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ON DIMENSIONAL REDUCTION OF GAUGE THEORIES:

SYMMETRIC FIELDS AND HARMONIC EXPANSION

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A B S T R A C T

We compare the four-dimensional symmetric fields obtained by the coset space dimensional reduction scheme to the infinite tower of fields given by the harmonic expansion in a $4+N$ dimensional gauge theory coupled to fermions on a space-time $M^4 \times S/R$.

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1. - INTRODUCTION

During the past few years much effort has been devoted to derive physically interesting four-dimensional models starting from theories defined in more than four dimensions as in this way, usually, one can eliminate the amount of arbitrariness in the model. One can view the extra dimensions in these theories in two different ways: either one can regard them as merely mathematical tools used for model building or alternatively, one can assume that they have physical significance. In the former case ("dimensional reduction") generally one obtains only a finite number of fields in four dimensions while in the latter, if the extra dimensions form a spacelike compact space, they manifest themselves in the form of "infinite towers" of four dimensional fields.

One of the multidimensional theories that received a lot of attention recently is a Yang-Mills theory with gauge group G coupled to fermions in $4+N$ dimensions where the $4+N$ dimensional space-time is assumed to be the direct product of Minkowski space and a compact coset space S/R . Applying to this model the dimensional reduction scheme of Ref. 1)-3) - we call it coset space dimensional reduction (CSDR) scheme - one obtains an attractive framework for the unification of gauge and Higgs fields^{4)-6)*}). In the CSDR scheme one keeps only those fields of the $4+N$ dimensional theory that depend in a special way on the extra co-ordinates, i.e., those that are invariant under S operating on the coset S/R . These fields constitute in four dimensions a gauge Higgs fermion theory whose gauge group H is some subgroup of G and whose Higgs potential is completely determined and is frequently of the symmetry breaking type. Some of the four-dimensional fields survive the dimensional reduction with zero mass, the others acquire masses of the order of R , where R is the size of the extra dimensions. In some models⁶⁾, one assumes that at the end $R \rightarrow 0$ thus only the zero-mass fields are interesting, but there are models⁵⁾ where R must be $O(G_F^{1/2})$.

As a first step towards regarding the extra dimensions in this theory to be physical ones in this paper we investigate the relation between the S symmetric fields of the CSDR scheme and the infinite tower of fields one obtains from the same theory by keeping the geometry $(M^4 \times S/R)$ fixed but allowing arbitrary fluctuations in the $4+N$ dimensional gauge and fermion fields, i.e., by allowing them to depend, in an arbitrary way, on the extra co-ordinates. An especially interesting question is the relation between the masses of the symmetric CSDR modes and of the non-symmetric ones and in particular the existence of non-symmetric zero mass fields.

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A geometrical approach to the symmetry breaking in gauge theories was also considered by M.E. Mayer⁷⁾.

We take into account the arbitrary dependence of the $4+N$ dimensional fields on the extra co-ordinates by a harmonic expansion on the coset space as suggested in Ref. 8). To set up the harmonic expansion we have to determine a solution ("vacuum") of the gauge theory in $4+N$ dimensions around which we expand. As in the most interesting cases in the CSDR scheme, one has two stages (stage one: after the dimensional reduction but before the symmetry breaking, stage two: after the symmetry breaking), we determined the appropriate "vacuum" solutions for both of them and showed that the symmetric fields of the CSDR scheme appear in a systematic way in the harmonic expansions around them.

Furthermore, we investigated the masses of these fluctuations by using the harmonic expansions in the linearized field equations. For the "vacuum" describing the first stage of the CSDR scheme, we found tachyons in the mass spectrum signalling the spontaneous symmetry breaking. For the fluctuations around the "vacuum" corresponding to the second stage of the CSDR scheme, we found no tachyons but we found more zero mass vectors than predicted by the CSDR scheme; i.e., we found that the symmetry breaking pattern is modified by the non-symmetric fluctuations.

The paper is organized in the following way: in Section 2 we describe briefly the coset space geometry and the CSDR scheme as applied to the pure G gauge theory in $4+N$ dimensions. In Section 3 we work out in detail the harmonic expansion for both stages of the CSDR scheme in this case and analyze the masses of the fluctuations. Section 4 contains the application of both procedures for fermions. We make our conclusions in Section 5.

2. - DESCRIPTION OF THE COSET SPACE S/R AND OF THE CSDR SCHEME

We start with a pure gauge theory with gauge group G in $4+N$ dimensional space-time, where the extra N dimensions are assumed to form a compact coset space S/R . (We also suppose that all groups G, S, R are compact and semi-simple.) For the co-ordinates on this space-time, we write $z^M = (x^i, y^\alpha)$ where x^i ($i = 0, \dots, 3$) denotes the Minkowski co-ordinates and y^α ($\alpha = 1, \dots, N$) label the left cosets of S with respect to the subgroup R . This means that from each coset we have a representative element $L(y)$ such that all $g \in S$ can be written as $g = L(y)h$ where $h \in R$. (We shall denote the tangent space ("flat") indices on S/R by early alphabet Latin suffixes a, b , etc.). Following Ref. 8), we define a covariant basis on S/R with the aid of the one form

$$e(y) = L^{-1}(y) dL(y) = e_{\hat{\alpha}}^{\alpha}(y) Q_{\hat{\alpha}} dy^{\alpha} \quad (1)$$

Here $Q_{\hat{a}}$ ($\hat{a} = 1, \dots, \dim S$) are the generators of S satisfying the appropriate commutation relations: $[Q_{\hat{a}}, Q_{\hat{b}}] = f_{\hat{a}\hat{b}\hat{c}} Q_{\hat{c}}$ with $f_{\hat{a}\hat{b}\hat{c}}$ being the totally antisymmetric structure constants of S . As R is a subgroup of S , the index $\hat{a}$ splits into two sets: $\bar{a}$ ($\bar{a} = 1, \dots, \dim R$) corresponding to the generators of R (i.e., $f_{\bar{a}\bar{b}\bar{c}}$ are the structure constants of R) and a ($a = 1, \dots, N$) corresponding to the coset space generators and - as a further consequence - some of the structure constants vanish: $f_{\bar{a}b\bar{c}} = 0 = f_{\bar{a}b\bar{c}}$. Furthermore, in this paper we assume that S/R is a symmetric space, this implies that $f_{abc} = 0$.

The coefficients $e_{\alpha}^a(y)$ constitute an invertible $N \times N$ matrix, the columns of which form the various vectors of the covariant basis on S/R . The inverse of this matrix, e_b^{α} , ($e_b^{\alpha} e_{\alpha}^a = \delta_b^a$, $e_a^{\beta} e_{\beta}^{\alpha} = \delta_a^{\alpha}$) defines the elements of the contravariant basis in a similar way.

