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LARGE N UNIVERSALITY OF VARIANT ACTIONS

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A B S T R A C T

Corrections to large N universality in mixed action lattice gauge theories imply constraints on the cut-off dependence. We state the form of these constraints and show that they are satisfied up to and including two loops (weak coupling). This implies that the second coefficient of the function on the lattice is indeed universal as expected. We then evaluate all the corrections to large N universality up to and including two loops. They give a sizeable contribution and determine qualitative aspects of equal string tension data like the bending seen for SU(2).

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1. - INTRODUCTION

In this paper we discuss the universal behaviour of lattice gauge theories¹⁾ as the gauge group $SU(N)$ becomes very large: N tends to infinity.

We are inspired by the arguments of Makeenko, Polikarpov and Samuel²⁾: they showed how in the $N = \infty$ limit a mixed lattice action consisting of a fundamental and an adjoint representation gives the same quantum theory as the fundamental representation alone, but with an effective coupling constant.

An immediate consequence of this fact at $N = \infty$ is the following: if confinement persists in the region of weak coupling (as is strongly suggested by computer simulation at low N ^{3),4)}), then all mass ratios are bound to be the same whatever mixture is taken. This is so because we need only one mass-parameter⁵⁾ and the effective coupling to describe our results in weak coupling in the mixed theory.

Obviously we want to have a clear idea of what the corrections to this effective coupling are when N is finite in order to compare with existing computer data for $N = 2$. Before we delve into the technical aspects of the large N expansion, we would like to give some motivations for doing so.

The key quantity in $SU(N)$ lattice gauge theory (no quarks) is the string tension¹⁾ ρ . This quantity parametrizes the area behaviour of averages of large loops C with minimal area $\Sigma(C)$ and perimeter $|C|$ for which $|C|/\Sigma(C) \rightarrow 0$:

$$\left\langle \frac{\text{Tr } U(C)}{N} \right\rangle = \exp[-g \Sigma(C)]. \quad (1.1)$$

Our notation is the standard one¹⁾: $U(C)$ stands for the ordered product of the link matrices $U(l)$ along C . The average is taken with respect to a Boltzmann factor

$$\exp \left[\frac{\beta_f}{2} \sum_P (\text{Tr } U(P) + \text{c.c.}) + \beta_A \sum_P |\text{Tr } U(P)|^2 \right] \quad (1.2)$$

where we took for definiteness a mixture of fundamental and adjoint actions, with couplings β_f and β_A , respectively.

Thus, the string tension ρ will be a function of β_f and β_A , with N fixed.

The computer data⁶⁾ for $N = 2$ show (see Fig. 1) a set of approximately straight lines on which the string tension is constant.

Conventional strong coupling expansion methods reproduce this simple "straight line behaviour" only when calculating to high order (11th order)⁷⁾ in β_F and β_A , for fixed N ($N = 2$).

On the other hand, the calculational effort with the large N method (see Section 3) is comparatively small, when we want to reproduce the data for $N = 2$.

The situation in the region where conventional weak coupling methods - fixed N - are applied, is much more dramatic. Based on such methods, equal string tension curves are predicted as in Fig. 1 for $N = 2$. Notice that the curves start in the weak coupling region of the pure Wilson theory ($\beta_A = 0$) and dive for positive and negative β_A into what was identified⁷⁾ as the strong coupling region. This curve is therefore far from what one would call an "equal physics curve".

Clearly a different approach is needed, or, sticking to the conventional approach one should calculate higher order weak corrections. This latter option was taken by Sharatchandran and Weisz⁸⁾ who calculated part of the two-loop diagrams.

In this paper, the large N method is applied to the weak coupling regime, up to and including two loops (Sections 4 and 5). Like in the strong coupling case, the calculational effort is much less than in the conventional case, partly because we calculate only curves of equal string tension, not the string tension itself.

Apart from this, we discover the penalty to be paid for the consistency of the approach (Section 4), which up to now has seemed to escape attention. In Section 6 we discuss the comparison with the data.

2. - A BRIEF REMINDER OF $N = \infty$ UNIVERSALITY AND A GENERALIZATION

We first recall how the original Makeenko-Polikarpov argument^{2a)} works.

Consider the mixed action with fundamental (Wilson) action and adjoint action:

$$S = \frac{\beta_F}{2} \sum_P (\text{Tr } U(P) + \text{c.c.}) + \beta_A \sum_P |\text{Tr } U(P)|^2. \quad (2.1)$$

The sum is over unoriented plaquettes P ; that is, a sum over all sites n and pairs of directions (μ, ν) with $\mu < \nu$.

Consider now a loop C with no repeated links and form the ordered product of the link variables $U(\ell)$ [$U(\ell) \in SU(N)$] with ℓ on C :

$$\prod_{\ell \in C} U(\ell) = U(C).$$

Then, the average $\langle \text{Tr } U(C)/N \rangle$, as defined in the introduction, obeys the Schwinger-Dyson equations^{2a)}, with respect to a small variation of the link variable $U(\ell_0)$ on some fixed link ℓ_0 on C . We shall omit here the derivation of these equations and recall only their most salient features. In the limit when $N \rightarrow \infty$ the form of the Schwinger-Dyson equations, for a system described by the mixed action (2.1), becomes identical to that of the pure Wilson action ($\beta_A = 0$) but with rescaled fundamental coupling $b_f = \beta_f/N$ replaced by an effective coupling B_0 satisfying

$$\frac{B_0}{2} = \frac{b_f}{2} + \beta_A \left\langle \frac{\text{Tr } U(P)}{N} \right\rangle (B_0). \quad (2.2)$$

