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IN QUANTUM CHROMODYNAMICS

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CONFORMALLY COVARIANT COMPOSITE OPERATORS
IN QUANTUM CHROMODYNAMICS *

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ABSTRACT

Conformal covariance is shown to determine renormalization properties of composite operators in QCD and in the $\mathcal{C}_6^3$ -model at the one-loop level. Its relevance to higher order (renormalization group improved) perturbative calculations in the short distance limit is also discussed. Light cone operator product expansions and spectral representations for wave functions in QCD are derived.

INTRODUCTION

I. PRELIMINARIES ON CONFORMAL TECHNIQUES

- 1.A The conformal group of a 2n-dimensional space time
- 1.B Basic and derivative fields

II. CONSERVED CONFORMAL TENSORS AS BILINEAR FUNCTIONS OF FREE FIELDS

- 2.A Composite tensor operators out of massless fields of spin 0 and spin 1/2
- 2.B Two- and three-point functions of basic conformal fields
- 2.C Light cone OPE for free 0-mass fields

III. CONFORMAL OPERATORS IN THE LIGHT CONE OPE

- 3.A One-loop renormalization of composite operators
- 3.B Diagonalization of the anomalous dimensions matrix
- 3.C Beyond the leading order in the $\mathcal{C}_6^3$ -model
- 3.D Spectral representations for the Wilson coefficients

IV. CONCLUDING REMARKS

APPENDIX A - Two point functions of composite fields

- A.1 General form of conformal invariant two-point functions
- A.2 Properties of normalized Gegenbauer polynomials

APPENDIX B - Conformal invariant three-point functions for free constituent fields

- B.1 Scalar constituent fields
- B.2 Spin 1/2 constituent fields

REFERENCES

The experimental observation of scaling phenomena has greatly enhanced the interest in conformal quantum field theory (QFT). It has been noted [1] that a critical point $g = g_{cr}$ in a massless renormalizable QFT corresponds to non-trivial conformal invariant Green's functions involving anomalous dimensions provided that the Callan-Symanzik β function has a simple zero at $g = g_{cr}$. Asymptotically free theories like quantum chromodynamics (QCD) or the $\mathcal{C}^* \otimes \mathcal{C}^*$ model in six dimension (hereafter denoted as the $\mathcal{C}_6^3$ model for short) are not conformal covariant away from the critical point $g_{cr} = 0$ (at which they are just trivial free field theories), the Green functions involving logarithmic terms already at the one-loop level. (Note that $\beta(g) \sim -g^3$ for these theories.) Nevertheless, it turns out that that techniques developed in the study of conformal QFT provide a useful tool in studying an asymptotically free theory at least in the leading logarithmic order. It was pointed out in Ref.2 (see also Ref.3) that the conformal covariant composite tensor fields constructed in Refs.4-9 provide a diagonalization of the renormalization group matrix. Indirectly, this was also observed in a Bethe-Salpeter approach in the light cone gauge in QCD [10] and made use of in a not altogether clear way in Ref.11 when analysing the Wilson expansion in this gauge.

In writing this article we have several objectives. Firstly, to collect and further develop conformal group techniques used in writing light cone operator product expansions (OPE) for free massless fields that appear to provide a non-trivial starting point for QCD calculations. Secondly, to clarify some points in the literature, concerning the link between conformal invariance and the light cone expansion *) of vertex functions in asymptotically free theories.

In order to make the paper accessible to readers with little (or no) experience with the conformal group we include some basic definitions and an elementary introduction to the relevant notions in Sec.I (for reviews see Refs.12,13, 6). In Sec.II we write down the composite minimal-twist conformal operators for scalar and spinor constituent fields and study their two- and three-point functions. (Part of the calculations needed in the derivation

*) The term light cone expansion in asymptotically free theories is understood in the sense of the approximation used in the Wilson OPE, where both arbitrary high dimensional operators (i.e. twist) and spin can be neglected.

of the results of Sec.II are presented in Appendices A and B.) Although Sec.II is basically a review (and a selection of useful formulae) of previous work [4-6] it also contains some new material, including a systematic use of a natural normalization for composite operators in evaluation of Wightman and time-ordered Green functions and in writing down explicit light cone OPE formulae for both scalar and spinor free fields related to the Taylor expansion in z of the normal products: $\Phi(x + \frac{z}{2}) \Phi^*(x - \frac{z}{2})$: and $\bar{\psi}(x - \frac{z}{2}) i(\gamma_5 \gamma_\mu \psi(x + \frac{z}{2}))$. Sec.III is devoted to the conformal structure of the OPE at small distances in asymptotically free theories. We deduce in two ways the important fact that conformal operators do not mix under renormalization at the one-loop level avoiding the calculation of the anomalous dimension matrix. We also propose an approximation scheme in the short-distance limit in $\mathcal{C}_6^3$. We also show in this section how the free field calculations of Sec.II can be used to derive spectral relations for vertex and wave functions.

The physical interest in this subject arises from the fact that the following vertex function is directly observable in QCD:

$$\bar{\Gamma}_{AB}(q, p) = \int d^4z e^{-iqz} \langle 0 | T \{ J_A(\frac{z}{2}) J_B(-\frac{z}{2}) \} | m(p) \rangle \quad (1)$$

where J_A, J_B refer for example to axial or vector currents and $|m(p)\rangle$ is a pion or other low lying meson state. For large q^2 and $q \cdot p/q^2$ fixed, this vertex function is determined by the Wilson operator product expansion extended along the light cone [14]. Further, this vertex function is related to the wave function of the meson m at short distances (see Ref.3 for details). The latter is defined by

$$\psi_m(z; p) = \langle 0 | \bar{\psi}(x_1) P \exp \left\{ i \int_{x_1}^{x_2} A_\mu^a(x) \lambda_a dx^\mu \right\} \psi(x_2) | m(p) \rangle, \quad (2)$$

$$x_1 = \frac{1}{2} z, \quad x_2 = -\frac{1}{2} z.$$

Although the integral depends on the path from x_1 to x_2 (which is, in principle, arbitrary), it can be shown [15] that in the leading twist and logarithmic order, the Wilson expansion does not depend on the contour. Further, it is shown in Refs.1 and 10 that the wave function (2) restricted to the light cone is directly related to the form factor of the meson m .

I. PRELIMINARIES AND CONFORMAL TECHNIQUES

1.A. The conformal group of a 2h-dimensional space time

The (proper) conformal group of the 2h-dimensional Minkowski space M^{2h} of signature $(- + \dots +)$ can be identified with the $(h+1) \cdot (2h+1)$ -parameter pseudo-orthogonal group

$$G = SO_0(2h, 2)/Z_2 \quad (h = 2, 3, \dots) \quad (1.1)$$

(The somewhat unusual notation, 2h, for the dimension of the underlying space, is justified by the fact that it is the half of the space-time dimension, h, that appears in most formulae.) It is compounded by Poincaré transformations $x \rightarrow \Lambda x + a$, dilations $x \rightarrow \rho x$, and special conformal transformations

$$X^\mu \longrightarrow X'^\mu = \frac{X^\mu - C^\mu X^2}{1 - 2CX + C^2 X^2} \quad (1.2)$$

The proper conformal transformations can be characterized by the fact that they preserve local causal order:

$$d'x^2 = \omega^2 dx^2 \text{ (and } dx^0 \geq |d\mathbf{x}| \text{ implies } d'x^0 \geq |d'\mathbf{x}|) \quad (1.3)$$

For the special conformal transformations (1.2) the conformal factor ω is $\omega(x, c) = (1 - 2cx + c^2 x^2)^{-1/2}$. Clearly, this conformal factor is singular on a cone (and so are the special conformal transformations (1.2)): global conformal transformations are not well defined in Minkowski space. In order to circumvent this difficulty one can either go to the conformal compactification of M^{2h} , or work in the framework of Euclidean QFT (both ways are expounded in Ref.6). For the practical purposes of the present paper it is sufficient to use infinitesimal special conformal transformations. That means, among other things, that we are dealing, in general, with representations of the infinite sheeted universal covering $\tilde{G}$ of G, which could be called the quantum mechanical conformal group.

I.B. Basic and derivative fields

The transformation properties of (finite component) conformal invariant fields [16,17,6] can be obtained from the study of induced representations of G with an inducing subgroup - the Weyl-type subgroup G_0 that leaves the

point $x = 0$ invariant. The group G_0 is generated by (homogeneous) Lorentz transformations, dilations and special conformal transformations; it has $h(2h+1) + 1$ parameters. We distinguish between basic fields and derivative fields. The basic fields transform under irreducible representations of the inducing subgroup G_0 (possibly extended by space reflections). In particular, they satisfy

$$[\psi(x), D]_{x=0} = -i d \psi(0), \quad [\psi(x), K_\mu]_{x=0} = 0, \quad (1.4)$$

where $D (= J_{2h+2, 2h+1})$ is the dilation generator, $K_\mu (= J_{\mu, 2h+2} + J_{\mu, 2h+1})$ is the infinitesimal operator for special conformal transformations (J_{ab} , $a, b = 0, \dots, 2h-1, 2h+1, 2h+2$, are the standard generators of $SO_0(2h, 2)$). In other words, the inducing representation of the (Abelian) subgroup of special conformal transformations is trivial for basic fields, while the inducing representation of dilations is one dimensional. The derivative fields are characterized by non-decomposable representations of the inducing subgroup.

Given the infinitesimal transformation law of basic fields for Poincaré transformations

$$[\psi(x), P_\mu] = -i \nabla_\mu \psi(x), \quad [\psi(x), J_{\mu\nu}] = (-i(x_\mu \nabla_\nu - x_\nu \nabla_\mu) + \tilde{S}_{\mu\nu}) \psi(x) \quad (1.5a)$$

(where $\tilde{S}_{\mu\nu} = \frac{1}{2} \gamma_{\mu\nu}$ and $S_{\mu\nu}$ provide a finite-dimensional (real) irreducible representation of spin $(2h-1, 1)$ such that $[S_{\mu\nu}, \gamma_\lambda] = 0 = (S_{\mu\nu}, \gamma^\lambda)$), their behaviour under dilations and special conformal transformations is expressed by

$$[\psi(x), D] = -i(d + x \nabla) \psi(x), \quad (1.5b)$$

$$[\psi(x), K_\mu] = i(2d x_\mu + 2x_\nu x^\nu \nabla_\mu - x^2 \nabla_\mu + 2i x^\nu \tilde{S}_{\mu\nu}) \psi(x). \quad (1.5c)$$

In particular, for a Dirac field, we have (in a γ_5 -diagonal basis)

$$S_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu] \left(= \frac{1}{2} \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \sigma_{\mu\nu} \end{pmatrix}, \sigma_{ij} = \varepsilon_{ijk} \sigma_k, \sigma_0 = -i\sigma_3 \right). \quad (1.5d)$$

The standard commutation relations of the Poincaré Lie algebra are to be complemented by the following additional relations for K_μ and D :

$$\begin{aligned} [K_\lambda, J_{\mu\nu}] &= i(\eta_{\lambda\mu} K_\nu - \eta_{\lambda\nu} K_\mu), \\ [D, P_\mu] &= i P_\mu, \quad [D, K_\mu] = -i K_\mu, \\ [K_\mu, P_\nu] &= -2i(\eta_{\mu\nu} D + J_{\mu\nu}). \end{aligned} \quad (1.6)$$

The infinitesimal transformation law under dilations (1.5b) can be integrated (unlike (1.5c)) without leaving Minkowski space; the result is

$$U(e^\alpha) \psi(x) U(e^\alpha)^{-1} = e^{-i\alpha D} \psi(x) e^{i\alpha D} = e^{\alpha d} \psi(e^\alpha x). \quad (1.7)$$

The real number $d = d_\psi$ is called the (scale) dimension of the field ψ (in mass units). Wightman positivity implies that $d \geq h-1$ for a spinless (scalar) field, $d \geq 2h + \ell - 2$ for a symmetric tensor field of rank $\ell \geq 1$ (see [6]). (The difference $d-\ell$ is called the twist of the corresponding conformal field.) For a Dirac field in $2h$ dimensions space-time positivity implies $d \geq h - \frac{1}{2}$.

Derivatives of basic fields (like gradients, divergences, curls) transform, in general, under non-decomposable representations of the conformal group and hence are not basic. Thus, for $d \neq 0$, the $(2h+1)$ -component field $(\nabla_\lambda \varphi, \varphi)$ transforms under a non-decomposable representation of the inducing subgroup (with (1.4) violated). Note that for the canonical dimension $d = h-1$ of a conformal vector potential $A_\mu(x)$ the field strength $F_{\mu\nu}(x) = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is also a basic conformal field.

II. CONSERVED CONFORMAL TENSORS AS BILINEAR FUNCTIONS OF FREE FIELDS

2.A Composite tensor operators out of massless fields of spin 0 and spin $\frac{1}{2}$

Let φ be a complex free 0-mass field in $2h$ -dimensional Minkowski space. By definition φ satisfies the d'Alembert equation

$$\nabla^2 \varphi(x) \equiv \nabla_\mu \nabla^\mu \varphi(x) = 0, \quad \nabla_\mu = \frac{\partial}{\partial x^\mu} \quad (2.1)$$

and the canonical equal-time commutation relations. Eq.(2.1) is only conformally invariant for a canonical dimension $d_\varphi = h-1$ (which is also implied by the equal-time commutation relations).