The group S acts on S/R by multiplying the representative elements, $L(y)$ from the left by $g \in S$. The product will belong to a coset, whose representative is $L(y')$, i.e., we can write

$$g L(y) = L(y') h \quad h \in R \quad (2)$$

The solution of this equation for y' and h as a function of y and g defines the left translation of S/R by g . In particular, the infinitesimal change in y , $y^{\alpha'} = y^{\alpha} + \delta y^{\alpha}$ caused by an infinitesimal left translation with $g = 1 + \delta g^{\hat{a}} Q_{\hat{a}}$ defines the Killing vectors, K_b^{α} associated with the left translations: $\delta y^{\alpha} = \delta g^{\hat{a}} K_b^{\alpha}(y)$. Solving (2) in the infinitesimal case (assuming $h = 1 + \delta h^{\bar{a}} Q_{\bar{a}}$) one obtains:

$$\begin{aligned} K_b^{\alpha}(y) &= D_{\hat{b}a}(L) e_a^{\alpha} \\ \delta h^{\bar{a}} &= \delta g^{\hat{c}} (D_{\hat{c}\bar{a}}(L) - D_{\hat{c}b}(L) \pi_b^{\bar{a}}) = \delta g^{\hat{c}} H_{\hat{c}\bar{a}} \end{aligned} \quad (3)$$

where $\pi_b^{\bar{a}} = e_b^{\alpha}(y) e_{\alpha}^{\bar{a}}(y)$ and $D_{\hat{a}\hat{b}}$ denotes the matrices of the adjoint representation of S defined by $g^{-1} Q_{\hat{a}} g = D_{\hat{a}\hat{b}}(g) Q_{\hat{b}}$.

Working out the transformation rules of $e_{\alpha}^a(y)$ and $e_{\alpha}^{\bar{a}}(y)$ under left translation, one can immediately see that e_{α}^a undergoes a rotation in its "Latin index" $e_{\alpha}^a(y') = e_{\alpha}^b(y) D_{ba}(h^{-1})$ (tangent space rotation) while $e_{\alpha}^{\bar{a}}$ transforms as a connection. The former fact means that R must be embedded into $SO(N)$ by $Q_{\bar{c}} = -\frac{1}{2} f_{cab} \Sigma^{ab}$ where Σ^{ab} are the $SO(N)$ generators⁸⁾ and e_{α}^a stays invariant under the combined action of left translation and tangent space rotation with $D(h)$, while the latter can be used to define a covariant derivative on S/R ⁹⁾:

$$\nabla_a = \frac{1}{R} e_a^{\alpha} (\partial_{\alpha} + e_{\alpha}^{\bar{b}} Q_{\bar{b}}) = \frac{1}{R} (\partial_a + \pi_a^{\bar{b}} Q_{\bar{b}}) \quad (4)$$

where R is a parameter with the dimension of length which represents the scale of S/R . As this is the only relevant length scale in the theory, from now on we set $R = 1$.

Putting all these together, finally the action of our theory has the form

$$A = \int d^4x d^N y |\det e_a^\alpha| \frac{1}{2} \text{Tr} F_{MN}^2 \quad (5)$$

$$F_{MN} = F_{MN}^k T_k = \nabla_M A_N - \nabla_N A_M - [A_M, A_N]$$

where T_k are the generators of G ($\text{Tr} T_k T_\ell = -2\delta_{k\ell}$), $\nabla_M = (\partial_i, \nabla_a)$, $A_M = (A_i, A_a)$ and the signature is $(-, +, \dots, +)$.

If on the coset space S/R we have gauge vectors A_α (taking their values in the Lie algebra of G) and scalar fields ϕ (transforming under G as some representation D_ϕ) then they change under infinitesimal left translation as

$$\delta_\beta \phi = -K_\beta^\alpha \partial_\alpha \phi; \quad \delta_\beta A_\alpha = -K_\beta^\rho \partial_\rho A_\alpha - (\partial_\alpha K_\beta^\rho) A_\rho \quad (6)$$

As A_α and ϕ transform non-trivially under G we say - following Refs.1)-3), and 10) - that they are S symmetric if the changes, (6), caused by the action of S instead of vanishing, are equivalent to an infinitesimal, compensating gauge transformation, W_β associated with each Killing vector K_β^α :

$$\delta_\beta A_\alpha \equiv -K_\beta^\rho \partial_\rho A_\alpha - (\partial_\alpha K_\beta^\rho) A_\rho = \partial_\alpha W_\beta - [A_\alpha, W_\beta] \quad (7)$$

$$\delta_\beta \phi \equiv -K_\beta^\alpha \partial_\alpha \phi = D_\phi(W_\beta) \phi \quad (8)$$

As the commutator of two left translations is also a left translation $[\delta_\alpha, \delta_\beta] = f_{\alpha\beta\gamma} \delta_\gamma$ the gauge transformations, W_β , must satisfy the constraint

$$K_\alpha^\rho \partial_\rho W_\beta - K_\beta^\rho \partial_\rho W_\alpha + [W_\alpha, W_\beta] + f_{\alpha\beta\gamma} W_\gamma = 0 \quad (9)$$

In the CSDR scheme of Refs. 1)-3), one assumes that both the Minkowski (A_i) and the coset space (A_a) components of the $4+N$ dimensional gauge potential depend on both x^i and y^α , however, their y dependence is restricted by requiring them to be S symmetric in the sense of Eqs. (7) and (8). For such fields these equations guarantee that gauge invariant scalars made of them, (e.g., the Lagrangian) are constant over the coset space S/R and hence can be computed

at any point on S/R . In the CSDR scheme, one adopts the parametrization $L(y) = \exp\{y^\alpha \delta_\alpha^a Q_a\}$ and solving Eqs. (7)-(9) at $y = 0$ determines the Lagrangian here, which after integrating it over S/R gives the effective Lagrangian of the dimensionally reduced symmetric fields in four dimensions.

Indeed using this parametrization and working in the $W_a (y = 0) = 0$ ($a = 1, \dots, N$) gauge [for more details see Ref. 6)], Eq. (9) yields at $y = 0$:

$$[W_{\bar{a}}, W_{\bar{b}}] + f_{\bar{a}\bar{b}\bar{c}} W_{\bar{c}} = 0 \quad (10a)$$

$$\partial_a W_{\bar{b}} = 0 \quad (10b)$$

$$\partial_a W_{\bar{b}} - \partial_b W_{\bar{a}} = -f_{ab\bar{c}} W_{\bar{c}} \quad (10c)$$