This result is derived using the property of factorization of loop averages in the limit $N \rightarrow \infty$, i.e.,

$$\begin{aligned} \left\langle \frac{\text{Tr } U(C_1)}{N} \frac{\text{Tr } U(C_2)}{N} \right\rangle &= \left\langle \frac{\text{Tr } U(C_1)}{N} \right\rangle \left\langle \frac{\text{Tr } U(C_2)}{N} \right\rangle \\ &+ \frac{1}{N^2} \left\langle \text{Tr } U(C_1) \text{Tr } U(C_2) \right\rangle_C \end{aligned} \quad (2.3)$$

with $\langle \text{Tr } U(C_1) \text{Tr } U(C_2) \rangle_C = O(1)$ for any two-loops C_1 and C_2 . Equation (2.3) means that for $N \rightarrow \infty$ loops become uncorrelated and correlation terms in Schwinger-Dyson equations can be dropped. In this limit $\langle \text{Tr } U(C)/N \rangle$ depends only on B_0 , in particular when $C = P$, which is why B_0 appears also in the argument of $\langle \text{Tr } U(P)/N \rangle$ on the right-hand side of Eq. (2.2).

Of course, the string tension [Eq. (1.1)] will stay constant when B_0 is constant. B_0 is constant on straight lines in the (b_f, β_A) plane according to Eq. (2.2) in striking qualitative agreement with computer $SU(2)$ experiment (see Fig. 1). Our main effort is to see how finite N corrections do affect Eq. (2.2) and under what conditions (Sections 3, 4 and 5).

Let us just note an amusing generalization of (2.2) for a large class of actions. Should the action be in terms of the irreducible characters $\chi_\Lambda(U)$ [$d_\Lambda \equiv \chi_\Lambda(1)$ is the dimension of the representation], then:

$$S = \sum_P \sum_\Lambda S(\Lambda) \frac{\text{Re } \chi_\Lambda(U)}{d_\Lambda}. \quad (2.4)$$

The $S(\Lambda)$ are the couplings $\beta_f N, \beta_A(N^2 - 1), \dots$, as Λ runs through the fundamental, adjoint, ..., representations. Obviously, we keep $\beta_f/N, \beta_A, \dots$, [in general $S(\Lambda)/N^2$] fixed, as $N \rightarrow \infty$. Then, using the asymptotic formula:

$$\frac{\chi_\Lambda(u)}{d_\Lambda} = \left\langle \frac{\text{Tr } u}{N} \right\rangle^{\Lambda_+} \left\langle \frac{\text{Tr } u^\dagger}{N} \right\rangle^{\Lambda_-} + O\left(\frac{1}{N}\right)$$

(where $\Lambda_\pm$ are two non-negative integers determined by Λ ⁹⁾) one can go through the same reasoning and find an effective coupling B_0 , satisfying an equation similar to (2.2), i.e., B_0 can be expressed as a polynomial in $\langle \text{Tr } U(P)/N \rangle(B_0)$ with coefficients linear in rescaled couplings $S(\Lambda)/N^2$.

Let us note an important point: at $N = \infty$ the effective coupling is the same, whatever the loop C . If, however, we want to take $1/N^2$ corrections into account, then correlations in the Schwinger-Dyson equations do survive. Thus, one can expect that if these corrections are included as $1/N^2$ corrections to the effective coupling B_0 they will, coming through correlations, depend in general on the observable (loop C) in question. This is what we indeed find in Section 3, where we show that in strong coupling a universal $1/N^2$ corrected form of Eq. (2.2) makes sense only for a limited class of loops, for which $|C|/\Sigma(C)$ goes to zero.

3. - STRONG COUPLING

In strong coupling the calculation of the $1/N^2$ corrections to the effective coupling B_0 is reduced to computing the $1/N^2$ corrections to the loop average:

$$\left\langle \frac{\text{Tr } u(C)}{N} \right\rangle(\beta_f, \beta_A, N^2) = \left\langle \frac{\text{Tr } u(C)}{N} \right\rangle(B_0) + \frac{1}{N^2} \Delta(C). \quad (3.1)$$

The left-hand side of Eq. (3.1) is an average with the mixed action (2.1); the first term on the right-hand side is an average with the pure Wilson action and coupling B_0 [see Eq. (2.2)]. In this first term, there are also $1/N^2$ corrections, due to the pure Wilson action. They are not of interest to us. Rather the $1/N^2$ corrections, due to the fact that β_A in Eq. (2.1) is not zero, are of interest and they are embodied in the second term on the right-hand side of (3.1).

By adding and subtracting in the mixed action (2.1) the term

$$\sum_f \beta_A \left\langle \frac{\text{Tr } u(P)}{N} \right\rangle \frac{\text{Tr } u(P)}{N} + \text{c.c.}$$

we obtain

$$S = N^2 \left\{ \sum_P \frac{B_0}{2} \left(\frac{\text{Tr } U(P)}{N} + \text{c.c.} \right) + \beta_A \sum_P \left| \frac{\text{Tr } U(P)}{N} - \left\langle \frac{\text{Tr } U(P)}{N} \right\rangle \right|^2 - \beta_A \left\langle \frac{\text{Tr } U(P)}{N} \right\rangle^2 \right\} \quad (3.2)$$

where the last term in (3.2) can be dropped in averages.

For notational convenience we will write for the average plaquette:

$$w \equiv \left\langle \frac{\text{Tr } U(P)}{N} \right\rangle.$$

The average is understood to be an average with the Wilson action with coupling B_0 , in order that the coupling B_0 in (3.2) obeys (2.2).

Using Eq. (3.2) in the left-hand side of Eq. (3.1), we find easily:

$$\begin{aligned} \frac{1}{N^2} \Delta(C) = N^2 \beta_A \sum_P \left\langle \left(\frac{\text{Tr } U(C)}{N} - \left\langle \frac{\text{Tr } U(C)}{N} \right\rangle \right) \left| \frac{\text{Tr } U(P)}{N} - w \right|^2 \right\rangle (B_0) \\ + O(\beta_A^2) \end{aligned} \quad (3.3)$$

where all averages are taken with respect to the Wilson action, the first term in Eq. (3.2).

