U(2) FOUR-DIMENSIONAL SIMPLICIAL LATTICE GAUGE THEORY

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A B S T R A C T

Monte Carlo simulations for pure U(2) gauge theory on a four-dimensional lattice with six sites in each direction are reported. Wilson loops and the string tensions for squares and triangles are presented. A first-order phase transition, similar to that found for the hypercubical lattice, is observed and found to confine SU(2) colour and define U(1) charge.

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In a recent paper⁽¹⁾, gauge theories on a simplicial lattice were introduced and applied to the gauge group $SU(2)$. In the present paper, we extend the analysis to the gauge group $U(2)$ which is known to have a first-order phase transition when formulated on a hypercubical lattice⁽²⁾. In order to decide if the first-order phase transition survives on the simplicial lattice, we study the Wilson loops for squares and triangles and the associated string tensions. This study shows that $SU(2)$ colour is confined while $U(1)$ charge is deconfined on the simplicial lattice.

The system is defined in a simplicial lattice of four Euclidean space-time dimensions⁽¹⁾. Sitting on any link on the lattice joining nearest-neighbour lattice sites denoted by i and j , is a matrix U_{ij} of the $U(N)$ gauge group with

$$U_{ij} = \exp(i\theta_{ij}) \bar{U}_{ij} ,$$

where $\bar{U}_{ij}$ is an $N \times N$ unitary unimodular matrix of $SU(N)$ and θ is the angle defining the group $U(1)$. We note that the reverse path on the lattice gives the inverse group element, i.e.

$$U_{ji} = (U_{ij})^{-1} .$$

We define the partition function by

$$Z(\beta) = \int \left(\prod_{\{i,j\}} dU_{ij} \right) \exp(-S[U])$$

where the action S is defined as a sum over all unoriented plaquettes such that

$$\begin{aligned} S[U] &= \beta \sum_{\Delta} S_{\Delta} \\ &= \beta \sum_{\Delta} \left(1 - \frac{1}{N} \text{Re Tr } U_{\Delta} \right) \end{aligned}$$

where U_{Δ} is the product of link variables around a triangle, β is the inverse coupling constant squared which is related to the bare coupling constant g_0 by

$$\frac{\sqrt{5}}{8} \beta = \frac{1}{g_0^2}$$

and the measure in the above integral is the normalized invariant Haar measure for the group. We used periodic boundary conditions throughout our calculations. Our lattice was equilibrated by the method of Metropolis et al.⁽³⁾

Following Ref. 1 we find that the strong-coupling expansion for the free energy for pure $U(2)$ gauge theory on a four-dimensional simplicial lattice is

$$F = 20\left(\frac{\beta}{4}\right)^2 + 5\left(\frac{\beta}{4}\right)^4 + \frac{5}{18}\left(\frac{\beta}{4}\right)^6 - \frac{5}{96}\left(\frac{\beta}{4}\right)^8 + \frac{14}{15}\left(\frac{\beta}{4}\right)^{10} + O(\beta^{12}) .$$

From this we obtain, using standard techniques⁽⁴⁻⁶⁾, the average action for triangles to be

$$\langle E_{\Delta} \rangle = 1 - \frac{\beta}{8} - \frac{\beta^3}{256} - \frac{\beta^5}{49152} + \frac{\beta^7}{3145728} - \frac{7\beta^9}{15728640} + O(\beta^{11}) . \quad (1)$$

and the average action for squares to be

$$\langle E_{\square} \rangle = 1 - \frac{\beta^4}{2048} - \frac{\beta^6}{16384} + O(\beta^8) . \quad (2)$$

The strong-coupling expansions for the string tension for triangles and squares are derived to be

$$K_{\Delta} = -\frac{4}{\sqrt{3}} (\ln \beta - 3 \ln 2 + \frac{\beta^2}{32} + \frac{\beta^4}{1536}) + O(\beta^6) , \quad (3)$$

and

$$K_{\square} = -4 \ln \beta - 4 \ln \left(\frac{\Gamma(\frac{1}{4})}{16\Gamma(\frac{3}{4})} \right) - \frac{\beta^2}{12} + O(\beta^4) , \quad (4)$$

respectively. Again following the methods of Ref. 1 we find the strong-coupling expansions for the determinant of triangles and squares to be

$$\langle \bar{E}_{\Delta} \rangle = 1 - \frac{\beta^2}{32} - \frac{\beta^4}{768} - \frac{\beta^6}{32768} + \frac{37 \beta^8}{94371840} + O(\beta^{10}), \quad (5)$$

(see appendix for details) and

$$\langle \bar{E}_{\square} \rangle = 1 - 33 \left(\frac{\beta}{8}\right)^8 + O(\beta^{10}), \quad (6)$$

respectively.