Since $W_{\bar{a}}$ are in the Lie algebra of G and $f_{\bar{a}\bar{b}\bar{c}}$ are the structure constants of R , (10a) means that R must be a subgroup of G and $W_{\bar{a}} (y = 0) = -\psi_{\bar{a}}$ where the constant $\psi_{\bar{a}}$'s are the generators of this R subgroup of G . As the Minkowski components of the gauge vector, A_i , are obviously scalars on S/R belonging to the adjoint representation of G , from (8) we obtain at $y = 0$:

$$\partial_a A_i = 0 \quad ; \quad [A_i, \psi_{\bar{a}}] = 0 \quad (11)$$

Equation (11) implies that A_i commutes with the R subgroup of G , i.e., the surviving gauge symmetry in four dimensions is H , the centralizer of R in G . Finally, from Eq. (7) one gets for the coset space components $A_a = e_a^\alpha A_\alpha$ at $y = 0$:

$$f_{c\bar{a}\bar{b}} A_b - [A_c, \psi_{\bar{a}}] = 0 \quad (12a)$$

$$\partial_a A_b = -\partial_b W_a \quad (12b)$$

After the integration over S/R , A_b 's will become Minkowski scalar fields restricted by (12a). As $f_{c\bar{a}\bar{b}}$ ($\bar{a} = 1, \dots, \dim R$, $c, b = 1, \dots, N$) are the representations of the generators of R over S/R and $\psi_{\bar{a}}$ form a representation of these generators in the Lie algebra of G , (12a) may be solved for the unconstrained fields by using Schurr's lemma¹⁾. However, without doing this explicitly, we can study the self-interaction of the scalars as their potential is proportional to the trace of the sum of the squares of $F_{ab} = \nabla_a A_b - \nabla_b A_a - [A_a, A_b]$. Using Eqs. (12b), (10c) and (10a) one gets at $y = 0$

$$F_{ab} = f_{abc} \psi_c - [A_a, A_b] \quad (13)$$

From Eq. (13) we see that, in general, the scalar potential of the four-dimensional theory will not vanish in the "naive" ground state, (i.e., for $A_a \equiv 0$). However, in the very interesting special case - which applies to almost all actual models built by the CSDR scheme - when S is also a subgroup of G from Eqs. (10)-(13) we conclude that the scalar potential can vanish in the "true" ground state when $A_a \equiv A_a^g$ satisfy

$$[A_a^g, A_b^g] = f_{abc} \psi_c, \quad [A_c^g, \psi_a] = f_{cab} A_b^g \quad (14)$$

Therefore, in this case the resulting scalar potential is necessarily of the symmetry breaking type, and its minimum breaks H down to the centralizer of S in G . Clearly we can distinguish two stages of the model building; the first one - corresponding to the "naive" ground state - occurring after the dimensional reduction but before the symmetry breaking and the second one occurring after the spontaneous symmetry breaking.

Thus, we see that the CSDR scheme provides us in four dimensions a small set of fields [gauge vectors A_i restricted by (11) and scalars A_a restricted by (12)] with definite symmetry properties and couplings to each other.

3. - HARMONIC EXPANSION ON THE COSET SPACE

Next, we want to find the place of these fields among the infinitely many four-dimensional fields one obtains by assuming that the extra N dimensions are physical ones, and accounting for this fact by expanding both A_i and A_a in a harmonic expansion on the coset S/R . To establish this harmonic expansion, we must first determine the ground state configuration ("vacuum") of the $4+N$ dimensional gauge theory around which we expand. This configuration must be a solution of the field equations in $M^4 \times S/R$:

$$\partial_i F_{ai} - [A_i, F_{ai}] + \nabla_b F_{ab} - [A_b, F_{ab}] = 0 \quad (15)$$

$$\partial_j F_{ij} - [A_j, F_{ij}] + \nabla_a F_{ia} - [A_a, F_{ia}] = 0$$

which is consistent with the symmetries of $M^4 \times S/R$; i.e., with Poincaré $\times S$. This means that in this background configuration A_i must vanish and $A_a^B(x, y) \equiv A_a^B(y)$ with A_a^B being S symmetric. Furthermore, to make contact with the

two stages of the CSDR scheme, the field strength of this configuration F_{ab}^B at $y = 0$ must be the same as F_{ab} of Eq. (13), i.e., $F_{ab}^B(y = 0) = f_{abc} \bar{\psi}_c^-$ for the first stage and $F_{ab}^B(y = 0) = 0$ for the second.

Let us concentrate primarily on the first stage in the CSDR scheme. Then, a possible candidate for $A_a^B(y)$ is the solution found recently by Randjbar-Daemi and Percacci¹¹⁾ in the context of the spontaneously compactified Einstein-Yang-Mills system:

$$A_a^B = -\pi_a^B \bar{\psi}_B \quad (16)$$

where - as previously - $\bar{\psi}_a^-$ generate an R subgroup of G : $[\bar{\psi}_a^-, \bar{\psi}_b^-] = f_{abc}^- \bar{\psi}_c^-$. Indeed, it is easy to see using the Maurer-Cartan equations satisfied by $e_a^{\bar{\alpha}}$ that for this A_a^B , $F_{ab}^B = f_{abc}^- \bar{\psi}_c^-$ all over the coset S/R and together with $A_i = 0$ it really solves (15). Note that using the parametrization of the CSDR scheme $[L(y) = \exp\{y^\alpha \delta_\alpha^a Q_a\}]$, one obtains $A_a^B(y = 0) = 0$. Furthermore, A_a^B changes under infinitesimal left translation with $g = 1 + \delta g^a Q_a$ as

$$A_a^{B'} = A_a^B + \delta g^d H_{dB}^a f_{abc} A_c^B + \delta g^d \left\{ e_a^\alpha \partial_\alpha (H_{dB}^a \bar{\psi}_B) - [A_a^B, H_{dB}^a \bar{\psi}_B] \right\}$$

where H_{dB}^a is defined in Eq. (3). The second term is an infinitesimal rotation in the tangent space [determined by δh of Eq. (3)] while the third is a gauge transformation. Therefore, the action of the left translation on the background solution can be compensated by an infinitesimal rotation in the tangent space and by an infinitesimal gauge transformation $\Lambda = -\delta g^d H_{dB}^a \bar{\psi}_B$. This means that A_a^B is really S symmetric and the compensating gauge transformations, W_B , associated to the Killing vectors K_B are $W_B = -H_{dB}^a \bar{\psi}_B$. It is straightforward to show that this W_B really satisfies (9), and furthermore, using the CSDR parametrization for $L(y)$, one finds that at $y = 0$, $W_a = 0$, $W_a^B = -\bar{\psi}_a^-$ and (12b) is indeed satisfied. From all these, we conclude that A_a^B , as given by Eq. (16), is a gauge field hidden in the first stage of the CSDR scheme, that gives the right background around which we must set up the harmonic expansion.

Following the general strategy of Ref. 8), we use the matrices of the unitary irreducible representations of S for the harmonic expansion on S/R :

$$A_i(x, y) = \sum_{(m)} D_{\ell j}^{(m)}(L^{-1}(y)) a_i^{(m) \ell j}(x) \quad (17)$$

$$A_a(x, y) = A_a^B + \sum_{(n)} D_{pq}^{(n)}(L^{-1}(y)) \phi_a^{(n) pq}(x) = A_a^B + \bar{A}_a$$