To evaluate $\Delta(C)$ we resort to the familiar character expansion⁸⁾. It is then useful to write the triple correlation (3.3) as:

$$\frac{1}{N^2} \Delta(C) = \frac{\beta_A}{N^2} \sum_P \frac{\partial}{\partial B_{0P}} \frac{\partial}{\partial \bar{B}_{0P}} \left\langle \frac{\text{Tr } U(C)}{N} \right\rangle (\{B_{0P}, \bar{B}_{0P}\}) \Big|_{B_{0P} = \bar{B}_{0P} = B_0} \quad (3.4)$$

Now the average in (3.4) is computed with the action

$$S = N^2 \sum_P \left(\frac{\bar{B}_{0P}}{2} \frac{\text{Tr } U(P)}{N} + \frac{B_{0P}}{2} \frac{\text{Tr } U^\dagger(P)}{N} \right)$$

and the reader can verify easily that (3.3) and (3.4) are identical. From (3.4) we learn that to order β_A/N^2 we only need to calculate diagrams that are of order one for N large and contain at least one plaquette P that carries as

weight $|B_{op}|^2$. This means that the adjoint representation must occur at least once in the graph. However, all graphs with just one adjoint plaquette are cancelled by the corresponding excluded volume graph as $N \rightarrow \infty$. It turns out that the lowest order graph is given by two neighbouring plaquettes carrying the adjoint representation, all the remaining plaquettes carrying the fundamental representation¹⁰⁾ (see Fig. 2). The contribution of this graph is:

$$\frac{1}{N^2} \Delta(C) = \frac{\beta_A}{N^2} 12(\alpha-2) \left(2\Sigma(C) - \frac{|C|}{2}\right) \left(\frac{B_0}{2}\right)^8 \left(\frac{B_0}{2}\right)^{\Sigma(C)} \quad (3.5)$$

$\Sigma(C)$ and $|C|$ are respectively minimal area and perimeter of the loop.

Now we are ready to identify the correction ΔB to the effective coupling B_0 :

$$\frac{B_1}{2} = \frac{B_0}{2} + \Delta B_0 \quad (3.6)$$

from (3.1) with the left-hand side of (3.1) calculated with the action

$$N \frac{B_0}{2} \sum_P (\text{Tr } U(P) + c.c.). \quad (3.7)$$

This gives the relation

$$\Delta B_0 = \frac{1}{N^2} \frac{\Delta(C)}{\sum_P (\langle CP \rangle_c + \langle C\bar{P} \rangle_c)} \quad (3.8)$$

where we used in the denominator of (3.8) the shorthand notation:

$$\sum_P \langle CP \rangle_c = \sum_P N^2 \left\langle \left(\frac{\text{Tr } U(C)}{N} - \left\langle \frac{\text{Tr } U(C)}{N} \right\rangle \right) \left(\frac{\text{Tr } U(P)}{N} - w \right) \right\rangle (B_0).$$

This quantity is to leading order in strong coupling:

$$\sum_P \langle CP \rangle_c + \langle C\bar{P} \rangle_c = \Sigma(C) \left(\frac{B_0}{2}\right)^{\Sigma(C)-1} (1 + O(B_0^4)). \quad (3.9)$$

Combining (3.5), (3.8) and (3.9) we get:

$$\Delta B_0 = \frac{\beta_A}{N^2} \frac{(2\Sigma(C) - \frac{|C|}{2})}{\Sigma(C)} 12 \left(\frac{B_0}{2}\right)^9 (1 + O(B_0^4)). \quad (3.10)$$

As anticipated in the introduction, the result is loop dependent. The string tension is obtained by taking large loops with $|C|/\Sigma(C) \rightarrow 0$. In that limit the effective coupling B_1 obeys the corrected equation (2.2):

$$\begin{aligned} \frac{B_1}{2} = \frac{b_f}{2} + \beta_A \left\langle \frac{\text{Tr } U(P)}{N} \right\rangle (B_1) + 24(d-2) \frac{\beta_A}{N^2} \left(\frac{B_1}{2} \right)^3 (1 + O(B_1^4)) \\ + O\left(\frac{\beta_A^2}{N^2}\right). \end{aligned} \quad (3.11)$$

This is the main result of this section. It is clear that in the region where strong coupling is supposed to be applicable¹¹⁾ (see the table) for $d = 4$ the corrections to the straight line system are negligibly small.

The corrections of order β_A^2/N^2 are even more suppressed by powers of B_1 . In fact, we expect the formula (3.11) to work everywhere in the (b_f, β_A) plane where the straight lines do not intersect and where $B_1/2 \leq (B_{1t}/2)$ as given in the table.

4. - WEAK COUPLING

In the weak coupling régime we will evaluate finite N corrections to the effective coupling B_0 (2.2) with the string tension ρ as observable; ρ is obtained from:

$$\left\langle \frac{\text{Tr } U(C)}{N} \right\rangle = \exp[-\rho \Sigma(C)] \quad (4.1)$$

for large enough loops.

This section will only describe the tactics needed to arrive at these corrections. Details about the actual evaluation of the relevant graphs are found in Section 5.

The string tension ρ will, at $N = \infty$, only depend on B_0 . To obtain the finite N corrections to B_0 we will, like in the preceding section, work with the mixed action as written in (3.2):

$$S = N^2 \left\{ \sum_P \frac{B_0}{2} \left(\frac{\text{Tr } U(P)}{N} + c.c \right) + \beta_A \sum_P \left| \frac{\text{Tr } U(P)}{N} - w \right|^2 \right\}. \quad (4.2)$$

The behaviour of ρ is determined by the way the coupling renormalizes in weak coupling. Therefore, we will ask the question: how does the coupling B_0 renormalize due to the presence of the second term in (4.2) ?