We look for basic conformal composite fields of the type:

$$Q_\ell(x, \vec{y}) = : \varphi^*(x) D_\ell^{h-1} (i\vec{y}\vec{\nabla}, i\vec{y}\vec{\nabla}) \varphi(x) : \quad (2.2)$$

where $\vec{\nabla}$ (or $\vec{\nabla}$) is the gradient acting to the left (or right), $D_\ell^{h-1}(\alpha, \beta)$ is a homogeneous polynomial of degree ℓ , and $\vec{y}$ is a (real) light-like vector (used as a device for writing symmetric traceless tensors in an analytic form - cf. [4,18]). The condition (1.4) for conformal invariance reads:

$$0 = [Q_\ell(x, \vec{y}), K_\mu]_{x=0} = \left(: \varphi^*(x) D_\ell^{h-1} [\varphi(x), K_\mu] : + : [\varphi^*(x), K_\mu] D_\ell^{h-1} \varphi(x) : \right)_{x=0} \quad (2.3a)$$

or, using (1.5c)

$$\left\{ d \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \right) + \alpha \frac{\partial^2}{\partial \alpha^2} + \beta \frac{\partial^2}{\partial \beta^2} \right\} D_\ell^d(\alpha, \beta) = 0. \quad (2.3b)$$

For canonical dimensions ($d = h-1$ implying $d_\ell = 2h-2+\ell$) and free fields φ (satisfying (2.1)) this condition turns out to be equivalent to the conservation law

$$\nabla_\mu Q_\ell^{\mu\mu_1 \dots \mu_{\ell-1}}(x) = 0 \quad (2.4a)$$

which reduces to the following differential equation for Q_ℓ :

$$\nabla \delta Q_\ell(x, \vec{y}) \equiv [(h-1+\vec{y}\vec{\nabla}) \nabla \partial - \frac{1}{2} \vec{y} \nabla \partial^2] Q_\ell(x, \vec{y}) = 0 \quad (2.4b)$$

$\delta_\mu^\mu \frac{\partial}{\partial x}$ being the interior derivative on the light cone $^*) \left[\partial \equiv \frac{\partial}{\partial \vec{y}}, \partial^2 = \partial_\mu \partial^\mu, \nabla = \frac{\partial}{\partial x} \right]$. Indeed, taking into account the conditions:

$^*)$ If $f_1(\vec{y})$ and $f_2(\vec{y})$ are two polynomials in $\vec{y}$ that coincide on the cone $\vec{y}^2 = 0$, so that $f_1(\vec{y}) - f_2(\vec{y}) = \vec{y}^2 f(\vec{y})$, then $\delta_\mu^\mu (f_1 - f_2) \big|_{\vec{y}^2=0} = 0$ since $\delta_\mu^\mu \vec{y}^2 = \vec{y}^2 (\delta_\mu^\mu + 2\partial_\mu)$ (see [16]). First order interior differentiations coincide with tangent vector fields on the cone $\vec{y}^2 = 0$; an example is provided by the intrinsic Lorentz generators $S_{\mu\nu} = -i(\delta_\mu^\nu \partial_\nu - \delta_\nu^\mu \partial_\mu)$ of O_h .

$$\vec{z} \cdot \vec{\partial} D_0(\vec{z}_R, \vec{z}_R) = 1 D_0(\vec{z}_R, \vec{z}_R) ; P_i^2 = 0 = P_i^2 \text{ for } \vec{V}^2 \varphi = 0 \quad (2.5)$$

we reduce both (2.4) and (2.3b) (for $d = h - 1$) to the form

$$\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} - \frac{\alpha + \beta}{k + \ell - 2} \frac{\partial^2}{\partial \alpha \partial \beta} \right) D_0^{h-1}(\alpha, \beta) = 0 \left(= \left(\alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} - \ell \right) D_0^{h-1}(\alpha, \beta) \right). \quad (2.6)$$

The general polynomial solution of (2.6) is proportional to a Gegenbauer polynomial *)

$$D_0^{h-1}(\alpha, \beta) = x_0^{(\alpha+\beta)} C_0^{h-3/2} \left(\frac{\beta - \alpha}{\beta + \alpha} \right). \quad (2.7)$$

It is convenient to choose the normalization constant κ_0 in such a way that

$$D_0^{h-1}(-1, 1) \left(= x_0^2 \frac{2}{\ell!} \frac{d^\ell}{d\xi^\ell} C_0^{h-3/2}(\xi) \right) = 1, \quad (2.8a)$$

which gives

$$x_0^{-1} = \binom{2k + 2\ell - 1}{\ell}. \quad (2.8b)$$

With this normalization

$$\begin{aligned} Q_0(x, \vec{z}) &= : \varphi^*(x) D_0^{h-1}(i\vec{z}\vec{V}, i\vec{z}\vec{V}) \varphi(x) : = \\ &= (i\vec{z}\vec{V})^\ell : \varphi^*(x) \hat{C}_0^{h-3/2} \left(\frac{\vec{z}(\vec{V} - \vec{V})}{\vec{z}(\vec{V} + \vec{V})} \right) : \end{aligned} \quad (2.9a)$$

where $D_0^{h-1} = \kappa_0^{-1} D_0$ and $\hat{C}_0^N(\xi)$ is the Gegenbauer polynomial normalized in such a way that the coefficient to the highest power in ξ is 1. The normalized polynomials $\hat{C}_0^{h-3/2}$ are related to the standard Gegenbauer polynomials $C_0^{h-3/2}$ by

$$C_0^{h-3/2}(\xi) = N_0^{h-3/2} \hat{C}_0^{h-3/2}(\xi),$$

*) For products of different (or non-free) fields it becomes necessary to use (also) the more general Jacobi polynomials. For a summary of properties of these polynomials, see Appx.A.2.

where

$$N_0^{h-3/2} = \frac{4^\ell \Gamma(k + \ell - 3/2)}{\ell! \Gamma(k - 3/2)}. \quad (2.9b)$$

(More properties of $\hat{C}_0^N$ are listed in Appx.A.2). With this normalization

$$Q_0(x) = : \varphi^*(x) \varphi(x) : , \quad Q_1(x) = \frac{i}{2} : \varphi^*(x) \vec{z}\vec{V} \varphi(x) : = \frac{1}{2} \vec{z}^\mu \hat{q}_\mu(x), \quad (\vec{V} = \vec{V} - \vec{V}),$$

$$Q_2(x, \vec{z}) = : \varphi^*(x) \frac{k \vec{z}\vec{V} \vec{z}\vec{V} - \frac{1}{2}(k-1) [(\vec{z}\vec{V})^2 + (\vec{z}\vec{V})^2]}{2k-1} \varphi(x) : = \vec{z}^\mu \theta_{\mu\nu}(x) \vec{z}^\nu,$$

$$Q_3(x, \vec{z}) = \frac{i}{8k+4} : \varphi^*(x) \{ 2(k+4) \vec{z}\vec{V} \vec{z}\vec{V} - \frac{1}{2}(k+1) [(\vec{z}\vec{V})^2 + (\vec{z}\vec{V})^2] \} \vec{z}\vec{V} \varphi(x) : , \text{ etc.} \quad (2.9c)$$

The advantage of this normalization (which includes the normal product $\varphi^* \varphi$ and the free conformal stress energy tensor $\theta_{\mu\nu}$ with coefficients 1) will be exhibited in the context of the light-cone OPE in Subsec.2.C.

The above construction is trivially extended to generalized free constituent fields φ . One has just to replace the canonical dimension $h-1$ of a massless scalar field by the dimension $d > h-1$ of φ . Eqs.(2.2), (2.6) or (2.9) modified in this way define a basic conformal tensor field of dimension $2d + k$. The conserved canonical tensors Q_k have minimal dimensions $d_k = 2h - 2 + k$ (or minimal twists $d_k - k = 2h - 2$) consistent with positivity of corresponding two-point functions. Note that for a real scalar field the odd operators (2.9) (corresponding to $k = 1, 3, \dots$) vanish.

Although we shall only encounter spinor constituent fields in four-dimensional space time, it is useful to work here too in a $2h$ -dimensional framework so that one could readily apply dimensional regularization.

If $\psi(x)$ is a massless Dirac field (of dimension $h - \frac{1}{2}$) satisfying

$$\not{V} \psi(x) \left(= \not{V} \not{V} \psi(x) \right) = 0 \quad (2.10)$$

then the conserved tensor fields [7,8]

$$Q_k(x, \vec{z}) = (i\vec{z}\vec{V})^{\ell-1} : \bar{\psi}(x) \vec{z}\vec{V} \hat{C}_{\ell-1}^{h-k/2} \left(\frac{\vec{z}\vec{V} - \vec{z}\vec{V}}{\vec{z}\vec{V} + \vec{z}\vec{V}} \right) \psi(x) : \quad (2.11a)$$

where

$$\Gamma_{\mathcal{F}} = i\mathcal{F}\mathcal{F} \quad \text{or} \quad \Gamma_{\mathcal{F}} = i\mathcal{F}\mathcal{F}\mathcal{F} \quad (2.11b)$$

have identical conformal transformation properties as the tensors (2.9) built out of charged scalar fields. (Note that with this normalization the operator $O_1(x, z; i\nu\mathcal{F})$ coincides with $j_\mu \mathcal{F}^\mu$ while O_2 is the conformal stress energy tensor for a free Dirac field.) Note also that in general the composite operator constructed from two free constituent fields with dimension d and spin s involves the Gegenbauer polynomial of type $C_{d+s-1/2}^s$.

The above tensor fields are γ_5 -even, i.e. they are invariant under the discrete γ_5 transformation defined for constituent fields by

$$\psi \rightarrow i\gamma_5 \psi, \quad \bar{\psi} \rightarrow i\bar{\psi}\gamma_5, \quad (\gamma_5^2 = 1), \quad \varphi \rightarrow -\varphi.$$

(We are using four-dimensional notation whenever a γ_5 is involved.) γ_5 -odd conformal operators (including operators with a pair of antisymmetric tensor indices and higher twist operators) can also be constructed (cf. [9]).

2.B Two- and three-point functions of basic conformal fields

It is remarkable that conformal invariance fixes two- and three-point functions up to overall constant factors. The functional form of vacuum expectation values of two or three (basic) conformal fields is determined solely by the transformation properties of these fields (i.e. their dimension and tensor character) and does not depend on whether they are built out of scalar or spinor constituents.

The evaluation of two-point functions of basic fields using conformal invariance alone is sketched in Appx.A. One can, of course, derive the corresponding expressions starting from the definitions (2.9) (or (2.11)) of composite tensor operators and using the properties of the free (constituent) fields. The computation is particularly simple if we equate the auxiliary isotropic vectors $\mathcal{F}$ in the two O_ℓ fields. For scalar constituent fields we have (cf. Appx.A.2):

$$\langle O_\ell(x, \mathcal{F}) \bar{O}_\ell(-p, \mathcal{F}) \rangle_0 = \int \langle O_\ell(x, \mathcal{F}) O_\ell(y, \mathcal{F}) \rangle_0 e^{ip\mathcal{F}} d^4y =$$

$$= \int e^{ip\mathcal{F}} \omega_\ell(x, y) D_\ell^{k-1}(\vec{i}\mathcal{F}_\ell, \vec{i}\mathcal{F}_\ell) D_\ell^{k-1}(\vec{i}\mathcal{F}_\ell, \vec{i}\mathcal{F}_\ell) \omega_\ell(x, y) d^4y =$$

$$= (2\pi)^2 \Theta_+(p) e^{ipx} \int D_\ell^{k-1}(\tfrac{1}{2}p\mathcal{F} + q\mathcal{F}, \tfrac{1}{2}p\mathcal{F} - q\mathcal{F}) D_\ell^{k-1}(\tfrac{1}{2}p\mathcal{F} + q\mathcal{F}, \tfrac{1}{2}p\mathcal{F} - q\mathcal{F}) \cdot$$

$$\times \delta(2pq) \delta(\tfrac{1}{4}p^2 + q^2) d_{2k} q \quad (d_{2k} q = \frac{d^{2k} q}{(2\pi)^{2k}})$$

$$= \frac{\pi^{k-1} \Theta_+(p) e^{ipx}}{4 (2\pi)^{k-2} \Gamma(k-1)} \left(-\frac{p^2}{4}\right)^{k-2} \int \hat{C}_\ell^{k-3/2}(\xi) \hat{C}_\ell^{k-2}(\xi) (1-\xi)^{k-2} d\xi =$$

$$= \delta_{2k} \frac{2\pi \Theta_+(p) \left(-\frac{p^2}{2}\right)^{k-2+\ell} e^{ipx} \ell!}{(4\pi)^k (2k+2\ell-3)!!} \prod_{\ell} \prod_{\ell} (p; \mathcal{F}, \mathcal{F}). \quad (2.12)$$