In (17) $D^{(m)}$ and $D^{(n)}$ denote the various matrix representations of S and $a_i^{(m) \ell j}$, $\phi_a^{(n) pq}$ are x dependent fields taking their values in the Lie algebra

of G . The problem with (17) is that since we are working on the coset space S/R rather than on the group S itself, both expansions are restricted in the sense that not all of the inequivalent S irreps give a contribution and even those that contribute do this only with some of their matrix elements⁸⁾ (with rows corresponding to subspaces carrying representation of R). We determine the relevant matrix elements by requiring that the resulting infinite component four dimensional theory possesses the symmetry of the vacuum, i.e., $\text{Poincaré} \times S$. This means that $a_i^{(m)\ell j}$ and $\phi_a^{(n)pq}$ must be Lorentz vectors and scalars, respectively, while under a generalized S transformation - that contains together with the left translation the previously found compensating tangent space rotation and gauge transformation as well - they must transform according to some irrep of S . As under an infinitesimal S transformation with $g = 1 + \delta g \hat{Q}_C$, we have

$$A_i' = \sum_{(m)} \left[\left\{ D_{\ell j}^{(m)}(L^{-1}) + \delta g \hat{c} (H_{\hat{c}\bar{a}} D_{\ell j}^{(m)}(Q_{\bar{a}} L^{-1}) - D_{\ell j}^{(m)}(L^{-1} Q_{\hat{c}})) \right\} a_i^{(m)\ell j} + \delta g \hat{c} D_{\ell j}^{(m)}(L^{-1}) [a_i^{(m)\ell j}, \psi_{\bar{a}}] H_{\hat{c}\bar{a}} \right] \quad (18a)$$

$$A_a' = A_a^B + \sum_{(n)} \left[\left\{ D_{pq}^{(n)}(L^{-1}) + \delta g \hat{c} (H_{\hat{c}\bar{a}} D_{pq}^{(n)}(Q_{\bar{a}} L^{-1}) - D_{pq}^{(n)}(L^{-1} Q_{\hat{c}})) \right\} \phi_a^{(n)pq} + \delta g \hat{c} H_{\hat{c}\bar{a}} D_{pq}^{(n)}(L^{-1}) \left\{ -f_{a\bar{a}b} \phi_b^{(n)pq} + [\phi_a^{(n)pq}, \psi_{\bar{a}}] \right\} \right] \quad (18b)$$

our requirement amounts to the vanishing in both (18a) and (18b) the terms containing $H_{\hat{c}\bar{a}}$:

$$D_{\ell u}^{(m)}(Q_{\bar{a}}) a_i^{(m)\ell j} + [a_i^{(m)u j}, \psi_{\bar{a}}] = 0 \quad (19)$$

$$D_{ps}^{(n)}(Q_{\bar{m}}) \phi_b^{(n)pq} - f_{b\bar{m}a} \phi_a^{(n)sq} + [\phi_b^{(n)sq}, \psi_{\bar{m}}] = 0 \quad (20)$$

The restrictions in the expansions (17) stem from Eqs. (19) and (20) as only those $D_{\ell j}^{(m)} a_i^{(m)}$ and $D_{pq}^{(n)} \phi_a^{(n)}$ pairs are present in (17) that solve Eqs. (19) and (20). As $D_{\ell j}^{(m)}(Q_{\bar{a}})$ ($D_{pq}^{(n)}(Q_{\bar{a}})$) generate a representation of R in the S irrep $D^{(m)}$ ($D^{(n)}$) and $f_{a\bar{b}c}(\psi_{\bar{b}})$ are the representations of the generators of R in the coset S/R (respectively in the Lie algebra of G) for any $a_i^{(m)\ell j}$ $\phi_a^{(n)pq}$ solution of Eqs. (19) and (20), the various indices of these fields [S irrep index " ℓ " and " p ", "internal or G index" and S/R index a for $\phi_a^{(n)}$] will be restricted and connected to each other.

Comparing Eqs. (19) and (20) with Eqs. (11) and (12), we see that the x dependent fields of the CSDR scheme indeed appear in the harmonic expansion; they are the coefficients of the trivial representation of S in (17); i.e., those that stay invariant under S . Though for our present investigation we shall not need the actual solutions of Eqs. (19) and (20), it is worth describing briefly how their solution goes to shed some light on the nature of restrictions they impose.

The solution of Eqs. (19) and (20) can - at least in principle - be found by using the Schurr's lemma. (We assume - without loss of generality - that we use such matrix representations of S which become block diagonal when restricted to the subgroup R .) The first step one has to make is to decompose into R irreps the induced representation of R acting in the tangent space of S/R , (i.e., $\underline{v}$ in the decomposition $\text{adj } S = \text{adj } R + \underline{v}$), for separating the various R irreducible components of the $SO(N)$ vector $\phi_a^{(n)pq}$. The second step is to decompose into R irreps the induced representation of R acting in the Lie algebra of G to obtain the R irreducible components for both $a_i^{(m)}$ and $\phi_a^{(n)}$ in their "internal" or "G" index. Then, from (19), Schurr's lemma yields that for each R irrep part of a_i in the internal index we have a harmonic expansion that contains all S irreps $D^{(m)}$, which, when restricted to the subgroup R , contain this particular R irrep. [Of course, all those $D^{(m)}$'s contribute only with those of their rows that correspond to this R irrep, see Ref. 8).] The situation is more complicated for the ϕ field as here one has to decompose into R irreps the tensor product of the R irreps coming from the tangent space and internal indices and at the end one obtains a similar harmonic expansion for these final R irrep parts.

Having done all this, finally we must multiply all $a_i^{(m)} \phi_a^{(n)}$ fields present in (17) by $\sqrt{\dim(D^{(n)})/\dim(\tilde{D})}$ [where $\tilde{D}$ is that particular R irrep to which the component in question is assigned by Eqs. (19) and (20)] to ensure that all of them have the right coefficient in the kinetic term.

Next, we illustrate this procedure in a simple example we shall use later: a model where the extra dimensions from a two sphere $S^2 = SU(2)/U(1)$ and we take G to be $SU(2)$ as well. We parametrize the sphere by $L(\theta, y) = \exp(-\varphi Q_3) \exp(-\theta Q_2) \exp(\varphi Q_3)$ where $Q_{\hat{a}}$, $\hat{a} = 1, 2, 3$ are the generators of $S = SU(2)$; $y^1 = \theta$, $y^2 = \varphi$ and we identify R with the $U(1)$ generated by Q_3 . For the expansion we use the $D_{\lambda m}^{(\ell)}$ matrices of the $2\ell+1$ dimensional unitary irreps of $SU(2)$ with the phase convention

$$D_{\lambda m}^{(\ell)}(Q_3) = -i \text{diag}(\ell, \ell-1, \dots, -\ell)$$

$$D_{\lambda m}^{(\ell)}(Q_{\hat{a} \pm i 2} L^{-1}) = -i \sqrt{\frac{(\ell \mp \lambda)(\ell \pm \lambda + 1)}{2}} D_{\lambda \pm 1, m}^{(\ell)}(L^{-1})$$