This can be most conveniently answered in the framework of the background field method¹²⁾. This technique introduces a background gauge field $B_\mu(n) \equiv B_\mu^a(n) \lambda_a/2$ in the following way:

$$U_\mu(n) = \exp[i g_0 Q_\mu(n)] \exp[i a B_\mu(n)], \quad (4.3)$$

$$\frac{B_0}{2} = \frac{1}{g_0^2 N},$$

λ_a , $a = 1, \dots, N^2 - 1$ are the $SU(N)$ generators satisfying $\text{tr} \lambda_a \lambda_b = 2\delta_{ab}$. The quantum field Q_μ is integrated out in a loop expansion using background field gauge fixing¹²⁾. We are interested in the local part of the two-point function $\Gamma(B)$, which, as the lattice length a shrinks to zero compared to the length scale set by the background field, takes the form:

$$\Gamma(B) = -N \Gamma S_{\text{cont}}(B) + O(a) \quad (4.4)$$

where

$$S_{\text{cont}}(B) = \frac{1}{4} \int d^4x \sum_{\mu, \nu} \left[f_{\mu\nu}^a(x) \right]^2$$

and $f_{\mu\nu}^a(x)$ is the covariant curl of $B_\mu^a(x)$. The two-point function $\Gamma(B)$ can be represented as a sum of one particle irreducible diagrams with two external background field lines. To define Γ from (4.4), we shall write $\Gamma(B)$ in momentum representation and by fixing the background field momentum $p^2 = M^2$, where M^2 is a subtraction mass, define

$$\Gamma = \sum_{L=0}^{\infty} \Gamma_L(L) \quad (4.5)$$

as a renormalized coupling. $L = \log a^2 M^2$ contains the cut-off dependence of the coupling.

For the action (4.2), every term Γ_L in this expansion consists of two parts: one part W_L due to only the Wilson action in (4.2) and a correction C_L of order $1/N^2$ due to the second term

$$\Gamma_L = \eta^L (W_L + C_L), \quad \Gamma_0 = \frac{B_0}{2}. \quad (4.6)$$

The power of the parameter η gives the number of loops; at the end of the calculation we set $\eta = 1$.

The terms W_L are of the form:

$$W_L = w_L \left(\frac{B_0}{2} \right)^{-L+1}$$

and the coefficients w_ℓ are written as:

$$w_\ell = \Delta_\ell^1 L^\ell + \Delta_\ell^{1-1} L^{\ell-1} + \dots + \Delta_\ell^1 L + F_\ell.$$

For the case $\ell = 1$ the numbers Δ_1^1 and F_1 are known^{5),13),14)}

$$\Delta_1^1 = \frac{\beta_0}{N} = \frac{1}{(4\pi)^2} \frac{11}{3}. \quad (4.8)$$

The two-loop contribution is not yet calculated on the lattice. On the basis of the renormalization group¹⁵⁾ we expect:

$$\begin{aligned} \Delta_2^2 &= 0 \\ \Delta_2^1 &= \frac{\beta_1}{N^2} = \frac{1}{(4\pi)^4} \frac{34}{3}, \end{aligned} \quad (4.9)$$

Δ_2^1 is expected to be independent of the mixture of representations in the action (4.2). We will support this expectation by showing in Section 5 that the correction term C_2 is indeed cut-off independent.

The corrections $C_1, C_2, \dots$, are of the form

$$\begin{aligned} C_1 &= \frac{\beta_A}{N^2} \gamma_1^1 \left(\frac{B_0}{2}\right)^{-1} \\ C_2 &= \frac{\beta_A}{N^2} \gamma_2^1 \left(\frac{B_0}{2}\right)^{-2} + \frac{\beta_A^2}{N^2} \gamma_2^2 \left(\frac{B_0}{2}\right)^{-3} \\ C_3 &= \frac{\beta_A}{N^2} \gamma_3^1 \left(\frac{B_0}{2}\right)^{-3} + \dots \end{aligned} \quad (4.10)$$

as is shown by inspection of the diagrams in Fig. 4. The coefficients γ_ℓ^k are numerical constants, eventually depending on powers of N^{-2} and L , in such a way that as $N \rightarrow \infty$ the γ_ℓ^k become constant.

As in the case of strong coupling [Eq. (3.6) and below] we will now identify the corrections to the effective coupling:

$$\frac{B_1}{2} = \frac{B_0}{2} + \eta \Delta_1 B_0 + \eta^2 \Delta_2 B_0 + \eta^3 \Delta_3 B_0 + \dots \quad (4.11)$$

writing (4.5) as in a pure Wilson theory with the coupling B_1 :

$$\Gamma = \frac{B_1}{2} + \eta W_1 + \eta^2 W_2 + \eta^3 W_3 + \dots \quad (4.12)$$

The effective action in (4.5) must be identical to that in (4.12), and it follows that

$$\frac{B_1}{2} = \frac{B_0}{2} + \eta C_1 + \eta^2 C_2 + \eta^3 (C_3 + W_2 \Delta_1 B_0 (\frac{B_0}{2})^{-1}) + \dots \quad (4.13)$$

or

$$\begin{aligned} \Delta_1 B_0 &= C_1 \\ \Delta_2 B_0 &= C_2 \\ &\dots \end{aligned} \quad (4.14)$$

In order that the corrected effective coupling be sensible, the right-hand side of (4.13) should be independent of the cut-off a . As we show in this paper, this is indeed the case up to two-loop order. For three and more loops, however, cut-off independence in (4.13) implies a definite cut-off dependent structure of C_3, C_4 , etc., related to that of Wilson's theory. In particular, for C_3 the cut-off dependence in γ_3^1 must cancel that of $\gamma_1^1 w_2$ [where we used (4.14) and (4.10)]. Whether this is the case remains to be seen and it may happen that on the three-loop level the concept of effective coupling breaks down. This can be the penalty to be paid for the consistency of the approach.

