a_{n-1-k} - 1, a_{n-k} + 1, \dots, a_{n-1}; -A_k - 1\}$$

When $a_{n-k} = 0$ the term a_1 has to be modified as follows: define $0 \leq r \leq k$ by $a_{n-k} = a_{n-k+1} = \dots = a_{n-r-1} = 0, a_{n-r} \geq 1$. Then the new term a_1 has $a_{n-r} \Rightarrow a_{n-r} - 1$ and the last parameter is now $-A_r + 1$.

When $a_{n-1-k} = 0$ the term a_3 has to be modified. Defining now $k \leq s \leq n-1$ by $a_{n-1-k} = a_{n-2-k} = \dots = a_{n-s} = 0, a_{n-1-s} \geq 1$. Then the new term a_3 has $a_{n-1-s} \Rightarrow a_{n-1-s} - 1$ and the last parameter becomes $-A_s - 1$.

These results hold for any value of $k = 0, 1, \dots, n-1$. Let us give now a few simple examples

$$\text{i)} \quad a_2 = \{0, \dots, 0; 0\} \quad \text{we obtain} \\ a_1 = \{0, \dots, 0; 1-n\} \quad \text{and} \quad a_3 = \{0, \dots, 0, 1; 0\}$$

a_1 and a_3 are contragradient atypical representations and a_2 is a self-contragradient one. It follows that $\llbracket 0, \dots, 0; 0 \rrbracket$ is also self-contragradient and its dimension is 2^{n+1}

$$\text{ii)} \quad a_2 = \{1, 0, \dots, 0; -1\} \quad \text{we obtain} \\ a_1 = \{0, \dots, 0, 1; 1-n\} \quad \text{and} \quad a_3 = \{1, 0, \dots, 0, 1, 0; 0\}$$

Again a_1 and a_3 are contragradient atypical representations and a_2 is a self-contragradient one. It follows that $\llbracket 1, 0, \dots, 0; -1 \rrbracket$ is also self-contragradient and its dimension is $n 2^{n+1}$

$$\text{iii)} \quad a_2 = \{0, \dots, 0, 1; 0\} \quad \text{we obtain} \\ a_1 = \{0, \dots, 0, 2; 0\} \quad \text{and} \quad a_3 = \{0, \dots, 0; 0\}$$

The dimension of $\llbracket 0, \dots, 0, 1; 0 \rrbracket$ is $(n+1)2^n$

$$\text{iv)} \quad a_2 = \{1, 0, \dots, 0, 1; 0\} \quad \text{we obtain} \\ a_1 = \{1, 0, \dots, 0, 2; 0\} \quad \text{and} \quad a_3 = \{1, 0, \dots, 0; 0\}$$

The dimension of $\llbracket 1, 0, \dots, 0, 1; 0 \rrbracket$ is $(n^2 + n - 1)2^n$. We have illustrated these results in Figs. 20 - 23 by representing one generalized atypical super-tableau of $SU(n|1)$ for each example considered.

8. - CONCLUDING REMARKS

We have constructed the supertableaux of class I of $SU(n|1)$ and they can be classified as follows

- i) $c_1 + \bar{c}_1 < n$: we have the atypical covariant, contravariant and mixed supertableaux uniquely associated to an atypical irreducible representation of $SU(n|1)$;

- ii) $c_1 + \bar{c}_1 = n$: we have typical covariant, contravariant and mixed supertableaux uniquely associated to a typical irreducible representation of $SU(n|1)$;
- iii) $c_1 + \bar{c}_1 > n$: we have two possibilities
 - α) the covariant, contravariant and mixed typical supertableaux are associated to typical irreducible representations of $SU(n|1)$ and we have equivalences;
 - β) the mixed atypical supertableaux are bounded by pairs into generalized mixed supertableaux associated to non-fully reducible representations of $SU(n|1)$ with four atypical components. Again we have equivalences.

The reduction of a tensor product of representations of $SU(n|1)$ considered here can be obtained by using the supertableaux and the ordinary rules of Young tableaux can be applied provided we take care of the case of mixed atypical super tableaux appearing in pairs and re-interpreted as forming generalized mixed atypical supertableaux.

The problem of equivalence of supertableaux has been discussed and it appears only in the case $c_1 + \bar{c}_1 > n$. It happens that only the atypical irreducible representations are described by one supertableau. All the typical representations and the non-fully reducible representations with four atypical components are described by an infinity of equivalent supertableaux. Incidentally, we have not found supertableaux corresponding to non-fully reducible representations of $SU(n|1)$ containing only two or three atypical components or representations where the Cartan operator Q is not diagonalized⁹⁾.