Here $\Pi_{\ell}^{k\ell}(p)$ is the projection operator on maximal spin (k) in the space of symmetric traceless tensors of rank k (such that $\Pi_{\ell}^{k\ell}(p) v_1 \dots v_k = 0$)

$$\prod_{\ell} (p; \mathcal{F}_1, \mathcal{F}_2) = \left(\frac{2 p \mathcal{F}_1 \mathcal{F}_2}{-p^2}\right)^\ell \hat{C}_\ell^{k-3/2} \left(1 - \frac{p^2 \mathcal{F}_1 \mathcal{F}_2}{p \mathcal{F}_1 p \mathcal{F}_2}\right); \quad (2.13a)$$

for $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, $\mathcal{F}^2 = 0$ we have, in particular,

$$\prod_{\ell} (p; \mathcal{F}, \mathcal{F}) = \left(\frac{2 (p\mathcal{F})^2}{-p^2}\right)^\ell \hat{C}_\ell^{k-1/2}(1) = \left(\frac{(p\mathcal{F})^2}{-p^2}\right)^\ell \frac{(2k+\ell-4)!}{(2k-4)!! (2k+2\ell-5)!!}.$$

$$(2.13b)$$

Since two-point functions of basic conformal fields are determined (up to a factor) from conformal invariance alone the result (2.12) also has implications for two-point functions of basic fields built out of spinor constituents. Indeed, the tensors $O_\ell^\Gamma(x, \mathcal{F})$, (2.11) and $O_\ell(x, \mathcal{F})$, (2.9) have identical conformal transformation properties, therefore their two-point functions should be proportional to each other. This is verified by a direct calculation which also fixes the proportionality coefficient; we have (for both choices of $\Gamma_{\mathcal{F}}$ in (2.11b):

$$\begin{aligned}
\langle O_{\ell+1}^\Gamma(x, \vec{z}) \tilde{O}_{n+q}^\Gamma(-p, \vec{z}) \rangle_0 &= (2\pi)^4 \Theta_+(p) e^{ipx} \int D_q^4 \left(\frac{1}{2} p \vec{z} + q \vec{z}, \frac{1}{2} p \vec{z} - q \vec{z} \right) \times \\
&\times D_n^4 \left(\frac{1}{2} p \vec{z} + q \vec{z}, \frac{1}{2} p \vec{z} - q \vec{z} \right) \text{tr} \left[\bar{\gamma} \left(\frac{1}{2} p + q \right) \gamma \left(\frac{1}{2} p - q \right) \right] \delta(2p_0) \delta \left(\frac{1}{2} p^2 + q^2 \right) d_{2k} q = \\
&= \frac{2\pi \Theta_+(p) e^{ipx}}{(4\pi)^4 (2k-4)!!} \left(-\frac{p^2}{2} \right)^{k-2} (p \vec{z})^{2\ell+2} \int_{-1}^1 \bar{C}_\ell \left(\frac{\xi}{2} \right) C_n \left(\frac{\xi}{2} \right) (1-\xi^2)^{k-1} d\xi = \\
&= \delta_n \frac{2\pi \Theta_+(p) e^{ipx}}{(4\pi)^4 2^\ell} \frac{k(2k+\ell-2)\ell!}{(2k+\ell-1)!!} \left(-\frac{p^2}{2} \right)^{k+\ell-1} \prod_{i=1}^{\ell+1} (p, \vec{z}, \vec{z}) = \\
&= \frac{2k(2k+\ell-2)}{\ell+1} \langle O_{\ell+1}(x, \vec{z}) \tilde{O}_{n+1}(-p, \vec{z}) \rangle_0. \quad (2.14b)
\end{aligned}$$

We shall also write down the three-point functions of the fields

$\tilde{O}_k(-p)$ with two constituent fields that are needed in the subsequent applications. Again these functions can be evaluated either from conformal invariance (they appear as certain Clebsch-Gordan kernels - see Ref.5) or directly, using (2.9) and (2.11). The results for the time-ordered Green functions (respectively, the Euclidean-Schwinger functions) $\tau_q^{(E)}$ and for (Minkowski space) Wightman function w_q , in the case of a scalar constituent field, are (Eqs.(2.15)-(2.19) are derived in Appx.B):

$$\begin{aligned}
\left. \begin{aligned} \tau_q^{(E)}(x_1, x_2, -p, \vec{z}) &= \int \langle \psi(x_1) \psi^\dagger(x_2) O_q(y, \vec{z}) \rangle_{(m)}^{(E)} e^{ipx} d^{2k}y = \\ w_q(x_1, x_2, -p, \vec{z}) &= \frac{e^{ipx}}{(4\pi)^4} C_q^{k-3/2} \left(\frac{2i\vec{z}V}{\vec{z}P} \right) F_{(m)}^{(E)}(x_{12}, p), \quad (x = \frac{1}{2}(x_1+x_2), V = \frac{\partial}{\partial x_{12}}) \end{aligned} \right\} \quad (2.15)
\end{aligned}$$

in the Euclidean case

$$\frac{e^{ipx}}{(4\pi)^4} F_{(m)}(x_{12}, p) = \int \Delta_k(x_1-y) \Delta_k(x_2-y) e^{ipx} d^{2k}y, \quad (2.16a)$$

where Δ_k is the canonical Euclidean propagator for a scalar field in $2k$ dimensions (given by Eq.(B.2) of Appx.B) so that

$$F_{(m)}(x_{12}, p) = \int d\xi e^{-i\xi \frac{p x_{12}}{2}} \left[(1-\xi^2) \frac{p^2}{x_{12}^2} \right]^{\frac{1}{2}k-1} K_{k-2} \left(\sqrt{(1-\xi^2)p^2 x_{12}^2} \right) \quad (2.16b)$$

(K_{k-2} being the Macdonald-Bessel function - see Eq.(B.4)); for the Minkowski space Wightman function we have

$$\frac{e^{ipx}}{(4\pi)^4} F_M(x_{12}, p) = \int w_k(x_1-y) w_k(x_2-y) e^{ipx} d^{2k}y \quad (2.17a)$$

where in the free field Wightman function w_k is given by (B.6), with the result

$$\begin{aligned}
F_M(x_{12}, p) &= \pi \Theta_+(p) \int_{-1}^1 d\xi e^{-\frac{1}{2}\xi p x_{12}} \left[\frac{(1-\xi^2)p^2}{x_{12}^2} \right]^{\frac{1}{2}k-1} \int_{k-2} \left(\sqrt{(1-\xi^2)p^2 x_{12}^2} \right) = \\
&= 2\pi \Theta_+(p) \left(-\frac{p^2}{2} \right)^{k-2} f_{k-2}(\omega), \quad \omega = \frac{1}{2} \sqrt{(p x_{12})^2 - p^2 x_{12}^2} \quad (2.17b)
\end{aligned}$$

Here

$$f_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+\nu+1)} \left(-\frac{z^2}{4} \right)^n$$

is the Bessel function of the first kind, and

$$\begin{aligned}
f_{k-2}(\omega) &= \sqrt{\frac{\pi}{2}} \omega^{\frac{3}{2}k-1} J_{k-3/2}(\omega) = \\
&= \frac{1}{2(2k-4)!!} \int_{-1}^1 e^{-i\xi\omega} (1-\xi^2)^{k-2} d\xi \quad (2.18)
\end{aligned}$$

is an (elementary) entire analytic function of ω^2 (see B.8).

Similarly, for a Dirac constituent field in $2k$ dimensions we have

$$\begin{aligned}
\left. \begin{aligned} \tau_q^{\Gamma\Gamma'}(x_1, x_2, -p, \vec{z}) &= \int d^{2k}y e^{ipx} \langle \bar{\psi}(x_1) \bar{\gamma} \vec{z} \psi(x_2) O_q(y, \vec{z}) \rangle_{(m)}^{(E)} = \\ w_q^{\Gamma\Gamma'}(x_1, x_2, -p, \vec{z}) &= \frac{e^{ipx}}{(4\pi)^4} C_{k-1}^{k-1/2} \left(\frac{2i\vec{z}V}{\vec{z}P} \right) \text{tr} \left[\bar{\gamma} \vec{z} \left(\frac{1}{2} p + V \right) \gamma \vec{z} \left(\frac{1}{2} p - V \right) \right] F_{(m)}^{(E)}(x_{12}, p), \end{aligned} \right\} \quad (2.19)
\end{aligned}$$

where F_E and F_M are again given by (2.16) and (2.17) while $\bar{\gamma} \vec{z}$ and $\gamma \vec{z}$ are either $i\gamma_\mu \vec{z}^\mu$ or $i\gamma_5 \gamma_\mu \vec{z}^\mu$. [In effect the trace on the right-hand side of (2.19) vanishes unless $\bar{\gamma} \vec{z} = \gamma \vec{z}$.]

2.C Light cone OPE for free 0-mass fields

The knowledge of the two- and three-point functions displayed in the preceding subsection allows us to write down (at least on a formal level) a conformal OPE for a product of free massless fields. (Conformal covariant OPE were studied in Refs.19, 20, 13, 12, 21 etc. In effect, global conformal OPE can only be justified if they are applied to the vacuum vector - see Refs.5 and 6 as well as the discussion in Ref.22. Small distance OPE, on the other hand, are being used as operator identities.) We shall write down the corresponding light cone expansion for the free 0-mass fields starting with the scalar fields' case.

To this end we shall need the light cone behaviour of three-point functions. For $x_{12}^2 \rightarrow 0$ we have

$$F_M(x_n, p) \approx \frac{\pi \Theta_+(p)}{(2k-y)!!} \left(\frac{p^+}{2}\right)^{k-2} \int_{-1}^1 e^{-\frac{i}{2}\xi p x_n} (1-\xi^2)^{k-2} d\xi. \quad (2.20)$$

It leads to the following approximation for the scalar field Wightman function (2.15):

$$\begin{aligned} \mathcal{W}_0(x_1, x_2, -p, \xi) &\approx \\ &= \frac{\pi \Theta_+(p) e^{ipx}}{(4\pi)^k (2k-y)!!} \left(\frac{p^+}{2}\right)^{k-2} \int_{-1}^1 e^{-\frac{i}{2}\xi p x_n} \hat{C}_0^{k-3/2}(\xi) (1-\xi^2)^{k-2} d\xi = \\ &= \frac{2\pi \Theta_+(p) (2k+y-4)! e^{ipx}}{(4\pi)^k (2k-y)!! (2k+2\ell-5)!!} \left(\frac{p^+}{2}\right)^{k-2} \left(\frac{p^+}{2}\right)^{\ell} \left(\frac{1}{2}\right)^{\ell} p x_n \quad (2.21a) \end{aligned}$$

valid for x_{12}^2 close to the ray $\{x_2\}$, i.e. for

$$x_n^2 p^2 \rightarrow 0, \quad p^2 x_n \xi \ll p x_n p \xi \quad (2.21b)$$

(which is precisely the domain needed for the derivation of light cone OPE). The small x_{12}^2 behaviour of the τ function is obtained from (2.16) and Eqs.(B.3) and (B.4) (see Appx.B):

$$F_E(x_n, p) \approx \int_{-1}^1 d\xi e^{-\frac{i}{2}\xi p x_n} \begin{cases} \left(\frac{2\pi e^{-\delta}}{i^{(k-\xi^2)} p^k x_n^k} \right)^{k-2}, & k=2, \\ \left(\frac{\Gamma(2-k)}{2} \left(\frac{p^+}{2}\right)^{k-2} (1-\xi^2)^{k-2} \right)^{k-2}, & k \neq 2, 3, \dots \end{cases} \quad (2.22a) \quad (2.22b)$$

It is the second of these expressions (Eq.(2.22b) which displays a pole for $h=2, 3, \dots$) that will be used in subsequent computations involving dimensional regularization.

From (2.12) and (2.21) we deduce the following light cone OPE for a free scalar massless field:

$$\begin{aligned} \varphi(x+\frac{z}{2}) \varphi^*(x-\frac{z}{2}) &= \mathcal{W}_h(z) \mathbb{1} + (2k-3)!! \frac{z^2}{2!} \mathcal{O}_0(x) + \\ &+ \sum_{\ell=1}^{\infty} (-i)^{\ell} \frac{(2k+2\ell-3)!!}{\ell!} \frac{z^{\ell}}{2^{\ell}} \mathcal{O}_{\ell}(x, z) = \\ &= \mathcal{W}_h(z) + \left[1 + \frac{1}{2k-1} \frac{z^2}{8} + \dots \right] \mathcal{O}_0(x) + \\ &+ \sum_{\ell=1}^{\infty} \frac{(-i)^{\ell}}{\ell!} \left[1 + \frac{1}{2k+2\ell-1} \frac{(z^2)^2}{8} + \dots \right] \mathcal{O}_{\ell}(x, z), \quad (V = \frac{z}{\partial x}). \end{aligned} \quad (2.23)$$

It is instructive to verify that Eq.(2.23) represents a (re-arranged) Taylor expansion in z of the normal product

$$:\varphi(x+\frac{z}{2}) \varphi^*(x-\frac{z}{2}): = \varphi(x+\frac{z}{2}) \varphi^*(x-\frac{z}{2}) - \mathcal{W}(z).$$

The corresponding expansion for the product $:\bar{\psi}(x-\frac{z}{2}) \Gamma_2 \psi(x+\frac{z}{2}):$ of Dirac fields is derived in the same way with the result

$$\begin{aligned} &:\bar{\psi}(x-\frac{z}{2}) \Gamma_2 \psi(x+\frac{z}{2}): = \\ &= \sum_{\ell=1}^{\infty} \frac{(-i)^{\ell}}{\ell!} \frac{(2k+2\ell-3)!!}{(\ell-1)!} \frac{z^{\ell}}{2^{\ell}} \mathcal{O}_{\ell}^{\Gamma}(x, z) \end{aligned} \quad (2.24)$$

with exactly the same f -functions as in the scalar case.