The $SO(2)$ vector $\phi_a^{(\ell)mn}$ can be decomposed

$$\phi_{\pm}^{(\ell)mn} = \frac{1}{\sqrt{2}} (\phi_1^{(\ell)mn} \mp i \phi_2^{(\ell)mn})$$

transforming under R as $\pm i \phi_{\pm}^{(\ell)mn}$. As all the different rows of $D_{\lambda m}^{(\ell)}(L^{-1})$ carry different representations of R the expansion would look like:

$$A_i(x, \theta, \varphi) = \sum_{\ell \geq |2|} \sum_{m=-\ell}^{+\ell} D_{\lambda m}^{(\ell)}(L^{-1}) \sqrt{2\ell+1} a_i^{(\ell)m}(x)$$

$$A_{\pm}(x, \theta, \varphi) = \sum_{\ell \geq |2|} \sum_{m=-\ell}^{+\ell} D_{\lambda m}^{(\ell)}(L^{-1}) \sqrt{2\ell+1} \phi_{\pm}^{(\ell)m}(x) \quad (17')$$

and Eqs. (19) and (20) determine the λ value for the various elements of the matrices $a_i^{(\ell)m}$, $\phi_{\pm}^{(\ell)m}$:

$$-i\lambda a_i^{(\ell)m} + [a_i^{(\ell)m}, T_3] = 0 \quad (19')$$

$$-i\lambda \phi_{\pm}^{(\ell)m} \mp i \phi_{\pm}^{(\ell)m} + [\phi_{\pm}^{(\ell)m}, T_3] = 0 \quad (20')$$

Here, we took ψ_a to be T_3 [$T_i = 1, 2, 3$ are the generators of $G = SU(2)$]. Noting that any Lie algebra valued matrix M can be written as $M^{(+)}(T_1 + iT_2) + M^{(-)}(T_1 - iT_2) + M^{(0)}T_3$, we can read off from (19') the λ values of $a_i^{(\ell)m}[\pm 0]$: $\lambda^+ = 1$, $\lambda^- = -1$, $\lambda^0 = 0$, while from (20'), one gets for the λ 's of $\phi_{\pm}^{(\ell)m}[\pm 0]$: $\lambda_+^+ = 0$, $\lambda_+^- = -2$, $\lambda_+^0 = -1$, $\lambda_-^+ = 2$, $\lambda_-^- = 0$, $\lambda_-^0 = 1$. This then completes the harmonic expansion for this model. The fields belonging to zero λ values have their expansion starting with the trivial representation of $SU(2)$; the coefficients of $D^{(0)}$ in (17') are the symmetric fields in the first stage of the CSDR scheme applied to this model.

Now the goal in setting up the harmonic expansion is that using it, we determine the masses of all the coefficient fields not just that of the symmetric ones. We derive these masses from substituting the expanded form of A_i and A_a into the field equations (15) keeping only the terms linear in $a_i^{(m)}$ and $\phi_a^{(n)}$. In doing so, it proves very useful to introduce the general covariant derivative

$$D_a = \nabla_a - [A_a^B,] = \nabla_a + \pi_a^{\bar{B}} [\psi_{\bar{B}},] \quad (21)$$

Indeed, using D_a in (15) and imposing the $\partial_i A_i + D_a \bar{A}_a = 0$ covariant gauge condition on the fluctuations around the background, one obtains,

$$-\partial_i^2 \bar{A}_a - D_b D_b \bar{A}_a + [D_b, D_a] \bar{A}_b - [\bar{A}_b, F_{ab}^B] = 0 \quad (22a)$$

$$-\partial_j^2 A_i - D_a D_a A_i = 0 \quad (22b)$$

where $F_{ab}^B = f_{abc} \psi_c^-$. As

$$[D_b, D_a] = -f_{ba\bar{c}} (Q_{\bar{c}} + [\psi_{\bar{c}}, \cdot])$$

equation (22a) finally has the form:

$$-\partial_i^2 \bar{A}_a - D_b D_b \bar{A}_a - f_{ba\bar{m}} f_{b\bar{m}c} \bar{A}_c - 2 f_{ab\bar{c}} [\bar{A}_b, \psi_{\bar{c}}] = 0 \quad (22a')$$

The great virtue of the constraint Eqs. (19) and (20) is that they make it quite simple to determine the action of D_a on their solutions. In fact, combining them with

$$\partial_a D^{(m)}(L^{-1}) = -D^{(m)}(Q_a L^{-1}) - \pi_a^{\bar{b}} D^{(m)}(Q_{\bar{b}} L^{-1}) \quad (23)$$

following from (1) one readily computes that for any $D^{(n)} \phi_a^{(n)}$, $D^{(m)} a_i^{(m)}$ pair satisfying Eqs. (19) and (20):

$$D_b D_b D_{pq}^{(n)}(L^{-1}) \phi_a^{(n) pq} = D_{pq}^{(n)}(Q_b Q_b L^{-1}) \phi_a^{(n) pq} = [C_S(D^{(n)}) - C_R(\tilde{D})] D_{pq}^{(n)} \phi_a^{(n) pq} \quad (24a)$$

$$D_b D_b D_{\ell j}^{(m)}(L^{-1}) a_i^{(m) \ell j} = [C_S(D^{(m)}) - C_R(\tilde{D})] D_{\ell j}^{(m)}(L^{-1}) a_i^{(m) \ell j} \quad (24b)$$

Here $C_S(D^{(n)})$, $C_R(\tilde{D})$ denotes the values taken by the quadratic Casimir operators of S and R in the representations $D^{(n)}$, $D^{(m)}$ and $\tilde{D}$ respectively, and $\tilde{D}$ is that particular R irrep to which the components of the fields $a_i^{(m)}$, $\phi_a^{(n)}$ in question are finally assigned by Eqs. (19) and (20).

The difference $-[C_S(D^{(m)}) - C_R(\tilde{D})]$ determines - via (22b) - the mass squares of the Lorentz vectors $a_i^{(m)}$ that are present in the expansion. Under our assumptions (both S and R are compact and semi-simple and R is a subgroup of S) $C_S(D^{(m)}) \leq C_R(\tilde{D}) \leq 0$ with equality holding only for the trivial representation, thus we conclude that all but the lowest (the symmetric) vector modes obtain a positive mass square.

For the scalar fields we still have to cope with the last two terms in (22a'). It is straightforward to show that for any $D^{(n)} \phi_a^{(n)}$ solution of (20) they can be written as

$$- D_{pq}^{(n)} (Q_{\bar{m}} Q_{\bar{n}} L^{-1}) \phi_a^{(n) pq} + D_{pq}^{(n)} (L^{-1}) [[\phi_a^{(n) pq}, \psi_{\bar{m}}], \psi_{\bar{n}}]$$

thus the full mass term of a particular component of $\phi_a^{(m)}$ is

$$- C_S (D^{(n)}) \phi_a^{(n) pq} + [[\phi_a^{(n) pq}, \psi_{\bar{m}}], \psi_{\bar{n}}] \quad (25)$$

This clearly shows the presence of tachyons: fields belonging to the trivial representation of S in the expansion obtain negative square masses given by $C_R(D)$ where D is that R irrep that enters into the solution (20) in this case.

The appearance of tachyons signals the instability of the background solution. This is not surprising as it corresponds to the spontaneous symmetry breaking we found earlier working with the symmetric modes only in the CSDR scheme. However, here we derived the instability under slightly different assumption: so far we have not assumed that $S \subset G$, for the harmonic expansion we needed only the possibility of embedding R into $SO(N)$.