The extension of these considerations to the superunitary groups $SU(n|m)$ is in progress but the classification of supertableaux is entirely analogous to that found here starting from the constraints of legality (1) and (2).

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APPENDIX A

SU(n) $\otimes$ U(1) DECOMPOSITION OF ATYPICAL SUPERTABLEAUX

We consider an atypical supertableau with N rows of b_j covariant boxes and $\bar{N}$ rows of contravariant boxes, $N + \bar{N} < n$ as represented in Fig. 1. The highest weight of this representation belongs to the $SU(n) \otimes U(1)$ component having the lowest eigenvalue of the $U(1)$ generator Q . Such a component is obtained, from the original diagram, by transferring the first contravariant column of $\bar{N}$ boxes to the $U(1)$ part as a row of $\bar{N}$ contravariant boxes. The result is shown in Fig. 2. It is straightforward to determine the $(n-1)$ Kac-Dynkin parameters $a_1, a_2, \dots, a_{n-1}$ on the $SU(n)$ part of this component and we immediately get the relation (7) of the text. The computation of the lowest eigenvalue of Q is straightforward

$$q_{\min} = \frac{1}{n} \left\{ \sum_{j=1}^N b_j - \sum_{j=1}^{\bar{N}} (\bar{b}_j - 1) \right\} - \bar{N}$$

Using now the Eqs. (8) we compute the two sums

$$\begin{aligned} \sum_{j=1}^N b_j &= \sum_{j=1}^N s a_j \\ \sum_{j=1}^{\bar{N}} \bar{b}_j &= \sum_{j=1}^{\bar{N}} s a_{n-j} = n \sum_{j=1}^{\bar{N}} a_{n-j} - \sum_{j=1}^{\bar{N}} (n-j) a_{n-j} \end{aligned}$$

and we get

$$q_{\min} = \frac{1}{n} \sum_{j=1}^{n-1} s a_j - \bar{N} - \sum_{j=1}^{\bar{N}} a_{n-j} = \frac{1}{n} \sum_{j=1}^n s a_j$$

The value obtained for a_n is atypical

$$a_n = -\bar{N} - \sum_{j=1}^{\bar{N}} a_{n-j} = -A_{\bar{N}}$$

APPENDIX B

SU(n) ⊗ U(1) DECOMPOSITION OF TYPICAL SUPERTABLEAUX

1) For supertableaux of type I^0 the $SU(n) \otimes U(1)$ component containing the highest weight is shown in Figs. 4 and 6 [see also Ref. 5)]. We now compute for this component the eigenvalue of the $U(1)$ operator Q . The result is simply

$$q_{\min} = \frac{1}{n} \sum_{j=1}^n b_j + \Delta \quad (B.1)$$

with

$$\begin{aligned} \Delta &= c_1 - n && \text{for covariant supertableaux } I_1^0 \\ \Delta &= -\bar{c}_1 && \text{for mixed supertableaux } I_0^0 \end{aligned}$$

The set of parameters $a_1, a_2, \dots, a_{n-1}$ is then trivially obtained as in the text and using Eq. (17) we get

$$\sum_{j=1}^n b_j = \sum_{s=1}^{n-1} s a_s + n b_n \quad (B.2)$$

The Kac-Dynkin parameter a_n is given by

$$a_n = q_{\min} - \frac{1}{n} \sum_{s=1}^{n-1} s a_s$$

Using Eqs. (B.1) and (B.2) we get

$$a_n = b_n + \Delta$$

The co-ordinate y_1 is given by Δ .

2) For supertableaux of type I^n the $SU(n) \otimes U(1)$ component containing the highest weight is shown in Figs. 8 and 10 [see also Ref. 5)]. We now compute for this component the eigenvalue of Q . The result is

$$q_{\min} = -\frac{1}{n} \sum_{j=1}^n (\bar{b}_j - 1) + \Delta \quad (B.3)$$

with

$$\left\{ \begin{array}{ll} \Delta = -\bar{c}_1 & \text{for contravariant supertableaux } I_0^n \\ \Delta = c_1 - n & \text{for mixed supertableaux } I_1^n \end{array} \right.$$

The set of parameters $a_1, a_2, \dots, a_{n-1}$ is obtained as in the text and using Eq. (23) we get

$$\sum_{s=1}^n (\bar{b}_s - 1) = n(\bar{b}_1 - 1) - \sum_{s=1}^{n-1} s a_s \quad (B.4)$$