III. CONFORMAL OPERATORS IN THE LIGHT CONE OPE

3.A One-loop renormalization of composite operators

Going beyond free fields in QCD or in the $\mathcal{C}_6^3$ model, we have to modify both the definition of composite fields and the operator products under consideration.

The change of a composite operator $\mathcal{O}_{\ell}(x, z)$ in the presence of interaction $L_I(x)$ (corresponding to a scattering operator $S = T \exp(i L_I(x) dx)$) is given by

$$\delta Q_\ell(x, z) = T(Q_\ell(x, z) S)^{S^{-1}} - Q_\ell(x, z). \quad (3.1)$$

In the $\mathcal{C}_6^3$ model with interaction Lagrangian $L_I(x) = g : \varphi^*(x) \varphi(x) : \chi(x) :$ we can write (suppressing the variable z or the tensor indices)

$$\begin{aligned} \delta Q_\ell(x) = & \int d\lambda \int d\lambda' \chi(\lambda) d\lambda' + \\ & + \left\{ \left\{ \left[\varphi^* \varphi(x; y_1, y_2) : \varphi^*(y_1, y_2) : + \int d\lambda \chi(\lambda; y_1, y_2) \chi(y_1, y_2) \right] \right\} \right\} + \dots \end{aligned} \quad (3.2)$$

(the dots indicating terms that involve higher order polynomials in the constituent fields).

The first term (linear in χ) has a non-vanishing contribution in the one-loop approximation for $\ell = 0$ only (due to the orthogonality property of Q_ℓ). Moreover, it has no counterpart for gauge invariant (colourless) composite operators in QCD. Therefore, we shall concentrate our attention on the quadratic in the fields terms on the right-hand side of (3.2). The presence of two such terms indicates a mixing problem. It stems from the fact that operators with the conformal properties of Q_ℓ can be built up (for even ℓ 's) not only from $\mathcal{C}^*$ and $\mathcal{C}$ (call it $O_\ell^{\mathcal{C}}$) but also from two χ 's (call it O_ℓ^{χ}). That is also true for QCD where (in the flavour singlet sector) the χ 's are to be replaced by the gluon stress tensor $G_{\mu\nu}$, while the ordinary derivatives ∇_μ in the (unrenormalized) fields Q_ℓ should be changed by the covariant derivatives $D_\mu = \nabla_\mu - iA_\mu$.

Let us now consider the second ($\mathcal{C}^* \mathcal{C}$) term in the expansion (3.2). Inserting it in the expression for the three-point Green function $\langle T \delta Q_\ell(x) \mathcal{C}(x_1) \mathcal{C}^*(x_2) \rangle$ we find that the coefficient $\Gamma_\ell \mathcal{C}^* \mathcal{C}$ coincides with the computed vertex function

$$\Gamma_\ell \varphi^* \varphi(x; x_1, x_2) = \int d\lambda_1 \int d\lambda_2 \langle T \delta Q_\ell(x) \varphi(\lambda_1) \varphi^*(\lambda_2) \rangle \Delta_\ell^{-1}(\lambda_1 - x) \Delta_\ell^{-1}(x - \lambda_2), \quad (3.3)$$

where Δ_ℓ^{-1} stands for the inverse of the scalar propagator (B.2) in the sense of the convolution product. In the one-loop approximation $\Gamma_\ell \mathcal{C}^* \mathcal{C}$ is given by the sum of three diagrams displayed in Fig.1

Fig.1.1

One-loop diagrams contributing to the coefficient Γ_ℓ in the expansion of δQ_ℓ in the $\mathcal{C}_6^3$ model.

All these diagrams are ultraviolet divergent. We are using dimensional regularization and are only interested in the coefficient to $1/\epsilon$ ($\epsilon = 3 - h$) which is a homogeneous polynomial of degree ℓ in the momenta. Expanding it in the Gegenbauer polynomials of Subsec.2.A we derive the following relation:

$$\begin{aligned} \delta Q_\ell^0(x, z) = & \frac{1}{\epsilon} \sum_{n=0}^{\ell} \binom{\ell-n}{n} \int d\lambda \chi^{\mu\nu} O_n^0(x, z) + \chi^{\mu\nu} O_n^{\chi}(x, z) + \\ & + \text{finite terms (for } \epsilon \rightarrow 0) \end{aligned} \quad (3.4)$$

where the χ 's are constants defining the anomalous dimension matrix. A similar expression is derived for δQ_ℓ^{χ} . (If we do not distinguish between φ and χ lines since they both carry the massless scalar propagators then the same three diagrams that are displayed in Fig.1 appear in all vertex functions encountered in this calculation.) The graphs of Figs.1b, 1c contribute to the wave function renormalization (and hence to the anomalous dimension) of the constituent fields while the graph in Fig.1a corresponds to the field strength renormalization of the composite field. Such a separation however is not gauge invariant in the case of QCD. Moreover, in QCD we have also to take into account the diagrams in Fig.2 that reflect the presence of covariant derivatives in the composite operators. Only the sum of all five graphs provides a gauge invariant notion of anomalous dimension in that case.

(a) (b)

Fig. 2

QCD diagrams coming from covariant derivatives in the composite operator.

Before proceeding to the solution of the mixing problem (in Subsec. 3.B) we shall write down the gauge invariant counterpart for the product of two spinor fields in QCD that is encountered in the definition of Bethe-Salpeter wave functions for meson states. It depends, in general, on a path C which connects the two points x_1 and x_2 :

$$O(x_1, x_2; \Gamma, C) = \bar{\psi}_i(x_1) \Gamma U_{ij}(x_1 | C | x_2) \psi_j(x_2) \quad (3.5a)$$

where i and j are the indices and the path ordered phase matrix $U = (U_{ij})$ is given by

$$U(x_1 | C | x_2) = P \exp \left(-i \int_{x_1}^{x_2} dx^\mu A_\mu(x) \right), \quad A_\mu(x) = g \sum_{a=1}^8 A_\mu^a \frac{\lambda_a}{2}, \quad (3.5b)$$

$\lambda_a = (\lambda_{aif})$ being the $SU(3)_c$ Gell-Mann matrices. A simple Poincaré invariant choice of the path C for the bilocal operators (3.5) is given by the segment $[x_1, x_2]$ of the straight line that joins the two points. In the string interpretation of (3.5), however, it is gratifying to realize that in the short distance limit both the renormalization constants and the Wilson coefficients are path independent provided that the total Euclidean length of the contour C goes to zero when $x_1 - x_2 \rightarrow 0$ [15]. Moreover, in a special gauge, in which the vertex and the wave function renormalization constants Z_1 and Z_3 are equal the dependence on the exponential of A_μ can be neglected altogether as far as renormalization is concerned [15]. That is the QCD counterpart of the Yennie gauge in QED corresponding to the gluon propagator:

$$\Delta_{\mu\nu}(k, a) = \frac{1}{k^2} \left(\eta_{\mu\nu} - (1-a) \frac{k_\mu k_\nu}{k^2} \right) \quad \text{for } a = -3 + O(g^2).$$

All divergences generated by the contour integral cancel among themselves in this gauge, so that the overall renormalization constant of the composite operator O obtained from (3.5) for $x_1 - x_2 \rightarrow 0$ becomes $Z_{3F}(a = -3)$ (where $Z_{3F}^{1/2}(a)$ is the fermion field wave function renormalization in an arbitrary covariant gauge).

3.B Diagonalization of the anomalous dimension matrix

There are two matrix problems associated with the expansion (3.4) corresponding to the upper and lower indices of γ and we shall deal with them separately.

In the work of Efremov and Radyushkin [2] (see also Ref. 3) the conformal basis of composite operators was recovered without reference to conformal invariance by just diagonalizing the field renormalization matrix at the one-loop level. (The traditionally used operators

$$O_{(x,z)}^{(n,m)} = i^m (\bar{z} \nabla)^{m-n} : \varphi(x) (\bar{z} \vec{\nabla} - z \vec{\nabla})^n \varphi(x) : \quad (3.6)$$

have a triangular renormalization matrix.) The question arises can we deduce the non-mixing property of conformal operators under renormalization from what we already know about them without actually computing (the $1/\epsilon$ term of) Feynman diagrams. If so it could help us understand what is going on beyond the one-loop level.

We shall present in what follows two such arguments which only refer to some general properties of perturbative renormalization. The first method uses the triangular character of general renormalization matrices and the orthogonality property (expressed by (2.12) and (2.14)) of conformal composite operators. The second applies the same property (1.4) of infinitesimal conformal invariance that led to the differential equation (2.3) for O_k and for D_k . The role of conformal Ward identities for the multiplicative renormalizability of conformal operators at one-loop level was also discussed in Refs. 8 and 23 but no complete argument was given.

From the orthogonality relation of Subsec. 2.B it follows that the p -space two-point function corresponding to the diagram in Fig. 3a is:

$$\langle \tilde{O}_\ell(p, z) \tilde{O}_\ell(p', z') \rangle \sim \delta_{\ell\ell'} \delta(p-p').$$

Fig 3

Feynman diagrams for $\langle 0_\lambda 0_{\lambda'} \rangle_0$ and $\langle \delta 0_\lambda 0_{\lambda'} \rangle_0 = \langle 0_\lambda \delta 0_{\lambda'} \rangle_0$.

Assume now that

$$\delta \tilde{Q}_\ell(p, z) = \sum_{n=0}^l Z_{\ell n} (z p)^{\ell-n} \tilde{Q}_n(p, z). \quad (3.7)$$

Then the two-point function $\langle \delta \tilde{Q}_\lambda(p) \tilde{Q}_{\lambda'}(p') \rangle$ corresponding to the diagram of Fig. 2b would single out the term with $Z_{\lambda \lambda'}$ for $\lambda' \leq \lambda$. However, we could interpret the same diagram as contributing to the renormalization $\delta \tilde{Q}_{\lambda'}(p')$ of the second factor. Because of the obvious symmetry with respect to the exchange $(p, \lambda) \leftrightarrow (p', \lambda')$ we have

$$\langle \delta \tilde{Q}_\ell(p, z) \tilde{Q}_{\ell'}(p', z') \rangle_0 = \langle \tilde{Q}_\ell(p, z) \delta \tilde{Q}_{\ell'}(p', z') \rangle_0$$

The two-point function on the right-hand side will single out $Z_{\lambda \lambda'}$ with $\lambda' \geq \lambda$. Thus only the diagonal elements of the renormalization matrix do not vanish, as asserted.

Going to the second argument we shall work in the spinor case (i.e. in QCD) in order to face the full complications of the problem.