We perform, at the lowest order, a similar calculation for the string tensions for the determinant of triangles which gives

$$\bar{K}_{\Delta} = - \frac{4}{\sqrt{3}} (-2 \ln \beta + 5 \ln 2 + O(\beta^2)). \quad (7)$$

The square case is more delicate. The lowest-order diagrams look like a minimal surface with the representation (2,0) (see appendix) and with bubbles of the representation (1, 1/2); entropy factors have been estimated in Ref. 1. The calculation shows that the bubble is subdominant and thus

$$\bar{K}_{\square} = -8 \ln \beta - 4 \ln \left(\frac{\Gamma(\frac{1}{4})}{64 \Gamma(\frac{3}{4})} \right) + O(\beta^2). \quad (8)$$

We note that, for $U(2)$,

$$\int \exp\left(\frac{\beta}{2} \text{Re Tr } U\right) DU = I_0^2\left(\frac{\beta}{2}\right) - I_1^2\left(\frac{\beta}{2}\right),$$

where I_0 and I_1 are modified Bessel functions.

This allows the following weak-coupling mean-field calculations:

$$\frac{2 \ln Z}{Nd(d+1)} = \text{Max}_K \left[\frac{\beta(d-1)}{12} u'^3 + u - Ku' \right],$$

with $u = \ln(I_0^2(K) - I_1^2(K))$ and

$$\langle E_{\Delta} \rangle = 1 - \frac{1}{8} u'^3,$$

$$\langle E_{\square} \rangle = 1 - \frac{1}{16} u'^4,$$

and $1 - \langle \det U_{\Delta} \rangle = 1 - \left[\frac{I_1^2(K) - I_0(K) I_2(K)}{I_0^2(K) - I_1^2(K)} \right]^3.$

This last formula comes from the following calculation. Consider the integrals

$$\int \exp(K \operatorname{tr} U) DU \quad \text{and} \quad \int \det U \exp(K \operatorname{tr} U) DU ,$$

and compute them using the representation

$$U = \left(\cos \frac{\theta}{2} \mathbf{1} + i \sin \frac{\theta}{2} \underline{\sigma} \cdot \hat{\underline{n}} \right) \exp(i\phi) ,$$

$$DU = \frac{1 - \cos \theta}{2\pi} d\theta \frac{d^2 \hat{\underline{n}}}{4\pi} \frac{d\phi}{2\pi} ,$$

with $0 < \theta < 2\pi$ and $0 < \phi < 2\pi$. First we integrate over θ and the result is

$$[I_0(2K \cos \phi) - I_2(2K \cos \phi)] .$$

We then integrate over ϕ using

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \cos(2\mu x) I_{2\nu}(2a \cos x) dx \\ &= \frac{\pi}{2} I_{\nu-\mu}(a) I_{\nu+\mu}(a) . \end{aligned}$$

The above result then follows.

We obtain the string tension from the logarithmic ratios

$$x_1^\Delta = \frac{2}{\sqrt{3}} \ln \left| \frac{W_4^\Delta}{W_6^\Delta} \right| ,$$

$$x_2^\Delta = \frac{1}{\sqrt{3}} \ln \left| \frac{W_9^\Delta}{W_{13}^\Delta} \right| ,$$

(9)

and

$$x_3^\Delta = \frac{2}{\sqrt{3}} \ln \left| \frac{W_4^{\Delta 2}}{W_1^\Delta W_9^\Delta} \right| ,$$

for triangles and

$$\chi_1^{\square} = \ln \left| \frac{W_3^{\square}}{W_4^{\square}} \right| ,$$

and

$$\chi_2^{\square} = \frac{1}{2} \ln \left| \frac{W_4^{\square 2}}{W_1^{\square} W_9^{\square}} \right| , \quad (10)$$

for squares, respectively.