For the Kac-Dynkin parameter a_n we obtain from Eqs. (B.3) and (B.4) the value

$$a_n = 1 - \bar{b}_1 + \Delta$$

leading again to

$$y_1 = -\Delta$$

3) Now for mixed supertableaux of type I^j with $0 < j < n$ the $SU(n) \otimes U(1)$ component containing the highest weight is shown in Figs. 12 and 14. We compute for this component the eigenvalue of Q

$$q_{\min} = \frac{1}{n} \left[\sum_{s=1}^{n-j} b_s - \sum_{s=1}^j (\bar{b}_s - 1) \right] + \Delta \quad (B.5)$$

with Δ given by

$$\left\{ \begin{array}{ll} \Delta = -\bar{c}_1 & \text{for mixed supertableaux } I_0^j \\ \Delta = c_1 - (n-j) - j = c_1 - n & \text{for mixed supertableaux } I_1^j \end{array} \right.$$

Again we immediately obtain the set of parameters $a_1, a_2, \dots, a_{n-1}$ as in the text. We now use the Eqs. (30) to compute the sums

$$\begin{aligned} \sum_{s=1}^{n-j} b_s &= \sum_{s=1}^{n-j} s a_s + (n-j) \sum_{s=j+1}^{n-1} a_s + (n-j)(1 - \bar{b}_1) \\ \sum_{s=1}^j (\bar{b}_s - 1) &= -j(1 - \bar{b}_1) + (n-j) \sum_{s=j+1}^{n-1} a_s - \sum_{s=j+1}^{n-1} s a_s \end{aligned}$$

Combining both results

$$\sum_{s=1}^{n-j} b_s - \sum_{s=1}^j (\bar{b}_s - 1) = \sum_{s=1}^{n-1} s a_s + n(1 - \bar{b}_1) \quad (\text{B.6})$$

Using Eqs. (B.5) and (B.6) we get

$$a_n = (1 - \bar{b}_1) + \Delta$$

corresponding again to

$$y_1 = -\Delta$$

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FIGURE CAPTIONS

- Fig. 1 : Atypical supertableau $c_1 + \bar{c}_1 < n$.
- Fig. 2 : $SU(n) \otimes U(1)$ component of Fig. 1 containing the highest weight.
- Fig. 3 : Typical covariant supertableau $c_1 + \bar{c}_1 \geq n$.
- Fig. 4 : $SU(n) \otimes U(1)$ component of Fig. 3 containing the highest weight.
- Fig. 5 : Mixed supertableau of type I_1^0 .
- Fig. 6 : $SU(n) \otimes U(1)$ component of Fig. 5 containing the highest weight.
- Fig. 7 : Typical contravariant supertableau $c_1 + \bar{c}_1 \geq n$.
- Fig. 8 : $SU(n) \otimes U(1)$ component of Fig. 7 containing the highest weight.
- Fig. 9 : Mixed supertableau of type I_0^n .
- Fig. 10 : $SU(n) \otimes U(1)$ component of Fig. 9 containing the highest weight.
- Fig. 11 : Mixed supertableau of type I_0^j .
- Fig. 12 : $SU(n) \otimes U(1)$ component of Fig. 11 containing the highest weight.
- Fig. 13 : Mixed supertableau of type I_1^j .
- Fig. 14 : $SU(n) \otimes U(1)$ component of Fig. 13 containing the highest weight.
- Fig. 15 : Generalized atypical supertableaux of $SU(2|1)$ for $\llbracket 2;0 \rrbracket$.
- Fig. 16 : Generalized atypical supertableaux of $SU(2|1)$ for $\llbracket 1;0 \rrbracket$.
- Fig. 17 : Generalized atypical supertableaux of $SU(2|1)$ for $\llbracket 0;0 \rrbracket$.
- Fig. 18 : Generalized atypical supertableaux of $SU(2|1)$ for $\llbracket 0;-1 \rrbracket$.
- Fig. 19 : Generalized atypical supertableaux of $SU(2|1)$ for $\llbracket 1;-2 \rrbracket$.

- Fig. 20 : Generalized atypical supertableau of $SU(n|1)$ for $\llbracket 0, \dots, 0; 0 \rrbracket$.
- Fig. 21 : Generalized atypical supertableau of $SU(n|1)$ for $\llbracket 1, 0, \dots, 0; -1 \rrbracket$.
- Fig. 22 : Generalized atypical supertableau of $SU(n|1)$ for $\llbracket 0, \dots, 0, 1; 0 \rrbracket$.
- Fig. 23 : Generalized atypical supertableau of $SU(n|1)$ for $\llbracket 1, 0, \dots, 0, 1; 0 \rrbracket$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17

Fig. 18

Fig. 19

$$c_1 = \bar{c}_1 = n$$

Fig. 20

$$c_1 = \bar{c}_1 = n-1$$

Fig. 21

$$c_1 = n$$

Fig. 22

$$c_1 = n$$

Fig. 23