The basic premise now is the statement that conformal covariance is preserved in the one-loop $1/\epsilon$ term. The justification for such a statement is rather straightforward: the (massless) QCD Lagrangian is manifestly conformally invariant and so are the bare composite operators (2.11) (in zero order). Conformal invariance is only broken in the process of (perturbative) renormalization when one introduces dimensional parameter (like λ in QCD) in the finite parts of propagators and vertex functions and there is no such

parameter in the $1/\epsilon$ term of (3.11). In effect, this term has the form *)

$$\delta Q_{\ell+1}^\psi(x, z) = \frac{1}{\epsilon} \left(\alpha_\lambda \int dx_2 \cdot \bar{\psi}(x_2) [z \cdot \psi(x_2) \cdot D_\ell(z \bar{u}, z \bar{u}) \delta(x_1 - x) \delta(x_2 - x) + \right. \\ \left. + \text{finite terms (for } \epsilon \rightarrow 0) \right), \quad (3.8)$$

where $D_\ell(\alpha \beta)$ is some homogeneous polynomial of degree ℓ (which could be evaluated from the sum of five one-loop diagrams that contribute to $\delta 0$). The conformal invariance of $\delta 0_{\ell+1}^\psi$ leads (by again applying (1.4)) to the counterpart of Eq.(2.6) for Dirac fields

$$\left\{ \lambda \left(\frac{\partial^2}{\partial \alpha} + \frac{2}{\partial \beta} \right) + \alpha \frac{\partial^2}{\partial \alpha^2} + \beta \frac{\partial^2}{\partial \beta^2} \right\} D_\ell(\alpha, \beta) = \\ = \left\{ (\ell+1) \left(\frac{\partial^2}{\partial \alpha} + \frac{2}{\partial \beta} \right) - (\alpha+\beta) \frac{\partial^2}{\partial \alpha \partial \beta} \right\} D_\ell(\alpha, \beta) = 0. \quad (3.9)$$

Since this equation has a unique (up to a factor) polynomial solution we recover an expression proportional to D_ℓ^h :

$$D_\ell(\alpha, \beta) = \lambda_\ell D_\ell^h(\alpha, \beta) = \lambda_\ell (\alpha+\beta) C_\ell^h \left(\frac{\beta-\alpha}{\beta+\alpha} \right) \quad (3.10)$$

(δ_ℓ standing for the gauge invariant anomalous dimension of 0_ℓ). Thus we see that conformal invariance is directly responsible for the diagonalization of the anomalous dimension matrix with respect to the tensor type of the composite field. The problem is thus reduced to the diagonalization of a 2×2 matrix labelled by the type of constituent fields. The analogue of Eq.(3.4) in the case of QCD is

$$\delta Q_\ell^\psi(x, z) = \frac{1}{\epsilon} \left(\lambda_\ell^{\psi\psi} Q_\ell^\psi(x, z) + \lambda_\ell^{\psi G} Q_\ell^G(x, z) + \right. \\ \left. \delta Q_\ell^G(x, z) = \frac{1}{\epsilon} \left(\lambda_\ell^{G\psi} Q_\ell^\psi(x, z) + \lambda_\ell^{GG} Q_\ell^G(x, z) + \right) \right. \quad (3.11a) \\ \left. \right) \quad (3.11b)$$

where 0_ℓ^ψ here stands for 0_ℓ^Γ and

$$\delta Q_\ell^G(x, z) = \frac{1}{\epsilon} \left(\lambda_\ell^{G\psi} Q_\ell^\psi(x, z) + \lambda_\ell^{GG} Q_\ell^G(x, z) + \right) \quad (3.11b)$$

*) The subsequent argument has been worked out in collaboration with Dr. N. Karchev from the University of Sofia.

$$O_l^G(x, z) = \sum_a : z^\mu G_\mu^{a\sigma}(x) D_{l-2}^{k+1}(\vec{z}, \vec{z}) G_{\sigma\nu}^a(x) z^\nu : \quad (3.12)$$

is the conformal composite operator for a pair of gluon fields. Since the γ 's are constants they coincide with the forward anomalous dimension mixing matrix known from the study of deep inelastic scattering (see Ref.24 where also its diagonalization is carried out explicitly). Thus we end up with two multiplicatively renormalizable linear combinations of O_l^ψ and O_l^G . In the case of spin-2 operators the anomalous dimension matrix has an eigenvalue zero whose eigenvector is the stress energy tensor

$$z^\mu \Theta_{\mu\nu} z^\nu = O_2^\psi + O_2^G = \frac{i}{2} : \bar{\psi}(x) \not{\partial} z \not{\partial} z \psi(x) : + z^\mu G_\mu^{a\sigma}(x) G_{\sigma\nu}^a(x) z^\nu.$$

The orthogonal combination is

$$O_2^{(1)} = \frac{1}{T+2C} (-2C O_2^\psi + T O_2^G), \quad T \equiv T(N_F) = \frac{N_F}{2}, \quad C \equiv C_2(N_C) = \frac{N_C^2 - 1}{2N_C}$$

(it corresponds to the eigenvalue $T + 2C$).

Let us stress that we have not used the assumption of asymptotic freedom above. Thus we have shown that for any QFT with conformally invariant Lagrangian, multiplicative renormalizability of the composite operators O_l is a consequence of either their orthogonality (in the sense of vacuum expectation values) and triangularity of the renormalization matrix or their transformation properties with respect to infinitesimal special conformal transformations.

3.C Beyond the one-loop level in the $\mathcal{C}_6^3$ model

The arguments of the preceding section can be extended for the $\mathcal{C}_6^3$ model beyond the one-loop level provided that the breaking of conformal invariance due to the non-vanishing of the Callan-Symanzik β -function may be neglected.

Consider the Euclidean scalar propagator

$$\Delta_F(k) = \frac{Z_3(k^2, g)}{k^2} \quad (3.13a)$$

The wave function renormalization factor satisfies the renormalization group equation

$$\left[k^2 \frac{\partial}{\partial k^2} - \gamma(g) - \beta(g) \frac{\partial}{\partial g} \right] Z_3(k^2, g) = 0. \quad (3.13b)$$

If we set $\beta(g) = 0$ then Eq.(3.13) implies that the dressed propagator acquires an anomalous dimension

$$\Delta_F(k) = \left(\frac{1}{k^2} \right)^{1-\gamma(g)} \quad (3.14)$$

Anomalous dimensions do not destroy conformal invariance they just change the representation of the conformal group. It is straightforward to verify that the contribution of the graph in Fig.4a (with the dressed propagator (3.14)) preserves the orthogonality property of the composite operators provided that we change the dimension labelled by the upper index of the polynomial D_k (2.7) from its canonical value, $h-1$, to $h-1+\gamma(g)$. Indeed it is reduced to the orthogonality condition (A.16) (of Appx.A.2) with $v = h - \frac{3}{2} + \gamma(g)$.

Fig.4

Diagrams with dressed propagators that contribute to $\langle O_l O_n \rangle$ and to $\langle \delta O_l O_n \rangle = \langle O_l \delta O_n \rangle$ in the $\mathcal{C}_6^3$ model.

We now observe that if K is a sum of connected graphs with four external legs that contribute to the renormalization of either O_l or O_n corresponding to the diagram in Fig.4b, then the triangular mixing structure of Eq.(3.7) is still preserved. Combining it with the above-mentioned orthogonality relation we again reproduce the diagonalization property (established in Subsec.3.B at the one-loop level).

We shall now argue that our neglect of the effect of $\beta(g)$ can be justified in the applications to light cone OPE for some appropriate choice of the renormalization scale.

In a first approximation the effect of a non-vanishing $\beta(g)$ amounts to replacing $\gamma(g)$ by γ_0 , $g^2(k^2)/\mu^2$ where

$$g^2(k^2) = \frac{g^2(\mu^2)}{1 + \frac{g^2(\mu^2)}{4\pi} \int_0^1 \log k^2/\mu^2}$$

is the running coupling constant. This would lead to the appearance of powers of $\log(\log \Lambda^2 z^2)$ (where Λ is the asymptotic freedom mass scale) that would invalidate the above orthogonality argument. The conformal invariant calculation amounts to replacing this double log term by a constant. This approximation fails at very short distances, but that is precisely the region where the leading order is dominant in the OPE. Hence it should suffice to choose the renormalization scale μ large enough to minimize the effect of the intermediate region where our approximation becomes unreliable.

This argument is not directly applicable to QCD for two related reasons: (i) the anomalous dimension γ_ψ of the ψ -field is gauge dependent; (ii) the additional diagrams originating from the covariant derivatives (see Fig.2) have also to be taken into account. Our hope that an argument of this type can nevertheless be produced in QCD relies on the fact that the contribution to the anomalous dimension matrix coming from the "covariant derivative graphs" of Fig.2 commutes with the contribution from the three diagrams of Fig.1 (at the one-loop level - see [3]).

3.D. Spectral representation for the Wilson coefficients

We outline the application of multiplicative renormalizability of conformal composite fields to OPE.

In the $\mathcal{C}_6^3$ model the OPE reads

$$\begin{aligned} \varphi(x+\frac{z}{2})\varphi^*(x-\frac{z}{2}) &\approx \sum_{m=0}^{\infty} \sum_{z^2 \rightarrow 0} C_m(z^2, g, \mu)(izV)^m \chi(x) + \\ &+ \sum_{\ell=0}^{\infty} \sum_{m=\ell}^{\infty} C_{m\ell}(z^2, g, \mu)(izV)^m Q_\ell(x, z). \end{aligned} \quad (3.15)$$

Since Q_ℓ do not mix under one-loop renormalization, we have

$$Q_\ell^{1-loop}(x) = Z_\ell Q_\ell(x), \quad Z_\ell = 1 + \frac{1}{\epsilon} \gamma_\ell \mu^\epsilon, \quad (3.16)$$

where

$$\gamma_\ell = \frac{g^2}{16\pi^2} \left(\frac{C}{\ell(\ell+1)\ell^2} - 1 \right), \quad \ell = 0, 1, \dots, \quad (3.17)$$

in the case of $\mathcal{C}_6^3$ and

$$\gamma_\ell = \frac{g^2}{8\pi^2} \frac{N-1}{2N} \left(\frac{2}{\ell(\ell+1)} - 1 + 4 \sum_{j=2}^{\ell} \frac{1}{j} \right), \quad \ell = 1, 2, \dots, \quad (3.18)$$

in the case of N-colour QCD (see Ref. 24).

Remark: As explained in Subsec.3.B an additional mixing problem is to be solved in $\mathcal{C}_6^3$ for even n and in the flavour singlet sector of QCD for $\ell \geq 2$. The anomalous dimension γ_1 of the conserved current vanishes in both cases as it should since there is no fermion-gluon (or φ - χ) mixing for $\ell = 1$.

The Wilson coefficients $C_{m\ell}$ satisfy the renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) + \gamma_\ell \right) C_{m\ell}(z^2, g, \mu) = 0. \quad (3.19)$$

In an asymptotically free theory one can write

$$C_{m\ell}(z^2, g, \mu) = \left[\frac{g^2(z^2)}{g^2(\mu^2)} \right]^{\frac{\gamma_\ell}{2\beta}} C_{m\ell}^0(z^2) \left[1 + O(g^2(z^2)) \right], \quad (3.20)$$

where $\beta(g) = -\beta_0 g^3 + O(g^5)$ and $C_{m\ell}^0(z^2)$ is the Wilson coefficient corresponding to the free field theory (with $g^2 = 0$). It remains to determine these coefficients, which in fact are simply proportional to the inverse of the matrix $U_{m\ell}$, which diagonalizes the mixing matrix $Z_{m\ell}$ (cf. (3.7)). This

can be determined using recurrence relations as is done in Ref.3. However it is useful to derive some identities involving sums over the C_{ml}^0 , which directly reflect the spectral (i.e. support) properties of vertex and wave functions. To this end the orthogonality properties of conformal operators discussed in Sec.II can be utilized.

Let us write down the light cone expansion of the three-point Wightman function defined in Subsec.2.B (Eq.(2.15)) in the limit $g^2 = 0$. For the scalar case we have

$$\begin{aligned} W_\ell(x_1, x_2; -p, \xi) &= \langle \varphi(x_1) \varphi^*(x_2) \tilde{O}_\ell(-p, \xi) \rangle = \\ &= \sum_{k=0}^{\infty} \sum_{m=n}^{\infty} C_{mn}^0 < \hat{O}^{(n,m)}(x, z) \tilde{O}_\ell(-p, \xi) \rangle_0, \quad (x = \frac{x_1 + x_2}{2}). \end{aligned} \quad (3.21)$$

Using the orthogonality relationship in the form in Eq.(2.12) we obtain

$$\begin{aligned} W_\ell(x_1, x_2; -p, \xi) &= \\ &= \sum_{m \geq \ell} C_{m\ell}^0 \frac{\ell!}{2^\ell (2\ell + \ell - 3)!} \left(\frac{1}{2} p z\right)^{m-\ell} \frac{2\pi \Theta_+(p)}{(4\pi)^k} \left(\frac{-p^2}{2}\right)^{\ell+k-2} \prod_{\ell} (p, z, \xi), \end{aligned} \quad (3.22)$$

where $\Pi^{(k)}(p; z, \xi)$ is given by (2.13). Comparing the light cone approximation (2.21) for w_ℓ with (3.22) we obtain the spectral representation

$$\begin{aligned} \sum_{m=\ell}^{\infty} C_{m\ell}^0 \left(\frac{1}{2} p z\right)^m &= (2\ell + 2\ell - 3)! \left(\frac{1}{2} p z\right)^\ell \ell_{\ell+k-2} \left(\frac{1}{2} p z\right) = \\ &= \frac{(2\ell)^\ell (2\ell + 2\ell - 3)! (2\ell + 2\ell - 5)!}{2(2\ell + \ell - 4)!} \int_{-1}^1 d\xi (1 - \xi^2)^{k-2} \ell_{\ell+k-2} \left(\frac{1}{2} p z\right) e^{-\frac{i}{2} \xi p z} \end{aligned} \quad (3.23)$$

(cf. (A.18)). In the case of fermion fields in QCD, we apply the light cone expansion to the Euclidean τ -function defined in Eq.(2.19). [We could have, of course, again used the Wightman functions as in the above calculation for the scalar case. The argument sketched below should appeal to readers with an experience in Euclidean space calculations involving dimensional regularization.]