Fig. 1 shows the average action for triangles $\langle E_{\Delta} \rangle$ as a function of the inverse coupling constant squared β . For these calculations, we first performed 16 iterations through the lattice. For each iteration we employed 20 Monte Carlo upgrades per link. By this means the space-time lattice was equilibrated. We then averaged over the next 72 iterations through the lattice. However, in order to avoid correlations between events we used every second iteration in our average, i.e. 36 lattice configurations were used in our averages. Disordered starting lattices were used for $\beta \leq 2.90$ while ordered starting lattices were used for $\beta \geq 2.90$. Also shown in Fig. 1 are the first, seventh and ninth order strong-coupling expansions of Eqn.1 along with the weak-coupling mean-field analysis. The weak-coupling mean-field calculation gives the end of the metastability of the weak-coupling phase at $\beta_{\text{end}} = 2.44$ while the location of the critical inverse coupling constant squared for the first-order phase transition is given as $\beta_c = 2.91$.

In Fig. 2 some mixed phase runs for the average action for triangles $\langle E_{\Delta} \rangle$ are shown. In our mixed phase⁽⁷⁾ starting lattice runs where the fourth component of the euclidean lattice, the time component, was equally divided with half the link variables disordered and the other half ordered. We see from Fig. 2 that we have a first-order phase

transition with a critical inverse coupling constant squared $\beta_c = 2.90 \pm 0.05$ which agrees well with the weak-coupling mean-field calculation. Fig. 2 presents the Wilson loops constructed out of triangles. The numbering scheme for the Wilson loops follows that of Fig. 3 of Ref.1. The strong-coupling behaviour of Eqn. 1 is obeyed for very strong coupling.

The logarithmic ratios χ_i^Δ ($i=1,3$) given by Eqn. 9 are shown as a function of the inverse coupling constant squared β in Fig. 4. Also shown are the strong-coupling expansions of first, second and fourth order given by Eqn. 3. The first-order discontinuity in the string tension is very distinct.

Fig. 5 shows the average action for squares $\langle E_\square \rangle$ for pure $U(2)$ gauge theory as a function of the inverse coupling constant squared β . In Fig. 5 the second and third order strong-coupling expansions of Eqn. 2 along with the weak-coupling mean-field calculation are also shown. Again the first-order phase transition is very pronounced. The Wilson loops of size up to 3×3 are shown in Fig. 6 where, again, we are following the numbering scheme of Fig. 3 of Ref. 1. The leading-order strong-coupling behaviour of the Wilson loops of Eqn. 2 is also shown.

In Fig. 7 we present the quantities $\chi_1^\square$ and $\chi_2^\square$ of Eqn. 10 as a function of the inverse coupling constant squared β . Also shown in Fig. 7 is the first and second-order strong-coupling expansion of Eqn. 4. The discontinuity in the string tension due to the first-order phase transition is evident.

We show the average action for the determinant of triangles $\langle \bar{E}_\Delta \rangle$ for pure $U(2)$ gauge theory as a function of the inverse coupling constant

squared β in Fig. 8. In this diagram the first-order phase transition is not as clearly visible as it was for the trace of triangles in Fig. 1. We also present in Fig. 8 the first and second-order strong coupling expansions of Eqn. 5 along with the weak-coupling mean-field calculation. The Wilson loops for the determinant of triangles are shown in Fig. 9 along with the leading-order strong-coupling behaviour of Eqn. 5. Fig. 10 presents the string tension measured by the quantities $\bar{\chi}_1^{-\Delta}$, $\bar{\chi}_2^{-\Delta}$ and $\bar{\chi}_3^{-\Delta}$ of Eqn. 9. The strong-coupling expansion of Eqn. 7 is also shown in Fig. 10.

In Fig. 11 we present the average action for the determinant of squares $\langle \bar{E}_{\square} \rangle$ for pure U(2) gauge theory as a function of the inverse coupling constant squared β . The leading-order strong-coupling expansion of Eqn. 6 along with the weak-coupling mean-field calculation are also shown.

