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THE WEAK COUPLING PHASE OF LATTICE SPIN AND GAUGE MODELS

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A B S T R A C T

We analyze the lattice weak coupling (w.c.) expansion of $O(N)$, CP^{N-1} and chiral spin models, and of large N reduced chiral and gauge models.

We find that the w.c. expansion always agrees with mean field results, whenever comparable, for arbitrary space-time dimensions, and that the expansion of the reduced models agrees with that of the original ones. However, w.c. results disagree with one-dimensional large N and (old and new) exact results. We explain this phenomenon as a failure of the analytic continuation from higher dimensions that defines lattice w.c. perturbation theory for massless models (even if infra-red singularities always cancel).

We use an improved version of the mean field (m.f.) technique suitable for reduced models. We compute the m.f. approximation of chiral models and use this result to determine the large d (m.f.) behaviour of reduced gauge models, finding agreement with standard Wilson theory results.

We give a new characterization of large N chiral models in terms of the single link integral for the adjoint representation of $SU(N)$.

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INTRODUCTION

The relevance of getting a better understanding of the weak coupling behaviour for lattice models cannot be overstressed.

From a field-theoretical point of view, lattice theory becomes relevant to the description of the continuum theory when we are in the neighbourhood of a critical point. The relevant critical point for physically interesting theories corresponds to $1/\beta \rightarrow 0$, where β is the strong coupling constant (inverse temperature).

The approach to weak coupling lattice models has followed by now a number of different competing routes: we mention (since we are interested in them) the mean field analysis, the $1/N$ expansion and standard perturbation theory. More recently, in the context of the $1/N$ expansion, new techniques have developed in connection with the introduction of reduced models and their quenched and twisted versions.

We decided to compare all the above-mentioned approaches, as far as possible, for a wide number of models [$O(N)$ and CP^{N-1} vector models, chiral models, gauge theories], and their reduced counterparts whenever available, in an arbitrary number of dimensions.

To our surprise we have found that, while there is always agreement between mean field and standard perturbation theory predictions, there exists a lower critical dimension ($d = 1$ for spin systems, $d = 2$ for gauge theories), for which both methods fail to reproduce results that have been obtained in exact, closed form by independent methods. On the other hand, the $1/N$ expansion confirms the exact results, and the same happens when considering the appropriate version of the reduced models.

This intriguing phenomenon is related (but not in an apparent way) to the more and more severe infrared pathologies of perturbation theory for the models considered in a low number of dimensions. Mean field and perturbation theory seem to share the nature of analytic continuations to a domain outside the radius of convergence of a $1/d$ expansion. In presenting our main results here, we follow the order in which this paper is organized.

In Section 1 we consider the perturbative series for the free and internal energy of $O(N)$ and CP^{N-1} vector models. Our computations are carried over to second order in β^{-1} . Infrared singularities are shown to cancel out and finite contributions are obtained.

We compare our results to previously computed mean field and large N results.

In Section 2 we consider one-dimensional versions of the previous models, presenting the known exact solution for $O(N)$ models and evaluating the exact partition function for CP^{N-1} models. We find disagreement with the perturbative results of Section 1 and discuss the origin of this phenomenon.

In Section 3 we develop the perturbative analysis of chiral models in d dimensions, by considering in particular the reduced version of the models, which we show to be equivalent to a single external field integral for the adjoint representation of $SU(N)$.

In Section 4, in order to compare with the results of Section 3 we develop a mean field analysis of the chiral models and we show that the same phenomenology occurs as in the case of vector models. We also compute the one-loop correction to the mean field result for weak and strong coupling. Chiral models in one dimension are briefly reviewed and discussed in Section 5.

In Section 6 we develop the perturbative analysis of large N (reduced) Wilson lattice gauge theories. We compute the free energy to $O(\beta^{-1})$ and show how it compares with known results (in the continuum and for $d = 2$).

In Section 7 we consider the mean field approach to reduced models. Since standard mean field techniques are very ineffective and plagued with zero-mode problems we develop a new and more appropriate (quadratic) mean field analysis, which we apply to the original Eguchi-Kawai (EK) reduced model as a testing ground, finding a strange relationship to $d = 1$ (!) lattice gauge theories.