Combining the assumption of cut-off independence of B_1 with (4.12), we see that $a(d/da)\Gamma|_{B_1} = \beta_W(B_1)$: a very strong statement !

From now on we concentrate on the one- and two-loop corrections, so neglect η^3 terms. Stimulated by the form of (2.2) satisfied by B_0 , we shall rewrite (4.13) in terms of B_1, b_f and β_A

$$\frac{B_1}{2} = \frac{b_f}{2} + \beta_A w(B_1) + C_1(B_1) + C_2'(B_1) \quad (4.15)$$

where $C_2'(B_1)$ differs from $C_2(B_1)$ by the terms coming from replacement of B_0 by B_1 in the argument of $w(B_0)$ and C_1 . These terms cancel respectively the diagrams $C_2^2(3)$ and $C_2^2(4)$ in Fig. 4. In (4.15) the value $\eta = 1$ is restored. The result of the evaluation of the graphs in Fig. 4 (see also Section 5 for details) gives

$$\begin{aligned} C_1 + C_2' &= \\ &= -\frac{\beta_A}{N^2 - 1} (1 - w^2(B_1)) + \frac{\beta_A}{32 N^4} \left(\frac{B_1}{2} \right)^{-2} \end{aligned} \quad (4.16)$$

$$\begin{aligned}
 & - \frac{\beta_A}{48 N^2} \left(1 - \frac{3}{2N^2} \right) \left(\frac{B_1}{2} \right)^{-2} - \frac{1}{64} \frac{N^2+1}{N^4} \beta_A^2 \left(\frac{B_1}{2} \right)^{-3} \\
 & - \frac{\beta_A}{2 N^2} \frac{1}{(4\pi)^2} \left(\frac{B_1}{2} \right)^{-2}.
 \end{aligned} \tag{4.16}$$

cont.

In (4.16), $w(B_1)$ is the mean action evaluated¹⁶⁾ up to and including two loops. The lines corresponding to equal B_1 on the $(b_f \beta_A)$ plane are plotted in Fig. 3. for $N = 2$. The form of $w(B_1)$ was taken from Ref. 16) as

$$w(B_1) = 1 - \frac{N^2-1}{8 N^2} \left(\frac{B_1}{2} \right)^{-1} - \frac{N^2-1}{4 N^2} \left(0.0203 - \frac{1}{32 N^2} \right) \left(\frac{B_1}{2} \right)^{-2}. \tag{4.17}$$

5. - EVALUATION OF THE ONE- AND TWO-LOOP GRAPHS

In this section, we will explain in more detail the calculation of the effective action. In Section 5a), we explain the rules to get all possible graphs. In Section 5b), we show why the corrections up to two loops are indeed finite and in Section 5c), we show how to calculate some typical diagrams.

5a) Our definition of the effective action in the presence of a background field is identical to that in Ref. 17), i.e., it is the Legendre transform of the connected vacuum functional.

To the action (4.2) we have to add the gauge fixing term which to the order we are interested in can be written as

$$S_{g.f.} = - \frac{1}{\alpha} \sum_n \text{Tr} \left[\sum_\mu D_\mu^- Q_\mu(n) \right]^2. \tag{5.1}$$

The gauge parameter α is chosen to equal one. $D_\mu^-(B)$ is the lattice covariant derivative^{13),14)}. The ensuing ghost term is called S_{gh} , and the measure term⁵⁾ S_m .

Using the background field parametrization of the link variable (4.3), the trace of the plaquette variable $U_{\mu\nu}(P)$ at a position described by an integer four vector n is written as

$$\text{Tr} U_{\mu\nu}(P) = \text{Tr} \left[\exp \left[i a^2 f_{\mu\nu}(n) \right] \exp \left[i g_0 F_{\mu\nu}(n) \right] \right]. \tag{5.2}$$

The quantity $f_{\mu\nu}(n)$ is the lattice form of the background field tensor

$$f_{\mu\nu}(n) = \frac{1}{a} (\Delta_\mu^+ B_\nu(n) - \Delta_\nu^+ B_\mu(n)) + i [B_\mu, B_\nu] + O(a) \quad (5.3)$$

which for $a \rightarrow 0$ goes into the continuum version of the curl of the background field. $F_{\mu\nu}(n)$ is the field tensor of the quantum field

$$F_{\mu\nu}(n) = F_{\mu\nu}^{(1)}(n) + g_0 F_{\mu\nu}^{(2)}(n) + g_0^2 F_{\mu\nu}^{(3)}(n) + \dots \quad (5.4)$$

where

$$\begin{aligned} F_{\mu\nu}^{(1)}(n) &= D_\mu^+ Q_\nu(n) - D_\nu^+ Q_\mu(n) \\ F_{\mu\nu}^{(2)}(n) &= i \left\{ [Q_\mu, Q_\nu] - \frac{1}{2} [D_\nu^+ Q_\mu, D_\mu^+ Q_\nu] \right. \\ &\quad \left. - \frac{1}{2} [D_\nu^+ Q_\mu, Q_\mu] + \frac{1}{2} [D_\mu^+ Q_\nu, Q_\nu] \right\}. \end{aligned} \quad (5.5)$$

In (5.5) usual lattice derivatives are replaced by background covariant derivatives^{13),14)}. The term $F_{\mu\nu}^{(3)}$ in (5.4) can be expressed in terms of triple commutators of the quantum fields and their covariant derivatives. We note here that the background field dependence in (5.2) is either by the coupling with $f_{\mu\nu}(n)$, Eq. (5.3), or by the commutator terms resulting from the definition of the background covariant derivative.