The Wilson loops of size up to 3×3 are presented in Fig. 12 along with the leading-order strong-coupling behaviour of Eqn. 8. Fig. 12 shows the logarithmic ratios $\bar{\chi}_1^{-\square}$, $\bar{\chi}_2^{-\square}$ and $\bar{\chi}_3^{-\square}$ as given by Eqn. 8.

Thus, we see that the results obtained for the gauge group U(2) on a four-dimensional simplicial lattice are similar to those obtained on a four-dimensional hypercubical lattice.⁽¹⁾

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APPENDIX - STRONG COUPLING EXPANSIONS

Eqn. A-1 of Ref. 1 gives the formula for the free energy F in terms of β_r . We Fourier analyze the exponentiated character

$$\begin{aligned} \exp[\chi(U)] &= \sum_r d_r \tilde{\beta}_r \chi_r(U) \\ &= A[1 + \sum_{r \neq 0} d_r \beta_r \chi_r(U)] , \end{aligned}$$

where d_r is the dimension of r , $A = \tilde{\beta}_0$ and

$$\beta_r = \frac{\tilde{\beta}_r}{\tilde{\beta}_0} .$$

For $U(2)$, r is a double index : the representations of $U = \exp(i\theta)\bar{U}$ ($\bar{U} \in SU(2)$) are (n, j) :

$$\exp(in\theta) D_j(\bar{U}) , \quad (n = 0, \pm 1, \pm 2, \dots ; j = 0, \frac{1}{2}, 1, \dots) .$$

We want to compute

$$\langle \det U_\Delta \rangle = \langle \chi_t(U_\Delta) \rangle , \quad t = (2, 0) .$$

To each diagram contributing to F , with the representation r on a given triangle, let us associate the diagram obtained by replacing the representation r on this triangle by the representation s , with $r = s \otimes t$: this diagram obviously contributes to $\langle \det U_\Delta \rangle$. The important thing is that these diagrams are one-to-one correlated, since r determines s uniquely and conversely (if $(r = n, j)$, $s = (n-2, j)$).

Now in the computation of the diagrams, we perform all the link integrations, but the three links bounding the triangle: the contribution of the diagram, except the triangle r , is a function of U_{Δ} expanded on the character basis $\sum_{\lambda} C_{\lambda}^{-} \chi_{\lambda}(U_{\Delta})$. Hence the contribution to Z is

$$\int \tilde{\beta}_r d_r \chi_r(U_{\Delta}) \sum_{\lambda} C_{\lambda}^{-} \chi_{\lambda}(U_{\Delta}) D U_{\Delta} = C_r \tilde{\beta}_r d_r ,$$

while the transformed diagram $r \rightarrow s$ contributes to $\langle \det U_{\Delta} \rangle Z$ an amount

$$\int \chi_t(U_{\Delta}) \tilde{\beta}_s d_s \chi_s(U_{\Delta}) \sum_{\lambda} C_{\lambda}^{-} \chi_{\lambda}(U_{\Delta}) D U_{\Delta} = \sum_r \tilde{\beta}_s d_s N_{rst} C_r ,$$

where N_{rst} is the number of times the trivial representation is in $r \otimes s \otimes t$. Denoting the number of plaquettes by n_p we therefore find

$$Z \langle \det U_{\Delta} \rangle = \sum_{rs} \frac{\tilde{\beta}_s d_s}{d_r} N_{rst} \frac{\partial Z}{n_p \partial \tilde{\beta}_r} .$$

But we have

$$Z = \exp(NF) = A^{n_p} \exp(n_p \mathcal{K})$$

with $\mathcal{K} = \mathcal{K}(\beta_r = \frac{\tilde{\beta}_r}{\beta_0})$ being independent of $A = \tilde{\beta}_0$.