In Section 8 we extend our mean field analysis to the twisted (TEK) reduced model and connect this problem to that of solving exactly large N d -dimensional chiral models.

In Section 9 we use our mean field solution of chiral models to estimate the mean field of the TEK model, and in the large d limit we recover the standard mean field equations for the Wilson model, thus confirming the overall consistency of our approach.

We leave a number of computations and results to the Appendices. In particular, in Appendix A we study one-dimensional vector models, comparing different formulations of equivalent models. In Appendices B and C we give details of our perturbative expansions for chiral and TEK models. In Appendix D we write down the random field transformed version of reduced models, suitable for an expansion around the mean field that emerges as a saddle point solution. Finally, we devote Appendix E to group integrals, solving exactly the large N Schwinger-Dyson equation for $O(N)$ groups and discussing the external field problem for the adjoint representation of $SU(N)$.

1. PERTURBATION THEORY FOR VECTOR MODELS

In order to obtain a better understanding of the properties of weak coupling perturbation theory on the lattice, we start with the simplest available class of non-trivial models: the $O(N)$ and CP^{N-1} vector models (in d dimensions).

The usual lattice Lagrangian for $O(N)$ vector models is

$$S = -N\beta \sum_{n,\mu} \vec{S}_n \cdot \vec{S}_{n+\mu} \quad N\beta = \frac{1}{g} \quad (1.1)$$

and the partition function takes the form

$$Z_N = \int \prod_n \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}} \delta(\vec{S}_n \cdot \vec{S}_n - 1) d\vec{S}_n \exp -S \quad (1.2)$$

In order to perform a weak coupling expansion we eliminate a (dependent) field component and rescale the remaining ones by the factor $g^{\frac{1}{2}}$:

$$\begin{aligned} S^i &= g^{\frac{1}{2}} \pi^i & 1 \leq i \leq N-1 \\ S^N &= \left(1 - g \sum_{i=1}^{N-1} \pi^i \pi^i \right)^{\frac{1}{2}} \end{aligned} \quad (1.3)$$

Integrating over ds^N we therefore obtain [up to $O(e^{-1/g})$]

$$Z_N = \int \prod_n \frac{\Gamma(\frac{N}{2})}{2 \pi^{\frac{N}{2}}} g^{\frac{N}{2}} \frac{d\vec{\pi}_n}{\sqrt{1-g\vec{\pi}_n^2}} \exp \sum_{n,\mu} [\vec{\pi}_n \cdot \vec{\pi}_{n+\mu} + \frac{1}{g} \sqrt{(1-g\vec{\pi}_n^2)(1-g\vec{\pi}_{n+\mu}^2)}] \quad (1.4)$$

The integration domain can now be extended to infinity, making only a non-perturbatively small error, and we obtain

$$Z_N = \exp \left\{ \left(\frac{d}{g} + \frac{N-1}{2} \ln g N - \frac{N}{2} - \frac{1}{2} \ln 2 \right) V \right\} \cdot Z'_N$$

$$Z'_N = \int \prod_n \frac{d\vec{\pi}_n}{(2\pi)^{\frac{N}{2}}} \exp \sum_{n,\mu} [\vec{\pi}_n \cdot \vec{\pi}_{n+\mu} + \frac{1}{g} \sqrt{(1-g\vec{\pi}_n^2)(1-g\vec{\pi}_{n+\mu}^2)} - \frac{1}{g}] - \frac{1}{2} \sum_n \ln(1-g\vec{\pi}_n^2) \quad (1.5)$$

$$- \frac{N}{2} \cong \ln \frac{\Gamma(\frac{N}{2})}{(\pi N)^{\frac{N}{2}} (\frac{N}{2})^{\frac{N}{2}-1}}$$

(V is the space-time volume and we used Stirling's formula for the Γ function).