Using the light cone expansion in the $g = 0$ limit (that is, the representation (2.22b) for F_E) we can write down the $1/\epsilon$ approximation to (2.19) as

$$\begin{aligned} \tau_\ell^{\Gamma}(x_1, x_2; -p, \xi) &\stackrel{z^2 = x_{12}}{\approx} \\ &\approx \frac{\Gamma(2-k)}{(4\pi)^k} \frac{1}{2} \left(\frac{p^2}{4}\right)^{\ell+1} \left(\frac{p^2}{4}\right)^{\ell+1} \ell_{\ell+k-2} \int_{-1}^1 d\xi (1 - \xi^2)^{k-1} \ell_{\ell+k-2} \left(\frac{1}{2} p z\right) e^{-\frac{i}{2} \xi p z} = \\ &= \frac{\Gamma(2-k)}{(4\pi)^k} \frac{2\ell (2\ell + \ell - 3)!}{(2\ell + 2\ell - 5)!} \left(\frac{p^2}{4}\right)^{\ell+1} \ell_{\ell+k-2} \left(\frac{p^2}{4}\right)^{\ell+1} \ell_{\ell+k-2} \left(\frac{1}{2} p z\right). \end{aligned} \quad (3.24)$$

If we define the Wilson coefficients C_{ml}^ψ for the light cone OPE of a free Dirac field by

$$\begin{aligned} \bar{\psi}(x - \frac{z}{2}) \Gamma z \psi(x + \frac{z}{2}) &\approx \sum_{z^2=0}^{\infty} \sum_{\ell=0}^{\infty} C_{m\ell}^\psi \left(\frac{z \not{V}}{2i}\right)^{m-\ell} O_{\ell+1}^\Gamma(x, z) \\ & \quad (3.25) \end{aligned}$$

we derive using (3.22) the spectral representation

$$\begin{aligned} \sum_{m=\ell}^{\infty} C_{m\ell}^\psi \left(\frac{z \not{V}}{2i}\right)^m &= (2\ell + 2\ell - 1)! \left(\frac{z \not{V}}{2i}\right)^\ell \ell_{\ell+k-1} \left(\frac{z \not{V}}{2i}\right) = \\ &= \frac{(2\ell + 2\ell - 1)! (2\ell + 2\ell - 5)!}{2(2\ell + \ell - 2)!} (2i)^\ell \int_{-1}^1 d\xi (1 - \xi^2)^{k-1} \ell_{\ell+k-1} \left(\frac{z \not{V}}{2i}\right) e^{-\frac{i}{2} \xi p z} \end{aligned} \quad (3.26)$$

We note that for $2h = 4$ Eq.(3.26) coincides with the representation (3.23) for the $\mathcal{O}_6^3$ model for $2h = 6$, so that $C_{ml}^\psi(h = 2) = C_{ml}^0$ ($h = 3$).

It follows from (3.23) (or (3.26)) that the inverse of the matrix U_{ml} , which diagonalizes the mixing matrix $\hat{Z}_{ln}$ (3.7) can be represented by

$$\begin{aligned} U_{m\ell}^{-1} &= m! C_{m\ell}^0 = 2^{\ell-1} \ell_{\ell+k-1} \frac{(2\ell + 3)! (2\ell + 1)!}{(\ell + 2)!} \int_{-1}^1 d\xi (1 - \xi^2)^{\frac{3}{2}} \ell_{\ell+k-1}(\xi) d\xi = \\ &= \delta_{m, \ell+2k} \frac{(2\ell + 3)! (2k + \ell)!}{(2\ell + 2k + 3)! (2k)!} \end{aligned} \quad (3.27)$$

Using (3.23, 26) we can write the following spectral representation of meson wave function defined in Eq.(2) of the Introduction:

$$\psi(z, \rho) = \sum_{\ell} a_{\ell} \left[\frac{\alpha_3(z^{-2})}{\alpha_3(\mu^2)} \right]^{\frac{\ell}{2\ell_0}} \int_{-1}^1 d\xi (1-\xi^2)^{\frac{3}{2\ell}} C_{\ell}^{3/2}(\xi) e^{\frac{i}{2} \xi \rho z}, \quad (3.28a)$$

where the normalization constants are related to the matrix elements of the conformal operators $\langle 0 | \hat{O}_{\ell}(0, z) | p \rangle$. The variable ξ corresponds to relative momentum of the constituent quarks in the above function at short distances. If we take the Fourier transform of (3.28a) along a light-like plane $\underline{z}_- = \underline{z}_0 - \underline{z}_3 = 0$, we obtain for the pion wave function

$$\phi_{\pi}(\xi, Q^2) = \sum_{\ell} a_{\ell}(\mu^2) (1-\xi^2)^{\frac{3}{2\ell}} C_{\ell}^{3/2}(\xi) \left[\frac{\alpha_3(Q^2)}{\alpha_3(\mu^2)} \right]^{\frac{\ell}{2\ell_0/2\ell_0}} \quad (3.28b)$$

which is the light cone wave function defined in Refs.2 and 10.

One interesting consequence of (3.28b) is that it can be inverted, so one can express the dynamical constants $a_{2\ell}(\mu^2)$ in terms of moments of the pion wave function at the scale $Q^2 = \mu^2$, i.e.

$$a_{2\ell} = \frac{(2\ell+1)(2\ell+1)}{(2\ell+1/2)} \int_{-1}^1 d\xi (1-\xi^2)^{\frac{3}{2\ell}} C_{2\ell}^{3/2}(\xi) \phi_{\pi}(\xi, \mu^2). \quad (3.29)$$

For a ground state wave function, one expects the wave function not to have a node, so that $a_{2\ell}/a_0$ (where $a_0 = F_{\pi}$) should decrease rapidly with ℓ . In particular, $a_2 \sim 0.1 F_{\pi}$, in line with the value obtained in Ref.3.

The technique described in this section can be used for any vertex function involving products of local currents or gauge invariant operators in QCD. Further, if we are interested in vertex or wave functions involving large mass states, so that $\underline{z}^2 M^2$ is not small, then the conformal technique developed in Sec.II can be utilized to at least take into account the kinematic higher twist corrections.

IV. CONCLUDING REMARKS

The results of the paper can be stated as follows.

For any QFT with a conformally invariant Lagrangian (that is, essentially, for the massless limit of any renormalizable QFT) there is a distinguished set

of (local) composite (symmetric traceless) tensor operators O_{λ} (with an arbitrary number λ of tensor indices). These operators are (a) conformally covariant (obeying the simple infinitesimal transformation law (1.4) with respect to dilatations and special conformal transformations); (b) conserved

(for $\lambda \geq 1$) in the free field limit; (c) orthogonal in the sense of vacuum expectation values; (d) they are multiplicatively renormalized, i.e. they diagonalize the anomalous dimension matrix at the one-loop level. This latter property has been discovered (in [2]) by direct diagonalization of the renormalization mixing matrix without any use of conformal invariance. A central result of the present paper is the proof that (d) is a consequence of either (a) or (c) (using in the latter case also the triangular character of the renormalization matrix). This result is relevant physically for QCD and allows us to derive a light cone expansion in the (renormalization group improved) one-loop approximation along with a simple spectral representation for the Wilson coefficients and for meson wave functions. The realization that the diagonalization property is directly related to conformal invariance and to orthogonality also gives us an insight of what is going on beyond the one-loop level at least for the ϕ_6^3 model.

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Two-point functions of composite fields

The objective of this appendix is two-fold. First, in Sec.A.1 we provide a short guide and a pedagogical supplement to the literature concerned with the evaluation of two-point functions of basic tensor fields that only uses their conformal transformation properties. Secondly, in Sec.A.2, we present a list of relevant formulae for (normalized) Gegenbauer polynomials used in the chains of Eqs.(2.12) and (2.14a).

A.1 General form of conformal invariant two-point functions

Let $O_\ell(x, z; c)$ ($z^2 = 0$) be a (not necessarily conserved) symmetric tensor field of dimension $h + c$ (in mass units) in $2h$ dimensional space-time. The derivation of the general form of its two-point Wightman function proceeds in three steps.

Step 1: First, we evaluate the Euclidean (Schwinger) two-point function in coordinate space. The most elegant way to achieve that is to write down the kernel of the intertwining operator between the representations $[\ell, c]$ and $[\ell, -c]$ of the Euclidean conformal group $SO(2h+1)$. These (type I) representations are equivalent for almost all values of c and are partially equivalent for some exceptional integer values of the dimensions $h \pm c$. The intertwining operator appears as a representation of the conformal inversion

$$I_r x = \frac{x}{x^2} \quad (A.1)$$

(Recall that every special conformal transformations can be written as a translation sandwiched between two inversions.) If $V^\mu(x)$ is a conformal vector field of dimension d , then its transformation law under conformal inversion is

$$V^\mu(x) I_r \rightarrow \frac{1}{x^{2d}} r^\mu_\nu(x) V^\nu(I_r x), \quad (A.2a)$$

where

$$r^\mu_\nu(x) = x^2 \nabla^\mu x^\nu / x^2 = \delta^\mu_\nu - 2 \frac{x^\mu x_\nu}{x^2}$$

so that

$$r^\mu, r^\nu_\nu = \delta^\mu_\nu. \quad (A.2b)$$

(The reader is advised to derive (A.2) in the special case in which $v^\mu = \nabla^\mu S(x)$, where $S(x)$ is a dimensionless scalar field transforming as $S(x) \rightarrow S(I_r x)$ under conformal inversion, so that $d_S = 0$ and hence $d_V = 1$.) Using this approach one arrives at the following expression for the two-point Schwinger function of the field $O_\ell(x, z; c)$:

$$G_{\ell c}(x_{12}; \tilde{z}_1, \tilde{z}_2) = \langle O_{\ell c}(x_1, \tilde{z}_1; c) O_{\ell c}(x_2, \tilde{z}_2; c) \rangle_E = \frac{n(\ell, c)}{(2\pi)^h} \frac{(\tilde{z}_{1\mu} r^\mu_\nu(x_{12}) \tilde{z}_2^\nu)^\ell}{\left(\frac{1}{2} x_{12}^2\right)^{h+c}}, \quad (A.3)$$

where $n(\ell, c)$ is an undetermined normalization constant. The derivation of this result with a detailed study of all underlying notions can be found in the lecture notes' volume [5]. A summary is given in Appx.B to the Pisa lectures [6]. A more elementary (but clumsier) derivation of the same result for lower rank tensors in R^4 , using global $SO(5,1)$ invariance, is given in Sec.IV.1A of Ref.6. The result can also be verified by applying infinitesimal conformal invariance (see Sec.I). Each of these methods also implies the vanishing of the conformal invariant two-point function for two basic fields of different dimensions.

Step 2: Secondly, we write down the Fourier transform of (A.3) (still in the Euclidean $2h$ -dimensional space). There is a technical problem of expanding the result in terms of $SO(2h-1)_p$ spin projection operators $\Pi^\ell_s(p)$. (Here $SO(2h-1)_p$ stands for the subgroup of the "Euclidean Lorentz group" $SO(2h)$ which leaves the vector p invariant.) This problem has been solved in Ref.5 (see Sec.5). The result is

$$G_{\ell c}(p; \tilde{z}_1, \tilde{z}_2) = \int G_{\ell c}(x; \tilde{z}_1, \tilde{z}_2) e^{-ipx} d^h x = \frac{n(\ell, c) \Gamma(-c)}{(h+\ell+c-1)\Gamma(h+c-1)} \left(\frac{p^2}{2}\right)^c \sum_{s=0}^{\ell} \alpha_{\ell s}(c) \Pi^\ell_s(p; \tilde{z}_1, \tilde{z}_2), \quad (A.4a)$$

(A.4a)

where

$$\alpha_{\ell s}(c) = \frac{\Gamma(h+\ell-c-1)\Gamma(h+s+c-1)}{\Gamma(h+\ell+c-1)\Gamma(h+s-c-1)} =$$

$$= \frac{\binom{h+\ell-c-2}{\ell-s}}{\binom{h+\ell+c-2}{\ell-s}} \quad (\text{A.4b})$$

We shall not reproduce here the expressions for $\Pi_{\ell s}^{\lambda s}$ ($s < \ell$) in terms of Gegenbauer polynomials since we will not need them ($\Pi_{\ell \ell}^{\lambda \ell}$ is given by (2.13)). Regarded as operators in the space of rank- ℓ symmetric traceless tensors the projectors

$$\Pi^{\ell s}(p) = \left(\Pi_{(p)}^{\ell s} \mu_1 \dots \mu_\ell \nu_1 \dots \nu_\ell \right)$$

satisfy the following orthogonality and completeness relations:

$$\Pi^{\ell s}(p) \Pi^{\ell s'}(p) = \delta_{ss'} \Pi^{\ell s}(p), \quad (\text{A.5a})$$

$$\sum_{s=0}^{\ell} \Pi^{\ell s}(p) = 1. \quad (\text{A.5b})$$

The maximal spin projector $\Pi_{\ell \ell}^{\lambda \ell}$ is characterized by the transversality property

$$p_{\mu_1} \Pi_{(p)}^{\ell \ell} \mu_1 \dots \mu_\ell \nu_1 \dots \nu_\ell = 0 = \Pi_{(p)}^{\ell \ell} \mu_1 \dots \mu_\ell p_{\nu_\ell}. \quad (\text{A.6})$$