The action (4.2) consists of two parts: one containing the Wilson action S_W and the part considered as a perturbation, proportional to β_A . In the graphical expansion the role of w is to subtract certain diagrams. This will be explained in more detail later.

Let us now construct the diagrams in two stages. The first stage is to expand only the quantum field in S . For the Wilson and gauge fixing part, we get

$$S_w + S_{g.f.} = S_w(B) + \frac{1}{2} Q^2 (S_w^{(2)} + S_{g.f.}) + Q^3 S_w^{(3)} + \dots \quad (5.6)$$

where the first term is the Wilson action of the background field. A similar expansion is performed with the perturbation term in (4.2) and with the ghost and measure parts. Upon integration of the quantum fields we get the "mother graphs" of Fig. 4 (left column) where the double line propagators represent the full, background field dependent, propagator $(S_w^{(2)} + S_{g.f.})^{-1}$. The same is the case for the ghost propagator. The vertices $S_w^{(3)}$, $S_w^{(4)}$, etc., are cross hatched circles in Fig. 4. The perturbation part of the action delivers

vertices coming always in complex conjugate pairs. These are represented by two empty semi-circles. Diagrams contributing to C_1 and C_2 [Eq. (4.6)] contain necessarily at least one vertex of this type. This first stage ensures that we will not forget diagrams obtained in the second stage by expanding the double line propagators (see Fig. 5a) and vertices (see Fig. 5b) up to second order in the background fields.

Through this expansion, we obtain many "daughter graphs" from a given "mother graph". Single lines now represent usual gluon (solid lines) or ghost (broken lines) propagators. With the gauge fixing we choose, they are diagonal in colour and direction indices [cf. Refs 14) and 15)] and are exactly the same as in the calculation without the background field. The wavy lines represent the background field. They can end on a vertex without being attached to an outgoing gluon line and are then understood to couple through the linear part of $f_{\mu\nu}^a(n)$ [Eq. (5.3)]. If they are attached to an outgoing gluon line, it is understood that they couple through the commutator of the covariant derivative in $F_{\mu\nu}$ [Eq. (5.4)].

The set of "daughter diagrams" is considerably reduced by understanding the nature of the w subtraction in (4.2) and by colour arguments. $Nw(B_0)$ can be represented as a sum of all connected diagrams (background field independent) which contain $\text{tr } U(P)$ or $\text{tr } \bar{U}(P)$ as a vertex. All the daughter diagrams containing a subdiagram of this form [with one of the semi-circles playing the role of $\text{tr } U(P)$] will thus be subtracted (since they are already contained in the definition of the coupling B_0). Another source of reduction of the "daughter diagrams" comes from colour arguments. As can easily be shown from the diagonal (in colour and direction) form of the gluon propagator, all the subdiagrams shown in Fig. 5c are zero and thus yield the vanishing of all diagrams containing them. This reduces considerably the number of "daughter diagrams" connected with the first of $C_2^1(2)$ mother diagrams of Fig. 4. Similarly, by colour arguments one can show that among the daughter diagrams resulting from $C_2^1(3)$ only those drawn on the figure survive. For the diagrams $C_2^1(3)$ (b) the only cubic term from $S_W^{(3)}$ which couples to the perturbation vertex has the form

$$\frac{\alpha^2 g_0}{3} \sum_{\mu < \nu} T_\nu \left[f_{\mu\nu} (F_{\mu\nu}^{(1)})^3 \right]$$

where in $F_{\mu\nu}^{(1)}$ [Eq. (5.4)], the background field dependence is suppressed.

5b) The reduced set of the "daughter diagrams" can now be grouped into three parts.

1) The part containing diagrams in C_1 , $C_2^1(1)(a)$, $C_2^1(2)(a)$ and $C_2^1(3)(a)$. All these diagrams are characterized by the way they couple to the background field. One background field line in the form of $f_{\mu\nu}$ [Eq. (5.3)] couples to each of two semicircles forming the perturbation vertex. From (5.2), it follows that a sum of all diagrams of this form can be written in a closed form

$$\frac{1}{4} \beta_A a^4 \sum_{\mu < \nu} f_{\mu\nu}^a(n) f_{\mu\nu}^b(n) < \text{Tr } \lambda_a U_{\mu\nu}(n) \text{Tr } \lambda_b U_{\mu\nu}^+(n) > \quad (5.7)$$

where in calculating the average all the background field dependence has to be dropped. The average in (5.7) is of the form $\delta_{ab} X$. With the help of the formula

$$\sum_a (\lambda_a)_{ij} (\lambda_a)_{kl} = 2(\delta_{jk} \delta_{il} - \frac{1}{N} \delta_{ij} \delta_{kl}) \quad (5.7)'$$

X can be related to

$$\begin{aligned} X &= \frac{2N}{N^2-1} \left(1 - \left\langle \frac{\text{Tr } U_{\mu\nu}(n)}{N} \frac{\text{Tr } U_{\mu\nu}^+(n)}{N} \right\rangle \right) \\ &= \frac{2N}{N^2-1} \left(1 - w^2(B_0) - \frac{1}{N^2} \left\langle \text{Tr } U_{\mu\nu}(n) \text{Tr } U_{\mu\nu}^+(n) \right\rangle_c \right) \end{aligned} \quad (5.8)$$

where $w(B_0)$ is the average plaquette action in the Wilson theory. As can be seen (5.7) gives a contribution which is both finite and cut-off independent. Formula (5.8) contains as its part a one-loop contribution (C_1 of Fig. 3) calculated earlier [cf., e.g., Refs. 14) or 18)]. The connected part in (5.8) is easily evaluated on the two-loop level:

$$\frac{1}{N^2} \left\langle \text{Tr } U_{\mu\nu} \text{Tr } U_{\mu\nu}^+ \right\rangle_c = \frac{1}{32N^2} \left(1 - \frac{1}{N^2} \right) \left(\frac{B_0}{2} \right)^{-2}. \quad (5.9)$$

Together with the result for w of Ref. 16), we have the full result for these graphs.