The expansion of the action to first order in g is straightforward and leads to the following expression for the free energy

$$F_N = \ln Z_N = \left(\frac{d}{g} + \frac{N-1}{2} \ln g N - \frac{N}{2} - \frac{1}{2} \ln 2 \right) V +$$

$$+ g \int \prod_n \frac{d\vec{\pi}_n}{(2\pi)^{\frac{N}{2}}} \exp - \sum_{n,\mu} \frac{1}{2} (\vec{\pi}_n - \vec{\pi}_{n+\mu})^2 \left[\frac{1}{2} \sum_n \vec{\pi}_n^2 + \frac{1}{4} \sum_{n,\mu} [\vec{\pi}_n^2 \vec{\pi}_{n+\mu}^2 - (\vec{\pi}_n^2)^2] \right] \quad (1.6)$$

In the momentum representation the propagator is

$$\langle \pi^i(p) \pi^j(q) \rangle^{(0)} = \delta_{ij} \delta(p+q) \left[\sum_{\mu} (1-e^{ip_{\mu}})(1-e^{iq_{\mu}}) \right]^{-1}$$

$$= \delta_{ij} \delta(p+q) \frac{1}{4 \sum_{\mu} \sin^2 \frac{p_{\mu}}{2}} \quad (1.7)$$

Integration over momenta is, in principle, plagued by infrared divergences for low space-time dimensions, but singularities do actually cancel and the final result is

$$\frac{F_N}{V} = \frac{d}{g} + \frac{N-1}{2} \ln g N - \frac{N}{2} - \frac{1}{2} \ln 2 - \frac{N-1}{2} \int \frac{d^d p}{(2\pi)^d} \ln 2 \sum_{\mu} (1-\cos p_{\mu}) + \frac{N-1}{8d} g + O(g^2) \quad (1.8)$$

The internal energy per unit link is

$$\langle \vec{S}(0) \vec{S}(\mu) \rangle = -g^2 \frac{\partial}{\partial g} \frac{F_N}{dV} = 1 - \frac{N-1}{2d} g - \frac{N-1}{8d^2} g^2 + O(g^3) \quad (1.9)$$

This expression is in full agreement with the result obtained by Elitzur¹⁾ for $d=2$.

We encounter a very similar situation when we analyze CP^{N-1} models, described by the lattice action²⁾

$$S = -N\beta \sum_{n,\mu} (\bar{z}_{n+\mu} \cdot z_n)(z_{n+\mu} \cdot \bar{z}_n) \quad (1.10)$$

and partition function

$$Z_N = \int \prod_n \frac{\Gamma(N)}{(2\pi i)^N} dz_n d\bar{z}_n \delta(\bar{z}_n z_n - 1) \exp -S \quad (1.11)$$

Again we can eliminate the N -th component of the complex vector field [also fixing the $U(1)$ gauge] by

$$\begin{aligned} z_N = \bar{z}_N &= (1 - g \sum_{i=1}^{N-1} \bar{z}_i z_i) \\ z_i &= g^{1/2} \zeta_i \\ \bar{z}_i &= g^{1/2} \bar{\zeta}_i \end{aligned} \quad 1 \leq i \leq N-1 \quad (1.12)$$

The computation is straightforward and leads to

$$\begin{aligned} Z_N &= \exp \left\{ (N-1) \ln g N + \frac{1}{2} \ln 2\pi N - N \right\} V Z'_N \\ Z'_N &= \int \prod_n \frac{d\zeta_n d\bar{\zeta}_n}{(2\pi i)^{N-1}} \exp \left\{ \sum_{n,\mu} \frac{1}{g} \left| g \bar{\zeta}_{n+\mu} \zeta_n + \sqrt{(1-g \bar{\zeta}_n \zeta_n)(1-g \bar{\zeta}_{n+\mu} \zeta_{n+\mu})} \right|^2 \right\} \approx \\ &\approx \exp \left\{ \frac{dV}{g} - \sum_{n,\mu} (\bar{\zeta}_{n+\mu} - \bar{\zeta}_n)(\zeta_{n+\mu} - \zeta_n) + \frac{g}{2} \left[2(\bar{\zeta}_{n+\mu} \zeta_n)(\bar{\zeta}_n \zeta_{n+\mu}) + 2(\bar{\zeta}_{n+\mu} \zeta_{n+\mu})(\bar{\zeta}_n \zeta_n) \right. \right. \\ &\quad \left. \left. - (\bar{\zeta}_n \zeta_{n+\mu} + \bar{\zeta}_{n+\mu} \zeta_n)(\bar{\zeta}_{n+\mu} \zeta_{n+\mu} + \bar{\zeta}_n \zeta_n) \right] - \frac{g^2}{8} \left[(\bar{\zeta}_{n+\mu} \zeta_{n+\mu} - \bar{\zeta}_n \zeta_n)^2 (\bar{\zeta}_n \zeta_{n+\mu} - \bar{\zeta}_{n+\mu} \zeta_n) \right] + O(g^3) \right\} \end{aligned} \quad (1.13)$$