Step 3: Finally, we pass from (A.4) to the Minkowski momentum space expression for the corresponding Wightman function. That is achieved by recalling that $w_{\lambda c}(x) = w_{\lambda c}(x^0, \vec{x})$ is obtained from $G_{\lambda c}(x, x_{2n})$ by taking the limit $\epsilon \rightarrow 0$ for $x_{2n} = -ix^0 - \epsilon$. If we then shift in the Fourier integral in p_{2n} along the two sides of the negative imaginary axis we find

$$w_{\ell c}(p^0, \underline{p}) =$$

$$= i\theta(p^0) \lim_{\epsilon \rightarrow 0^+} \left[G_{\ell c}(\underline{p}, -ip^0 - \epsilon) - G_{\ell c}(\underline{p}, -ip^0 + \epsilon) \right]. \quad (\text{A.7})$$

Using the identities

$$(p^2 + i0)^c - (p^2 - i0)^c = 2i \theta(-p^2) (-p^2)^c \sin \pi c \quad (\text{A.8})$$

and

$$\Gamma(c) \Gamma(1-c) \sin \pi c = \pi \quad (\text{A.9})$$

we derive from (A.4) and (A.7) the expression

$$w_{\ell c}(p) = \frac{2\pi n(\ell, c) \theta_+(p)}{\Gamma(c+1) \Gamma(h+c-1) \Gamma(h+\ell+c-1) \left(\frac{-p^2}{2}\right)^c} \sum_{s=0}^{\ell} \alpha_{\ell s}(c) \Pi^{\ell s}(p)$$

$$(\theta_+(p) = \theta(p^0) \theta(-p^2)). \quad (\text{A.10})$$

In order to derive from here (2.12), we note that for

$$c = h + \ell - 2 \quad (\text{or} \quad d_\ell = 2h + \ell - 2) \quad (\text{A.11})$$

$\alpha_{\ell s}(c) = 0$ for $s < \ell$ (while $\alpha_{\ell \ell} = 1$) and set

$$n(\ell, c) = \frac{\ell! \Gamma(\ell+h-1) (2h+\ell-4)!}{(4\pi)^h 2^\ell (2h+2\ell-5)!}. \quad (\text{A.12})$$

The fact that the resulting $w_{\ell c}$,

$$W_{\ell, h+\ell-2}(p) = \frac{\ell!}{(4\pi)^h} 2^\ell \frac{2\pi \Theta_+(p)}{(2h+2\ell-3)!} \left(\frac{-p^2}{2}\right)^{\ell+h-2} \prod_{(p)}^{\ell\ell} \quad (\text{A.13})$$

indeed represents the vacuum expectation value of a conserved tensor, is implied by (A.6).

It is worth noting that the value (A.11) is one of those exceptional values of c for which the representations of dimension $h \pm c$ are only partially equivalent and the right-hand side of (A.4a) diverges (for integer h) unless $n(\ell, c)$ vanishes like $\frac{1}{\Gamma(-c)}$ (which would have led to the vanishing of the Wightman function). Thus it is important that we only set $c = h - 2 + \ell$ at the end of our calculation, in the expression for the Minkowski space Wightman function. On the other hand, it is precisely for these exceptional values of c that we have the expression (2.9) (or (2.11)) for O_ℓ in terms of normal products of free fields which enables us to give a direct derivation of (A.13). In order to reproduce (2.14) we need a different normalization $n(\ell, c)$ so that

$$\begin{aligned} W_{\ell\Gamma}(p; \zeta_1, \zeta_2) &= \langle O_\ell^\Gamma(c, \zeta_1) \tilde{O}_\ell^\Gamma(p, \zeta_2) \rangle_0 = \\ &= \frac{2h(2h+\ell-3)}{\ell} W_{\ell, h+\ell-2}(p; \zeta_1, \zeta_2). \end{aligned} \quad (\text{A.14})$$

A.2 Properties of normalized Gegenbauer polynomials

We write down here for the reader's convenience a few formulae involving the normalized Gegenbauer polynomials

$$\begin{aligned} \hat{C}_\ell^\nu(\xi) &= \frac{1}{N_\ell^\nu} C_\ell^\nu(\xi), \\ N_\ell^\nu &= \frac{2^\ell d^\ell}{\ell! d^2 \ell} C_\ell^\nu(\xi) = \frac{2^\ell}{\ell!} \frac{(2\nu+2\ell-2)!}{(2\nu-2)!} \end{aligned} \quad (\text{A.15})$$

used throughout the paper (in particular, in the computations of Subsec.2.B). $\hat{C}_\ell^\nu$ are simply related to the Jacobi polynomials:

$$P_\ell^{(\nu-\frac{1}{2}, \nu-\frac{1}{2})}(\xi) = \binom{2\nu+2\ell-1}{\ell} \hat{C}_\ell^\nu(\xi).$$

Hence from (2.3) we have a similar expansion formula.

Orthogonality and renormalization (used in the last equations (2.12) and (2.14a)) are summarized by

$$\begin{aligned} \int_{-1}^1 d\xi (1-\xi^2)^{\nu-\frac{1}{2}} \hat{C}_\ell^\nu(\xi) \hat{C}_n^\nu(\xi) &= \\ &= \frac{\ell! (2\nu+\ell-1)! \delta_{\ell n}}{2^{2\ell-1} (2\nu+2\ell-2)! (2\nu+2\ell)!} \end{aligned} \quad (\text{A.16})$$

The value of $\hat{C}_\ell^\nu$ at $\xi = 1$ which appears in (2.13b) is

$$\hat{C}_\ell^\nu(1) = \frac{1}{N_\ell^\nu} \binom{2\nu+\ell-1}{\ell} = \frac{(2\nu+\ell-1)!}{2^\ell (2\nu-1)! (2\nu+2\ell-2)!} \quad (\text{A.17})$$

The "Fourier transform" of $\hat{C}_\ell^\nu$ which appears in the spectral representations (2.21), (3.21)-(3.23) is given by

$$\begin{aligned} \int_{-1}^1 d\xi (1-\xi^2)^{\nu-\frac{1}{2}} \hat{C}_\ell^\nu(\xi) e^{-i\omega\xi} &= \\ &= 2 \frac{(\omega)}{(2i)} \frac{(2\nu+\ell-1)!}{(2\nu+2\ell-2)!} \frac{1}{\ell+\nu-\frac{1}{2}} \frac{(\omega)}{(2i)} = \\ &= \frac{(2\nu+\ell-1)!}{(2\nu+2\ell-1)!} \frac{(\omega)}{(2i)} \int_{-1}^1 d\xi (1-\xi^2)^{\nu+\ell-\frac{1}{2}} e^{-i\xi\omega} \end{aligned} \quad (\text{A.18})$$

where f_n (for integer n) are expressed in terms of Bessel functions of half integer rank according to (2.18). Eq. (A.18) can also be verified directly by using the representation

$$(1-\xi^2)^{\nu-\frac{1}{2}} \hat{C}_\ell^\nu(\xi) = \frac{(2\nu+\ell-1)!}{(2\nu+2\ell-1)!} \left(-\frac{1}{2} \frac{d}{d\xi}\right)^\ell (1-\xi^2)^{\ell+\nu-\frac{1}{2}} \quad (\text{A.19})$$

Conformal invariant three-point functions for free constituent fields

The three-point function of a basic tensor field and two scalar (or spinor) fields is determined from conformal invariance up to one (or two) real constants. The expressions (2.15)-(2.19) for $W_0(\Gamma)(x_1, x_2; -p, z)$ can again be obtained in three steps, matching the three steps in the derivation of the conformal invariant two-point function in Appx.A. We shall provide a direct elementary derivation for each of these steps using the explicit formula for Q_ℓ in terms of free fields.

B.1 Scalar constituent fields

The three-point τ -function in Euclidean x -space can be written in the form

$$\begin{aligned} \tau_\ell^E(x_1, x_2; x_3, \bar{z}) &= \langle \varphi(x_1) \varphi^*(x_2) O(x_3, \bar{z}) \rangle_E \\ &= D_\ell^{h-1}(-i\bar{z}V_2, -i\bar{z}V_2) \Delta_\ell(x_{13}) \Delta_\ell(x_{23}), \end{aligned} \quad (\text{B.1})$$

where $\Delta_\ell(x)$ is the canonical Euclidean propagator for a scalar field

$$\Delta_\ell(x) = \frac{\Gamma(h-1)}{4\pi(\pi x^2)^{h-1}} = \frac{1}{(4\pi)^h} \int_0^\infty d\alpha \alpha^{h-2} e^{-\frac{\alpha}{4}x^2} =$$

$$= \int \frac{e^{ipx}}{p^2} d_{2h}p \quad \left(d_n p = \frac{d^n p}{(2\pi)^n} \right). \quad (\text{B.2})$$

The transfer of the derivatives in Eq.(B.1) from x_3 to the arguments x_1 and x_2 allows us to evaluate the Fourier transform in x_3 :

$$\begin{aligned} \tau_\ell^E(x_1, x_2; -p, \bar{z}) &= \int d^{2h}y \, e^{ipV} \tau_\ell^E(x_1, x_2; y, \bar{z}) = \\ &= \frac{1}{(4\pi)^h} D_\ell^{h-1}(-i\bar{z}V_2, -i\bar{z}V_2) \int_0^\infty \frac{d\alpha}{\alpha^2} \int_0^\infty \frac{d\beta}{\beta^2} \left(\frac{\alpha\beta}{\alpha+\beta} \right)^h \exp \left\{ -\frac{\alpha\beta \frac{x_{12}^2}{2} + i(\alpha x_1 + \beta x_2)p + p^2}{\alpha+\beta} \right\} = \\ &= \frac{e^{ipx}}{(4\pi)^h} D_\ell^{h-1}(\tfrac{1}{2}\bar{z}p - i\bar{z}V, \tfrac{1}{2}\bar{z}p + i\bar{z}V) F_E(x_2, p) \quad \left(x = \frac{x_1+x_2}{2}, V = \frac{\partial}{\partial x_{12}} \right) \end{aligned} \quad (\text{B.3a})$$

where

$$\begin{aligned} F_E(x_{12}, p) &= \\ &= \frac{1}{2^{2h-3}} \int_{-1}^1 d\xi (1-\xi^2)^{h-2} e^{-\frac{i}{2}\xi x_{12}p} \int_0^\infty d\lambda \lambda^{h-3} e^{-\lambda \frac{x_{12}^2}{4} - \frac{p^2}{\lambda}} \end{aligned} \quad (\text{B.3b})$$

(we have used the α representation of Eq.(B.2) and have made the change of variables $\alpha = \lambda \frac{1+\xi}{2}$, $\beta = \lambda \frac{1-\xi}{2}$). Using a standard integral representation for the Macdonald Bessel function K_{h-2} we can perform the integration in λ with the result (2.16).

The transition from τ_ℓ^E to the three-point Wightman function involves analytic continuation in p (and in x_1, x_2) from Euclidean to (real) Minkowski space values and evaluation of the discontinuity for $p_{2h} = -ip^0 \mp \epsilon$ (cf. Eq.(A.7)). To do this we use the defining relations for the K function

$$K_{h-2}(z) = \frac{\pi}{2} \frac{I_{2-h}(z) - I_{h-2}(z)}{\sin \pi(h-2)},$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{1}{n! \Gamma(n+\nu+1)} \left(\frac{z^2}{4}\right)^n \quad (\text{B.4})$$

(for integer h the right-hand side of the first equation is understood as a limit from non-integer values). Inserting (B.4) in (B.3b) we find that only the second term (proportional to I_{h-2}) contributes to the discontinuity in p . Using again (A.8), we find

$$\begin{aligned} W_\ell(x_1, x_2; -p, \bar{z}) &= \\ &= \frac{1}{(4\pi)^h} e^{ipx} D_\ell^{h-1}(\tfrac{1}{2}\bar{z}p - i\bar{z}V, \tfrac{1}{2}\bar{z}p + i\bar{z}V) F_M(x_2, p) \end{aligned} \quad (\text{B.5})$$

($x = \frac{x_1+x_2}{2}$),

where F_M is given by (2.17b). The simpler expression (2.17c), on the other hand, can be obtained starting from the free field Wightman function

$$f_1(\omega) = \frac{1}{\omega^2} \left(\frac{\sin \omega}{\omega} - \cos \omega \right), \quad (\text{B.8b})$$

$$\begin{aligned} f_{h-2}(\omega) &= \left(-\frac{1}{\omega} \frac{d}{d\omega} \right)^{h-2} \frac{\sin \omega}{\omega} = \\ &= \frac{1}{(2h-3)!!} - \frac{1}{(2h-1)!!} \frac{\omega^2}{2!} + \frac{1}{(2h+1)!!} \frac{\omega^4}{4!} - \dots \end{aligned} \quad (\text{B.8c})$$