2) The second set of diagrams is drawn as R diagrams in Fig. 4. All of them result from the "mother diagrams" $C_2^1(2)$ and in the $a \rightarrow 0$ limit will contribute to $\Gamma(B)$ a quantity $-NRS_{\text{cont}}(B)$. In all these diagrams the two semicircles forming the β_A vertex are joined by the gluon line not coupled to the background field. The background field lines are attached either both to the same semicircle or one to a semicircle and one to the remaining gluon line, or finally both to the remaining gluon line (coupled either by $f_{\mu\nu}$ or by

covariant derivatives). The set R contains many diagrams which are individually divergent. However, the vertices, where a β_A insertion occurs, gives upon contraction of the internal gluon line an effective vertex, which turns out to be proportional through a factor $-\Delta$ where $\Delta = (1/4N^2)\beta_A(B_0/2)^{-2}$ to the corresponding vertex in $S_W^{(2)}$ (see Figs. 6b and 6c). Obviously, $S_{g.f.}$ cannot be contained in this effective vertex due to the β_A insertion. The β_A insertion in the gluon propagator can be viewed as a result of multiplying by the same factor the background field independent part in $S_W^{(2)}$ (or transverse part of the inverse propagator) (see Fig. 6d).

Therefore, if we rescale the vertices and the transverse part of the kinetic term in S_W

$$Q^2 S_W^{(2)} + Q^2 S_{g.f.} \rightarrow (1 - \Delta) Q^2 S_W^{(2)} + Q^2 S_{g.f.}$$

then, studying the one-loop diagrams in so rescaled Wilson theory, we get the sum of the gluon contribution $W_1^{gl}(\alpha = 1)$ (see Fig. 6a) and the set of diagrams R . On the other hand, we can absorb this rescaling factor in the Q fields, with the result that the gauge fixing parameter α changes from its value 1 to $1 - (1/4N^2)\beta_A(B_0/2)^{-2}$, and we have the equation

$$W_1^{gl}(\alpha = 1) + R = W_1^{gl}(\alpha = 1 - \Delta). \quad (5.10)$$

The ghost contribution to W_1 contains no gauge dependence, which means that the cut-off dependent terms on both sides of Eq. (5.10) must cancel since their coefficient is independent of α . R defined by (5.10) gives directly a contribution to Γ and is thus finite.

To avoid the tedious computation of the gauge dependence of the finite part of W_1 the following argument can be used: this gauge dependence must be the same as the gauge dependence of the finite part of W_1 calculated, say, in the dimensional scheme, because the difference between these finite parts determines the gauge independent quantity Λ_{min}/Λ_L ¹⁴⁾. The gauge dependence in this case was calculated by Abbott¹⁷⁾ and one finds easily

$$R = - \frac{\beta_A}{2N^2} \frac{1}{(4\pi)^2} \left(\frac{B_0}{2} \right)^{-2}. \quad (5.11)$$

3) The insertion diagrams $C_2^2(3)$ and $C_2^2(4)$ replace simply B_n by B_1 in the one loop [proportional to $(B_0/2)^{-1}$] part of $w(B_0)$ and C_1 , respectively. The remaining graphs are $C_2^1(3,b)$, $C_2^2(1)$ and $C_2^2(2)$. In all these diagrams both vertices are linear in $f_{\mu\nu}$ and cubic in $F_{\mu\nu}^{(1)}$. Their contribution to Γ is again cut-off independent.

5c) The only diagrams to be calculated are from the third set of the preceding subsection. They all involve an integral of the type

$$I = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} Y_{12}(p_1) Y_{12}(p_2) Y_{12}(p_1 + p_2) \quad (5.12)$$

where

$$Y_{12}(p) = \frac{|P_1(p)|^2 + |P_2(p)|^2}{\sum_s |P_s(p)|^2}$$

with $P_\mu(p) = e^{ip_\mu} - 1$, and p_μ the lattice momentum. To evaluate I , notice that it does not depend on the pair of directions (12). Write

$$Y_{12}(p) = 1 - Y_{34}(p) \quad (5.13)$$

for $p = p_1, p_2$ and $p_1 + p_2$ in (5.12) to get an equation

$$I = 1 - 3 \int \frac{d^4 p_1}{(2\pi)^4} Y_{34}(p_1) + 3 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} Y_{34}(p_1) Y_{34}(p_2) - I. \quad (5.14)$$

In obtaining (5.14) we changed variables where necessary, making use of the symmetry of (5.12) with respect to the momenta p_1, p_2 and $p_1 + p_2$. Obviously, from (5.14) and directional independence of the integrals we find

$$\int \frac{d^4 p_1}{(2\pi)^4} Y_{34}(p_1) = \frac{1}{2}$$

and, therefore, $I = 1/8$. Numerically one finds $I = 0.125^{(8)}$. The diagrams considered above yield for $a \rightarrow 0$

$$N \left[\frac{\beta_A}{48 N^2} \left(1 - \frac{3}{2N^2} \right) \left(\frac{B_0}{2} \right)^{-2} + \frac{1}{64} \frac{N^2 + 1}{N^4} \beta_A^2 \left(\frac{B_0}{2} \right)^{-3} \right] \times \quad (5.15)$$

$$\times \frac{1}{4} \sum_{\mu, \nu} \int d^4 x (f_{\mu\nu}^a(x))^2$$

where in deriving the N dependent factors use is made of Eq. (5.7)'.