Next, we turn to the investigation of the second stage of the CSDR scheme assuming $S \subset G$. Again first we have to determine the ground state configuration of the $4+N$ dimensional theory around which we expand A_i and A_a in this case. As explained above, this configuration must be a solution of (15) with vanishing field strength at $y = 0$. Furthermore, as the "true" ground state of the CSDR scheme is obtained from the "naive" one by a constant shift in A_a , we expect that our new configuration, A_a^V , is related to A_a^B in the same way. Therefore, we try if

$$A_a^V = -\pi_a^{\bar{b}} T_{\bar{b}} - T_a \quad (26)$$

(where the constant $T_a, T_{\bar{a}}$ form the generators of an S subgroup of G) satisfies these requirements. It is obvious from Eq. (1) that A_a^V is a pure gauge, thus its field strength vanishes all over S/R and trivially solves (15). In the CSDR parametrization at $y = 0$, Eq. (26) yields $A_a^V(y = 0) = -T_a$, thus it really satisfies (14). Furthermore, a simple computation shows that the action of an infinitesimal left translation on A_a^V is compensated by the same tangent space rotation and gauge transformation as for A_a^B . Therefore,

we conclude that by setting up a harmonic expansion around A_a^V , we shall be able to investigate the stability of the symmetric fields in the second stage of the CSDR scheme and to compare their masses with that of the non-symmetric modes.

Since A_a^V transforms under S just in the same way as A_a^B the matrices and fields present in the harmonic expansion around A_a^V

$$A_i(x, y) = \sum_{(n)} D_{ij}^{(n)} (L^{-1}) a_i^{(n) \ell j}(x) \quad (27)$$

$$A_a(x, y) = A_a^V + \sum_{(n)} D_{pq}^{(n)} (L^{-1}) \phi_a^{(n) pq} = A_a^V + \bar{A}_a$$

are restricted in the same way as before: namely they must solve, Eqs. (19) and (20). Just as in the previous case, the x dependent fields in the second stage of the CSDR scheme appear in (27) as the coefficients of the trivial representation of S .

Of course, the field equations for the present coefficient functions change as a result of the different background. The equations for $a_i^{(m)}$ and $\phi_a^{(m)}$ stemming from (15) in the linear approximation can now be written conveniently in terms of the derivative

$$D_a^V = \nabla_a - [A_a^V,] = D_a + [T_a,] \quad (28)$$

as

$$\begin{aligned} -\partial_i^2 \bar{A}_a - D_b^V D_b^V \bar{A}_a - f_{ba\bar{m}} f_{b\bar{m}c} \bar{A}_c &= 0 \\ -\partial_j^2 A_i - D_a^V D_a^V A_i &= 0 \end{aligned} \quad (29)$$

In deriving (29) we assumed that the $a_i^{(m)}$ $\phi_a^{(n)}$ fluctuations satisfy the $\partial_i A_i + D_a^V \bar{A}_a = 0$ gauge condition. The constraint Eqs. (19) and (20) together with Eq. (23) again make it possible to evaluate the action of $D_b^V D_b^V$ on their solutions and using it in (29) one obtains:

$$\begin{aligned} \sum_{(n)} \{ D_{\ell k}^{(n)} (L^{-1}) (-\partial_j^2 a_i^{(n) \ell k} - [T_a, [T_a, a_i^{(n) \ell k}]] - \\ - (C_s(D^{(n)}) - C_R(\tilde{D})) a_i^{(n) \ell k} + 2 D_{\ell k}^{(n)} (Q_a L^{-1}) [T_a, a_i^{(n) \ell k}] \} = 0 \end{aligned} \quad (30a)$$

$$\begin{aligned} \sum_{(n)} \{ D_{pq}^{(n)} (L^{-1}) (-\partial_i^2 \phi_a^{(n) pq} - C_s(D^{(n)}) \phi_a^{(n) pq} - [T_b, [T_b, \phi_a^{(n) pq}]] + \\ + 2 D_{pq}^{(n)} (Q_b L^{-1}) [T_b, \phi_a^{(n) pq}] \} = 0 \end{aligned} \quad (30b)$$

Eqs. (30) have some novel features. First of all, the last terms in them, in general, represent some mixing between components belonging to the same S irrep but assigned to different R irreps. Furthermore, it is not difficult to show - exploiting Eqs. (19) and (20) again - that the mass square matrices of $a_i^{(m)}$ and $\phi_a^{(n)}$ in Eqs. (30a) and (30b) are in fact the negatives of the square of an anti-Hermitian operator, thus in the present case, there are no tachyons in the mass spectrum. From Eq. (30a), we also see that, in general, the symmetric vectors of the CSDR scheme $(a^{(\text{trivi})}(x); D^{(m)}(Q_{\hat{a}}) \equiv 0)$ obtain some positive mass squares; only those of them stay massless for which $[T_{a,a_i^{(\text{trivi})}}] = 0$; i.e., putting this together with (19) those that belong to the centralizer of S in G . (We denote it by K .)

An important question from the point of view of the models built by using the CSDR scheme is whether these K gauge fields are the only zero mass vectors even when we allow non-symmetric fluctuations as well. Next, we show that, in general, this is not true, as a result of the mixing we mentioned earlier, certain combinations of vectors $a_i^{(m)}$ - that are the coefficients of a representation $D^{(m)}$, different from the trivial one - acquire vanishing mass. In particular, we show this when $D^{(m)} = \text{adj } S$.

Recalling that we are working under the assumption $S \subset G$, from (19) we see that all R irreps appearing in the decomposition $\text{adj } S = \text{adj } R + \underline{v}$ effectively contribute to the harmonic expansion. If, in addition, we suppose that $\underline{v}$ is itself an R irrep [this happens for example for $S = SO(N+1)$, $R = SO(N)$, $N > 2$], then the contribution of $D^{(n)} = \text{adj } S = D_{\hat{a}\hat{b}}$ to the harmonic expansion has the form

$$\sqrt{\frac{d(S)}{d(R)}} D_{\hat{a}\hat{b}} (L^{-1}) a_i^{\hat{b}}(x) T^{\hat{a}} + \sqrt{\frac{d(S)}{d(v)}} D_{a\hat{b}} (L^{-1}) A_i^{\hat{b}}(x) T^a$$

where $a_i^{\hat{b}}(x)$ and $A_i^{\hat{b}}(x)$ are two independent sets of Lorentz vectors, $T^{\hat{a}}$ are the generators of the S subgroup of G ; $D_{\hat{a}\hat{b}}$ denotes the matrix of the adjoint representation of S satisfying $D_{\hat{a}\hat{b}}(Q_{\hat{c}}) = f_{\hat{a}\hat{b}\hat{c}}$; $d(S)$, $d(R)$ stand for the dimensions of the group S and R respectively, and $d(v) = N$ is the dimension of S/R . Exploiting the orthogonality properties of the generalized harmonics $D^{(m)}(L^{-1})^8$, it is straightforward to isolate from (30a) the equations satisfied by $a_i^{\hat{b}}$ and $A_i^{\hat{b}}$:

$$\begin{aligned} -\partial_j^2 a_i^{\hat{b}} - 2 (c_s - c_R) (a_i^{\hat{b}} - \sqrt{\frac{d(R)}{d(v)}} A_i^{\hat{b}}) &= 0 \\ -\sqrt{\frac{d(R)}{d(v)}} \partial_j^2 A_i^{\hat{b}} + c_s (a_i^{\hat{b}} - \sqrt{\frac{d(R)}{d(v)}} A_i^{\hat{b}}) &= 0 \end{aligned} \quad (31)$$

where C_S and C_R are the values taken by the quadratic Casimir operators in the adjoint representation of S and R respectively, $\delta_{ab} C_S = f_{as\hat{m}} f_{ms\hat{b}}$, $\delta_{ab} C_R = f_{asm} f_{ms\hat{b}}$. The properties of the structure constants $f_{a\hat{b}\hat{c}}$ expressing that $R \subset S$ and S/R is a symmetric space also guarantee $3C_S - 2C_R = (d(S)/d(R))C_S$. Using this in (31), one easily gets that the combination $a_i^{\hat{b}} + \sqrt{d(v)/d(R)} A_i^{\hat{b}}$ has zero mass while $a_i^{\hat{b}} - \sqrt{d(R)/d(v)} A_i^{\hat{b}}$ acquires a mass square $m^2 = -C_S (d(S)/d(R))$. As both $a_i^{\hat{b}}$ and $A_i^{\hat{b}}$ transform under the action of the generalized S transformation as $\text{adj } S$, we see that taking into account all the modes in the expansion (not just the symmetric ones) increases the symmetry group of the zero mass vectors compared to that of those predicted by the CSDR scheme: it contains at least $S \times K$ (K is the centralizer of S in G).

Very precisely, we derived this here under the assumption that $\underline{v}$ in the decomposition $\text{adj } S = \text{adj } R + \underline{v}$ is an R irrep. To indicate the general nature of our finding, we show now that it holds at least in one case when this additional assumption is not valid: it is true for the model with S^2 as extra dimensions we described earlier. Indeed, using the expansion (17'a) [with the λ values determined from (19')] in (30a) after some algebra, one gets for $a_i^{(\ell)m[\pm 0]}$, ($\ell = 0, 1, \dots$; $m = -\ell, \dots, +\ell$)

$$-\partial_j^2 a_i^{(0)[0]} + 2 a_i^{(0)[0]} = 0 \quad \ell=0 \quad (32)$$

and for $\ell > 0$

$$\begin{aligned} -\partial_j^2 a_i^{(\ell)m[0]} + (\ell(\ell+1)+2) a_i^{(\ell)m[0]} + 2\sqrt{\ell(\ell+1)} (a_i^{(\ell)m[+1]} - a_i^{(\ell)m[-1]}) &= 0 \\ -\partial_j^2 a_i^{(\ell)m[+1]} + \ell(\ell+1) a_i^{(\ell)m[+1]} + \sqrt{\ell(\ell+1)} a_i^{(\ell)m[0]} &= 0 \\ -\partial_j^2 a_i^{(\ell)m[-1]} + \ell(\ell+1) a_i^{(\ell)m[-1]} - \sqrt{\ell(\ell+1)} a_i^{(\ell)m[0]} &= 0 \end{aligned} \quad (33)$$

Equation (32) means that the only symmetric vector has a non-zero mass, while the eigenvalues of the M^2 matrix in (33): $\ell(\ell+1)$, $(\ell-1)\ell$ and $(\ell+1)(\ell+2)$ give the masses of the appropriate combinations of $a_i^{(\ell)m[\pm 0]}$ (with ℓ, m fixed). Clearly, for $\ell = 1$ (i.e., for the adjoint of S) we have a triplet of massless vectors and for all ℓ these are the only ones with 0 mass. Thus, even in this case, the symmetry group of the zero mass vectors is $S = \text{SU}(2)$.

4. - INCLUDING FERMIONS

To build realistic models by using the CSDR scheme, one must incorporate fermions as well. To do this one has to add to the initial pure gauge Lagrangian a Dirac Lagrangian with the fermions transforming under some representation of G and having the standard minimal coupling to the gauge fields and has to extend the concept of generalized S symmetry for the fermions as well^{3),12)}. As we are working in $4+N$ dimensions with the extra N dimensions forming a compact coset space S/R the Dirac Lagrangian has the form

$$L_F = \frac{i}{2} \bar{\Psi} (\Gamma^i D_i + \Gamma^a D_a) \Psi + h.c.; \quad \Gamma^a = e_a^\alpha \Gamma^\alpha \quad (34)$$

where $\Psi(x,y)$ is an $SO(1,3+N)$ spinor having $4.2^{[N/2]}$ indices and $\Gamma = (\Gamma^i, \Gamma^a)$ are the generalized Dirac matrices satisfying $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$, $\eta^{MN} = \text{diag}(-1, 1, \dots, 1)$. Furthermore, D_i, D_a in (34) are the appropriate covariant derivatives

$$D_i = \partial_i - A_i^k G_k; \quad D_a = \partial_a - \frac{1}{2} \omega_{\alpha ab} \Sigma^{ab} - A_a^k G_k \quad (35)$$

Here G_k denote the generators of G in the representation of Ψ , $\Sigma^{ab} = -\frac{1}{4}[\Gamma^a, \Gamma^b]$ stand for the $SO(N)$ generators in the spinor representation (we can always find a representation of Γ^M 's such that the Lorentz generators $\Sigma^{ij} = -\frac{1}{4}[\Gamma^i, \Gamma^j]$ commute with Σ^{ab}) and $\omega_{\alpha ab}$ is the spin connection on S/R . Assuming vanishing torsion $\omega_{\alpha ab}$ is uniquely determined by $e_{\alpha}^{\hat{a}}$, in particular, using the Maurer-Cartan equations for our symmetric coset space we get $\omega_{\alpha ab} = e_{\alpha}^{\hat{c}} f_{\hat{c}ab}$.

Since Ψ , in addition to being an $SO(N)$ spinor, transforms under coordinate transformations as a scalar we say - following Refs. 3), 12) - that it is S symmetric if an infinitesimal left translation of Ψ can be compensated by an appropriate gauge transformation and rotation in the tangent space:

$$-K_{\hat{a}}^{\alpha} \partial_{\alpha} \Psi - \frac{1}{2} H_{\hat{a}\bar{m}} f_{\bar{m}bc} \Sigma^{bc} \Psi = W_{\hat{a}}^k G_k \Psi \quad (36)$$

where $W_{\hat{a}}$ is the same gauge transformation that appears in Eqs. (7) and (8).

In the CSDR scheme - just as for the vectors - one assumes that the y dependence of Ψ is restricted by (36). Using the CSDR parametrization for L and the results, we obtained for $W_{\hat{a}}$ Eq. (36) yields at $y = 0$

$$\frac{1}{2} f_{\bar{m}ab} \Sigma^{ab} \Psi - \psi_{\bar{m}}^k G_k \Psi = 0, \quad \partial_a \Psi = 0 \quad (37)$$

Substituting the solutions of Eqs. (10)-(12) and (37) into Eq. (34) and integrating it over the coset space yields the dimensionally reduced Lagrangian for the symmetric fermions coupled to the symmetric vectors A_i and scalars A_a .