B.2 Spin $\frac{1}{2}$ constituent fields

Proceeding to the evaluation of the three-point function of a spinor constituent field with a basic tensor field, we observe that there are two independent conformal invariant structures in this case. For example, the three-point Schwinger function of a free Dirac field and a conserved current has the general form

$$\begin{aligned} \langle \psi(x_1) \tilde{\psi}(x_2) j_\mu(x_3) \rangle_E &= C_1 S_h(x_{13}) i \gamma_\mu S_h(x_{23}) + \\ &+ C_2 S_{3/2}(x_{12}) \Delta_h(x_{13}) \Delta_h(x_{23}) \lambda_\mu, \end{aligned} \quad (\text{B.9})$$

where $S_{\frac{d+1}{2}}(x)$ is the conformal invariant Euclidean propagator of a Dirac field of dimension d (in $2h$ dimensional space time); $\Delta_h(x)$ is given by (B.2) while

$$\begin{aligned} S_h(x) &= \int \frac{-i \not{p}}{p^2} e^{ipx} d_{2h} p = \\ &= -\not{x} \Delta_h(x) = \frac{\Gamma(h/4)}{(4\pi)^{h/2}} \frac{x^\mu}{x^2}, \end{aligned} \quad (\text{B.10})$$

$$S_{3/2}(x) = -i \int \frac{\not{p} e^{ipx}}{p^{2h-1}} d_{2h} p = \frac{\sqrt{\pi}}{2(4\pi)^{h/2}} \left(\frac{x^\mu}{x^2} \right)^{3/2} \frac{x^\mu}{2}, \quad (\text{B.11})$$

$$\lambda_\mu = \frac{\partial}{\partial x_{3\mu}} \ln \frac{X_{23}^2}{X_{13}^2} = 2 \left(\frac{x_{13}^\mu}{X_{13}^2} - \frac{x_{23}^\mu}{X_{23}^2} \right). \quad (\text{B.12})$$

$$\begin{aligned} W_h(x) &= \langle \psi(x) \psi^*(0) \rangle_0 = \langle \psi^*(x) \psi(0) \rangle_0 = \\ &= \frac{\Gamma(h-1)}{(4\pi)^h} \left(\frac{4}{x^2 + i\epsilon x^0} \right)^{h-1} = \int d_{2h} p \delta_0^+(p) e^{ipx}, \end{aligned} \quad (\text{B.6a})$$

where

$$\delta_m^+(p) = 2\pi \theta(p^0) \delta(m^2 + p^2). \quad (\text{B.6b})$$

A straightforward calculation gives

$$\begin{aligned} F_h(x_{12}, p) &= (4\pi)^h \int W_h\left(\frac{x_{12}}{2} - y\right) W_h\left(-\frac{x_{12}}{2} - y\right) e^{ipy} d^{2h} y = \\ &= \frac{4\theta_+(p)}{\pi^{h-1}} \int \delta(2pq) \delta\left(\frac{p^2}{4} + q^2\right) e^{iqx_{12}} d^{2h} q \\ &= \frac{\theta_+(p)}{\pi^{h-1}} \sqrt{\frac{4}{p^2}} \int \delta(r^2 + \frac{p^2}{4}) e^{iqr} d^{2h-1} r \\ &= 2\pi \theta_+(p) (-p^2)^{h-2} f_{h-2}(\omega), \end{aligned} \quad (\text{B.7a})$$

where

$$f_{h-2}(\omega) = \sqrt{\frac{\pi}{2}} \omega^{\frac{3}{2}-h} \int_{h-\frac{3}{2}}^{\frac{3}{2}-h} J_{h-\frac{3}{2}}(\omega), \quad \omega = \frac{1}{2} \sqrt{(px_{12})^2 - p^2 x_{12}^2}. \quad (\text{B.7b})$$

The variable $\underline{r}$ in the intermediate integration in (B.7a) is the space part of x_{12} in the rest frame of p ; $J_{h-\frac{3}{2}}(\omega)$ and hence f_{h-2} are elementary functions for integer h :

$$f_0(\omega) = \frac{\sin \omega}{\omega}, \quad (\text{B.8a})$$

Clearly, only the first term (with $C_1 = 1$) would contribute to (B.9) if $j_\mu(x)$ is identified with the composite field : $\tilde{\psi}(x) i\gamma_\mu \psi(x)$: (as in (2.11)). Hence the use of the explicit expression (2.11) for the fields q not only simplifies the evaluation of the three-point function, but is actually necessary in order to recover its x and p dependence as given by (2.19).

The calculation of the Euclidean three-point τ -function involving a product of spinor fields and the conformal operators (2.11) follows closely the case of scalar constituent fields. In fact, it can be reduced to the latter by the following argument. Writing the product of spinor propagators (B.10) as derivatives of their scalar counterparts

$$S_k(x_1) \Gamma \tilde{z} S_k(x_2) = \not{X}_2 \Gamma \tilde{z} \not{X}_2 \Delta_k(x_1) \Delta_k(x_2) .$$

We can express the three-point function

$$\tau_{\theta\Gamma}^E(x_1, x_2; p, \tilde{z}) = \langle \psi(x_1) \tilde{\psi}(x_2) \tilde{O}_E(p, \tilde{z}) \rangle_E \quad (\text{B.13a})$$

in terms of derivatives of the same function Γ_E which appears in the scalar case (B.3)

$$\tau_{\theta\Gamma}^E(x_1, x_2; p, \tilde{z}) = (\tilde{z}p)^{\ell-1} \hat{c}^{h-\frac{1}{2}} \left(\frac{2i\tilde{z}\nabla}{\tilde{z}p} \right) \Gamma_{\Gamma}^{(h)}(x_1, x_2; p, \tilde{z}), \quad (\text{B.13b})$$

where

$$\begin{aligned} \Gamma_{\Gamma}^{(h)}(x_1, x_2; p, \tilde{z}) &= \int d^{2h} X_3 e^{ipX_3} S_k(x_1) \Gamma \tilde{z} S_k(x_2) = \\ &= \not{X}_2 \Gamma \tilde{z} \not{X}_2 \frac{e^{ipX}}{(4\pi)^h} \Gamma_E(x_{12}, p) = \\ &= \frac{e^{ipX}}{(4\pi)^h} \left(\frac{i}{2} \not{X} + \not{X} \right) \Gamma \tilde{z} \left(\frac{i}{2} \not{X} - \not{X} \right) \Gamma_E(x_{12}, p) \quad (\text{B.14}) \end{aligned}$$

$$(x = \frac{1}{2}(x_1 + x_2), \quad = \frac{\partial}{\partial x_{12}}). \quad \text{This proves (2.19).}$$

Note finally that the conservation law (2.7) implies the transversality relation

$$p \delta \omega_{\ell(\Gamma)}(x_1, x_2; -p, \tilde{z}) = 0 \quad (\delta_\mu = (1 - i + \tilde{z} \partial) \partial_\mu - \frac{1}{2} \tilde{z} \mu \partial^2) \quad (\text{B.15})$$

for both the scalar and spinor field three-point functions. A direct verification of (B.15) uses the relation

$$p \nabla F_M(x_{12}, p) = 0 \quad (\text{B.16})$$

which is manifest, since F_M depends on x_{12} through the variable

$$4\omega^2 = (p x_{12})^2 - p^2 x_{12}^2. \quad (\text{B.17})$$

REFERENCES

- [1] B. Schroer, "A necessary and sufficient condition for the softness of the trace of the energy momentum tensor", *Lettere al Nuovo Cimento* **2**, 867-872 (1971).
- [2] A.V. Efremov and A.V. Radyushkin, "Asymptotic behaviour of the pion form factor in quantum chromodynamics", *Teor. Mat. Fiz.* **42**, 147-166 (1980) [transl: *Theor. Math. Phys.* **42**, 97-110]; (see also Dubna preprint JINR E2-11983 (1978)).
- [3] N.S. Craigie and J. Stern, "What can we learn from sum rules involving vertex functions in QCD", *Nucl. Phys.* **B216**, 209 (1983).
- [4] V.K. Dobrev, V.B. Petkova, S.G. Petrova and I.T. Todorov, "Dynamical derivation of vacuum operator product expansion in Euclidean conformal quantum field theory", *Phys. Rev. D13*, 887-912 (1976).
- [5] V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova and I.T. Todorov, Harmonic Analysis in the n-Dimensional Lorentz Group and its Applications to Conformal Quantum Field Theory, *Lecture Notes in Physics* **63** (Springer, Berlin 1977).
- [6] I.T. Todorov, M.C. Mintchev and V.B. Petkova, Conformal Invariance in Quantum Field Theory, (Scuola Normale Superiore, Pisa 1978).
- [7] A.A. Migdal, "Multicolor QCD as a dual resonance theory", *Ann. Phys.* (N.Y.) **109**, 365-392 (1977) (see in particular Eq.(61) which is attributed to A.B. Zamolodchikov).
- [8] Yu.M. Makeenko, "On conformal operators in QCD", *Yad. Fiz.* **33**, 842-847 (1981) [transl: *Sov. J. Nucl. Phys.* **33**, 440-442 (1981)].
- [9] V.K. Dobrev, A.Ch. Ganchev and O.I. Yordanov, "Conformal operators from spinor fields: The symmetric tensor case", *Phys. Lett.* **119B**, 372-376 (1982);
V.K. Dobrev and A.Ch. Ganchev, "Conformal operators from spinor fields: The antisymmetric tensor case", *Dubna preprint JINR E2-82-881* (1982).
- [10] G.P. Lepage and S.J. Brodsky, "Exclusive processes in perturbative quantum chromodynamics", *Phys. Rev.* **22**, 2157-2198 (1980).
- [11] S.J. Brodsky, Y. Frishman, G.P. Lepage and S. Sachrajda, "Hadronic wave functions at short distances and the operator product expansion", *Phys. Lett.* **91B**, 239-244 (1980).
- [12] S. Ferrara, R. Gatto and A. Grillo, "Conformal algebra in space-time and operator product expansion", *Springer Tracts in Modern Physics* **67**, (Springer-Verlag, Berlin 1973);
S. Ferrara, R. Gatto, A. Grillo and G. Parisi, "General consequences of conformal algebra", in *Scale and Conformal Symmetry in Hadron Physics*, Ed. R. Gatto (Wiley, New York 1973), pp. 59-108.
- [13] G. Mack, "Conformal invariance and short distance behaviour in quantum field theory", *Lecture Notes in Physics* **17**: *Strong Interaction Physics* (Springer-Verlag, Berlin 1972), pp. 300-334; "Conformal invariant quantum field theory" in *Scale and Conformal Symmetry in Hadron Physics*, Ed. R. Gatto (Wiley, New York 1973), pp. 109-130.
- [14] R.A. Brandt and G. Preparata, "Operator product expansions near the light cone", *Nucl. Phys.* **B27**, 541-567 (1971).
- [15] N.S. Craigie and H. Dorn, "On the renormalization and short distance properties of hadronic operators in QCD", *Nucl. Phys.* **B185**, 204-220 (1981).
- [16] G. Mack and Abdus Salam, "Finite Component field representations of the conformal group", *Ann. Phys.* (N.Y.) **53**, 174-202 (1969).
- [17] G. Mack, "All unitary ray representations of the conformal group $SU(2,2)$ with positive energy", *Comm. Math. Phys.* **55**, 1-28 (1977).
- [18] V. Bargmann and I.T. Todorov, "Spaces of analytic functions on a complex cone as carriers for the symmetric tensor representations of $SO(n)$ ", *J. Math. Phys.* **18**, 1141-1148 (1977).
The homogeneous polynomial technique has been also used earlier in a similar context, see e.g. A.I. Oksak and I.T. Todorov, "On the covariant structure of the two-point function", *Comm. Math. Phys.* **14**, 271-304 (1969); I.T. Todorov and R.P. Zaikov, "Spectral representation of the covariant two-point function and infinite component fields with arbitrary mass spectrum", *J. Math. Phys.* **10**, 2014-2019 (1969).
- [19] S. Ferrara, R. Gatto and A. Grillo, "Manifestly conformal covariant operator product expansion", *Nuovo Cim. Lett.* **2**, 1363-1369 (1971);
"Manifestly conformal covariant expansion on the light cone", *Phys. Rev. D5*, 3102-3108 (1972).
- [20] L. Bonora, G. Sartori and N. Tonin, "Conformal covariant operator product expansion", *Nuovo Cim.* **10A**, 667-681 (1972).
- [21] V.K. Dobrev, E.H. Hristova, V.B. Petkova and D.B. Stamenov, "Conformal covariant operator product expansion of two-spin 1/2 fields", *Bulg. J. Phys.* **1**, 42-57 (1974).

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- [22] B. Schroer, J.A. Swieca and A.M. Volkel, "Global operator product expansions in conformal invariant relativistic quantum field theory", Phys. Rev. D11, 1509-1520 (1975).
- [23] Th. Ohmrodorf, "Constraints from conformal invariance on the mixing of operators of lower twist", Nucl. Phys. B198, 26-44 (1982).
- [24] H. Georgi and H.D. Politzer, "Electroproduction scaling in an asymptotically free theory of strong interactions", Phys. Rev. D9, 416-420 (1974);
D.J. Gross and F. Wilczek, "Asymptotically free gauge theories. II", Phys. Rev. D9, 980-993 (1974);
H.D. Politzer, "Asymptotic freedom: An approach to strong interactions", Phys. Rev. D14, 129-180 (1974);
D.J. Gross, "Applications of the renormalization group to high-energy physics", in Methods in Field Theory, Eds. R. Balian and J. Zinn-Justin, Les Houches, Session XXVIII, 1975, (North Holland, Amsterdam, 1976);
A.J. Buras, "Asymptotic freedom in deep inelastic processes in the leading order and beyond", Rev. Mod. Phys. 52, 199-276 (1980).
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