From the discussion in 5b) and 5c) the quoted value for the correction in Eq. (4.16) follows easily.

6. - DISCUSSION

In this paper, we showed that finite N corrections to large N universality only make sense for the string tension if we have constraints on the infinity structure of the theory [Eq. (4.13)]. These constraints were verified in Section 5b up to and including two loops.

Only if these constraints are verified, can we extend the string tension of the Wilson theory (the $\beta_A = 0$ axis) along the effective coupling curves (4.16) into the coupling constant plane, as shown in Fig. 3.

Comparison with the data is reasonably good; the β_A^2 contribution predicts correctly the bending of the data. Unfortunately, data in the weak coupling region are sparse.

It seems that the present expansion converges much faster than the conventional one plotted in Fig. 1. Let us remind the reader that the conventional approach and the one advocated here do coincide for B_1 large enough: both predict straight line equal physics curves with the direction given by $b_F/2 + \beta_A$.

The parametrization of the pure Wilson case should be pushed to terms of order g_0^2 . This necessitates a two- and three loop calculation of the pure Wilson theory (W_2 and W_3).

A few comments are in order. First we would like to emphasize that the universality of β_1 for other than the adjoint action can be shown along the same lines as in section 5b, since it is always a gauge invariant insertion into W_1 that is involved.

Secondly a calculation somewhat similar in spirit to the one presented here has been done in Ref. 18). Their result for the corrections is an expression linear in w the mean plaquette action. This coincides with our result if only the one loop corrections [proportional to $(B_1/2)^{-1}$] are considered. For the two loop, results differ since as we argued in the introduction, corrections necessarily involve correlations.

Thirdly, an exact $1/N^2$ calculation to any order in β_A for $d = 2$ ($d = 2$ is a one plaquette theory so an exact $1/N^2$ answer is possible) shows qualitatively the same features as in four dimensions¹⁹⁾, e.g., a negative β_A^2 correction in weak coupling, and an exact absence of strong coupling corrections. The only qualitative difference is a slower fall off for B_0 large, because in $d = 2$ the $U(1)$ component confines also for weak coupling.

Finally, let us emphasize the importance of finding out whether the constraints (4.13) are indeed verified beyond two loops. For this one needs powerful combinatorial arguments.

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Table 1

Values of $B_{it}/2$ ¹¹⁾ for various N

N	$B_{ot}/2$
2	0.28
3	0.31
4	0.32
5	0.33
6	0.33
∞	0.34 - 0.38

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FIGURE CAPTIONS

- Fig. 1 : Equal string tension data for $SU(2)$ from Ref. 6). The almost straight lines drawn are the result of an 11th order strong coupling expansion⁶⁾. The curved lines are conventional weak coupling predictions. Straight line marked $g_n^2 = \infty$ is the line $b_f + 2\beta_A = 0$.
- Fig. 2 : Lowest order graph in strong coupling expansion contributing to $\Delta(C)$ [Eqs. (3.1)-(3.4) in the main text]. The two shaded plaquettes are carrying the adjoint representation, the others the fundamental representations.
- Fig. 3 : Equal string tension lines [Eq. (4.16)].
- Fig. 4 : Weak coupling graphs contributing to Γ [Eq. (4.4)]. "Mother graphs" are shown in the left column, "daughter graphs" in the right column.
a) one loop graph C_1 ;
b) two loop graphs.
- Fig. 5 : a) Expansion of the full background field dependent propagator into gluon and ghost propagators;
b) expansion of vertices;
c) examples of subdiagrams vanishing by colour arguments.
- Fig. 6 : Cancellation of infinite two loop diagrams
a) graphs contributing to W_1^{g1} ;
b) vertex identity for one background vertex, $\Delta = \frac{1}{4} (\beta_A/N^2)(B_0/2)^{-2}$;
c) vertex identity for two-background line vertex;
d) vertex identity for transverse part π_t of gluon propagator.

The dotted vertices in b) and c) correspond to $S_W^{(2)}$ (no gauge fixing terms included).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

$$(a) \quad \text{====} = \text{---} + \text{---} \begin{array}{c} \text{wavy} \\ \uparrow \end{array} + \text{---} \begin{array}{c} \text{V} \\ \text{wavy} \end{array} + \text{---} \begin{array}{c} \text{wavy} \text{ wavy} \\ \uparrow \uparrow \end{array}$$

$$\text{=====} = \text{-----} + \text{-----} \begin{array}{c} \text{wavy} \\ \uparrow \end{array} + \text{-----} \begin{array}{c} \text{V} \\ \text{wavy} \end{array} + \text{-----} \begin{array}{c} \text{wavy} \text{ wavy} \\ \uparrow \uparrow \end{array}$$

$$(b) \quad \begin{array}{l} \text{D} = \begin{array}{c} \text{wavy} \\ \text{D} \end{array} + \dots \\ \text{D} = \begin{array}{c} \text{D} \\ \text{wavy} \end{array} + \dots \\ \text{D} = \begin{array}{c} \text{D} \\ \text{D} \end{array} + \begin{array}{c} \text{wavy} \text{ D} \end{array} + \begin{array}{c} \text{D} \text{ wavy} \end{array} + \dots \\ \text{D} = \text{---} \text{---} + \text{---} \text{---} \text{wavy} + \dots \end{array}$$

$$(c) \quad \begin{array}{c} \text{D} \text{---} \text{---} \text{---} \end{array} \quad \begin{array}{c} \text{wavy} \\ \text{D} \text{---} \text{---} \end{array} \quad \begin{array}{c} \text{D} \text{---} \text{---} \text{wavy} \end{array}$$

Fig. 5

(a)

(b) $-\Delta x$ = + +

(c) $-\Delta x$ = + +
+ + +

(d) $\Delta x \Pi_t$ =

Fig. 6