After some algebra we obtain

$$\langle \det U_{\Delta} \rangle = d_t \beta_t + \sum_{r \neq 0} \frac{\partial \mathcal{K}}{\partial \beta_r} \left[\left(\sum_s \frac{d_s \beta_s}{d_r} N_{rst} \right) - d_t \beta_t \beta_r \right] .$$

From Eqn. A-1 of Ref. 1 we have

$$F = \frac{d(d^2-1)}{3} \{ \ln A + \mathcal{K} \} .$$

Using the algebraic manipulation program⁽⁸⁾ we obtain

$$\langle \det U_{\Delta} \rangle = \frac{\beta^2}{32} + \frac{\beta^4}{768} + \frac{\beta^6}{32768} - \frac{37 \beta^8}{94371840} + O(\beta^{10}) .$$

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FIGURE CAPTIONS

- Fig. 1. The average action for triangles $\langle E_{\Delta} \rangle$ for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The curves (A), (B) and (C) represent the 1st, 7th and 9th order strong-coupling expansions of Eqn.1, respectively and curve (D) represents the weak-coupling mean-field calculation.
- Fig. 2. The evolution of the average action for triangles $\langle E_{\Delta} \rangle$ for pure U(2) gauge theory as a function of the number of iterations through the lattice for mixed phase starting lattices for various values of the inverse coupling constant squared β .
- Fig. 3. The Wilson loops for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent W_1^{Δ} , the open circles represent W_2^{Δ} , the full circles represent W_3^{Δ} , the crosses represent W_4^{Δ} , the open upward triangles represent W_6^{Δ} , the full downward triangles represent W_9^{Δ} and the open downward triangles represent W_{13}^{Δ} .
- Fig. 4. The string tension for triangles for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent χ_1^{Δ} , the full circles represent χ_2^{Δ} and the open circles represent χ_3^{Δ} . Also shown in the diagram is the behaviour of Eqn.3.

Fig. 5. The average action for squares $\langle E_{\square} \rangle$ for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The curves (A) and (B) represent the 4th and 6th order strong-coupling expansions of Eqn.2 while curve (C) represents the weak-coupling mean-field calculation.

Fig. 6. The Wilson loops for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $W_1^{\square}$, the open circles represent $W_2^{\square}$, the crosses represent $W_3^{\square}$, the open upward triangles represent $W_4^{\square}$, the full downward triangles represent $W_6^{\square}$ and the full circles represent $W_9^{\square}$.

Fig. 7. The string tension for squares for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $\chi_1^{\square}$ and the full circles represent $\chi_2^{\square}$. Also shown in the figure is the behaviour of Eqn 4.

Fig. 8. The average action for the determinant of triangles $\langle \bar{E}_{\Delta} \rangle$ for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The curves (A) and (B) represent the 2nd and 8th order strong-coupling expansion of Eqn. 5, respectively and curve (C) represents the weak-coupling mean-field calculation.

Fig. 9. The Wilson loops for the determinant for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $\bar{W}_1^\Delta$, the open circles represent $\bar{W}_2^\Delta$, the full circles represent $\bar{W}_3^\Delta$, the crosses represent $\bar{W}_4^\Delta$, the open upward triangles represent $\bar{W}_6^\Delta$, the full downward triangles represent $\bar{W}_9^\Delta$ and the open downward triangles represent $\bar{W}_{13}^\Delta$.

Fig. 10. The string tension for the determinant for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $\bar{\chi}_1^\Delta$, the full circles represent $\bar{\chi}_2^\Delta$ and the open circles represent $\bar{\chi}_3^\Delta$. Also shown in the diagram is the behaviour of Eqn. 7.

Fig. 11. The average action for the determinant of squares $\langle \bar{E}_{\square} \rangle$ for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The curve (A) represents the 8th order strong-coupling expansion of Eqn. 6 and curve (B) represents the weak-coupling mean-field calculation.

Fig. 12. The Wilson loops for the determinant for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $\bar{W}_1^\square$, the open circles represent $\bar{W}_2^\square$, the crosses represent $\bar{W}_3^\square$, the open upward triangles represent $\bar{W}_4^\square$, the full downward triangles represent $\bar{W}_6^\square$ and the full circles represent $\bar{W}_9^\square$.

Fig. 13. The string tension for the determinant for pure U(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $\chi_1^{-\square}$ and the full circles represent $\chi_2^{-\square}$. Also shown in the diagram is the behaviour of Eqn. 8.

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