Next, we briefly describe how one can set up a harmonic expansion for fermions. First, the $SO(1,3+N)$ spinor $\Psi(x,y)$ must be decomposed relative to the subgroup $SO(1,3) \times R$ [using the embedding of R into $SO(N)$]. Each piece in the decomposition yields an x dependent $\Psi(x)$ and obtains a harmonic expansion

$$\Psi(x,y) = \sum_{(m)} D^{(m)}(L^{-1}(y)) \bar{\Psi}^{(m)}(x) \quad (38)$$

However, further restrictions on the $D^{(m)}$'s and $\bar{\Psi}^{(m)}$'s come from the generalized S symmetry of the background. Requiring the S symmetry of the emerging infinite component theory in the sense of Section 3, one gets that only such $D^{(m)} \bar{\Psi}^{(m)}$ pairs are present in (38) that satisfy

$$D^{(m)}(Q_{\bar{a}}) \bar{\Psi}^{(m)} + \frac{1}{2} f_{\bar{a}bc} \sum^{bc} \bar{\Psi}^{(m)} - G_{\bar{a}} \bar{\Psi}^{(m)} = 0 \quad (39)$$

This equation can be solved along the same lines we described for Eqs. (19) and (20). Comparing Eqs. (39) and (37), we see that the symmetric fermions of the CSDR scheme are again the first x dependent terms in the expansion (38).

Turning to the determination of the masses of the four-dimensional fields, we note that combining Eq. (39) with Eq. (23) makes it easy to compute $\Gamma^\alpha D_\alpha^v D^{(m)}(L^{-1}) \bar{\Psi}^{(m)}$ appearing in Eq. (34) in the linear approximation for the fluctuations around the background in the second stage of the CSDR scheme:

$$\Gamma^\alpha D_\alpha^v D^{(m)}(L^{-1}) \bar{\Psi}^{(m)} = \Gamma^\alpha (-D^{(m)}(Q_a L^{-1}) + G_a D^{(m)}(L^{-1})) \bar{\Psi}^{(m)} \quad (40)$$

where $G_{\hat{a}} = (G_a, G_{\bar{a}})$ denote the generators of the S subgroup of G in the fermion representation. Since $\Gamma^\alpha D_\alpha^v$ appears on the right-hand side in the Dirac equation for $\bar{\Psi}^{(m)}(x)$ in the linear approximation, its square $\Gamma^\alpha D_\alpha^v \Gamma^\beta D_\beta^v D^{(m)}(L^{-1}) \bar{\Psi}^{(m)}$ is proportional to the M^2 matrix of $\bar{\Psi}^{(m)}$. Using again Eq. (39), we finally have

$$\begin{aligned} \Gamma^\alpha D_\alpha^v \Gamma^\beta D_\beta^v D^{(m)}(L^{-1}) \bar{\Psi}^{(m)} = & \left\{ (C_S(D^{(m)}) + C_R(\tilde{D}) + C_S(F) - \right. \\ & \left. - 3C_R(F)) D^{(m)}(L^{-1}) - 2D^{(m)}(Q_a L^{-1}) G_a \right\} \bar{\Psi}^{(m)} \end{aligned} \quad (41)$$

Here $C_S(F) = G_a G_a$, $C_R(F) = G_a G_a$ and $\tilde{D}$ denotes that particular R irrep to which the component of $\Psi^{(m)}$ in question is finally assigned by Eq. (39). The last term in (41) represents a similar kind of mixing we found in the case of the vector equations. From (41) we see that the symmetric Ψ 's [i.e., those for which $D^{(m)}(Q_a) \equiv 0$] obtain non-zero masses unless $C_S(F) - 3C_R(F) = 0$. The vanishing of $C_S(F) - 3C_R(F)$ for a solution of Eq. (37) is precisely the condition derived in Ref. 6) for the existence of massless fermions in the second stage of the CSDR scheme. In this context, it was also conjectured in Ref. 6) that if there are zero mass fermions among the fluctuations at all, then, they will be symmetric in the sense of Eq. (37). While being unable to decide if this conjecture is valid in general, we checked that it is true in at least one case: when we couple to the model with S^2 in the extra dimensions a spinor field belonging to the doublet representation of $G = SU(2)$. Indeed in this case the "isospin" of the spinors is coupled to their spin by Eq. (39) in such a way that some components have their expansion starting with $D^{(m)} \equiv 1$. The mass of these symmetric fermions is obviously zero, while an analysis of Eq. (41) for the non-symmetric modes yielded that there are no other vanishing mass eigenvalues. If, on the other hand, the fermions we couple to the model belong to the triplet representation of $G = SU(2)$, then Eq. (39) does not allow the trivial representation $D^{(m)}$ to contribute to the expansion of any of the components and from Eq. (41) we found that in this case there are no massless fermions at all.

5. - CONCLUSIONS AND DISCUSSION

In this paper we compared the four-dimensional fields one obtains by two kinds of dimensional reduction scheme applied to a $4+N$ dimensional gauge theory on a space-time $M^4 \times S/R$: the symmetric fields given by the CSDR scheme on the one hand and the infinite tower of excitations given by the harmonic expansion on the coset S/R on the other. We identified two "ground state" solutions of the $4+N$ dimensional theory that correspond to the two stages of the CSDR scheme and established a harmonic expansion describing the fluctuations around them. We showed that the x dependent CSDR fields appear in a systematic way in these harmonic expansions. Using the linearized field equations for the fluctuations, we analyzed their masses and found that some of the symmetric modes around the "vacuum" corresponding to the first stage of the CSDR scheme are tachyons corresponding to the symmetry breaking. For the ground state describing the second stage of the CSDR scheme, we found non-symmetric zero mass vectors in addition to the symmetric ones predicted by the CSDR scheme.

Underlying this model, one can conceive that $4+N$ dimensional gravity is coupled to the Yang-Mills fields and is the agent responsible for the "spontaneous compactification" of the extra N dimensions^{13),11)}. As we mentioned earlier, A_a^B (16) was derived in Ref. 11) as a part of the spontaneously compactified solution of the coupled Einstein-Yang-Mills equations, therefore, to test its stability in the framework of the $4+N$ dimensional Einstein-Yang-Mills theory one should include the fluctuations coming from the metric tensor as well. There is some indication that the inclusion of metric fluctuations, in general, will not stabilize the solution; quite recently this was found to happen in the case of an $SU(3)$ gauge theory coupled to gravity in six-dimensional space-time compactified over $M^4 \times S^2$ ¹⁴⁾. If this persists in the general case, one may wonder if the shift from A_a^B to A_a^V that eliminated the tachyons in the pure Yang-Mills theory can do this in the coupled Einstein-Yang-Mills theory as well. The answer is no, though A_a^V solve the Yang-Mills equations, as it is a pure gauge with its vanishing field strength, it is unable to compactify the extra N dimensions when substituted into the $4+N$ dimensional Einstein equations; i.e., A_a^V is not a part of a spontaneously compactified solution, therefore, it cannot be used as a ground state in the Einstein-Yang-Mills theory.

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