and therefore

$$F_N = \left[\frac{d}{g} + (N-1) \ln g N - N + \frac{1}{2} \ln 2\pi N \right] V + \\ + \frac{g}{4} \langle (\bar{\zeta}_{n\mu} \zeta_n)(\bar{\zeta}_n \zeta_{n\mu}) + (\bar{\zeta}_{n\mu} \zeta_{n\mu})(\bar{\zeta}_n \zeta_n) - \frac{1}{2} (\bar{\zeta}_n \zeta_{n\mu} + \bar{\zeta}_{n\mu} \zeta_n) \times \\ \times (\bar{\zeta}_{n\mu} \zeta_{n\mu} + \bar{\zeta}_n \zeta_n) \rangle^{(0)} \quad (1.14)$$

One can now use the momentum representation propagators

$$\langle \bar{\zeta}^i(p) \zeta^j(q) \rangle^{(0)} = \delta(p-q) \delta_{ij} \frac{1}{\left[4 \sum_{\mu} \sin^2 \frac{p_{\mu}}{2} \right]^{-1}} \quad (1.15)$$

to obtain (in first order in g):

$$\frac{F_N}{V} = \frac{d}{g} + (N-1) \ln g N + \frac{1}{2} \ln 2\pi N - N - (N-1) \left\{ \frac{d}{(2\pi)^d} \ln 2 \sum_{\mu} (1 - \cos p_{\mu}) \right\} + \frac{N(N-1)}{4d} g \\ + O(g^2) \\ - N \cong \ln \frac{\Gamma(N)}{N^{N-1} \sqrt{2\pi N}} \quad (1.16)$$

The internal energy per unit link is

$$\langle (\bar{z}(0) z(\mu)) (z(0) \bar{z}(\mu)) \rangle = 1 - \frac{N-1}{d} g - \frac{N(N-1)}{4d^2} g^2 + O(g^3) \quad (1.17)$$

in full agreement with the computation given by Di Vecchia et al.²⁾ for the case $d = 2$.

When we consider the large N limit of these expressions, we find that in the $O(N)$ case the series terminates and

$$f_{O(N)} = \lim_{N \rightarrow \infty} \frac{F_N}{NV} = \beta d - \frac{1}{2} \ln 2\beta d - \frac{1}{2} \quad (1.18)$$

while in the CP^{N-1} case

$$f_{CP^{N-1}} = \lim_{N \rightarrow \infty} \frac{F_N}{NV} \cong \beta d - \ln 2\beta d - 1 + \frac{1}{4\beta d} + O(\beta^{-2}) \quad (1.19)$$

A number of results are now available concerning the properties of vector models in the mean field (large d) and large N regimes. It is especially interesting to compare these results with our perturbative computations.

It has actually been found³⁾ that for all dimensions $d \geq 2$ and in the large N limit, the mean field result plus one loop correction agree exactly with the perturbative part of the exact (large N) result, while the non-perturbative part is completely lost.

In general we have³⁾

$$\begin{aligned} f_{O(N)}(\text{mean field}) &\xrightarrow{N \rightarrow \infty} f_{O(N)}(\text{perturbative}) = \\ &= \beta d - \frac{1}{2} \ln 2\beta d - \frac{1}{2} - \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{1}{d} \sum_{\mu} \cos p_{\mu} \right) \end{aligned} \quad (1.20)$$

$$\begin{aligned} f_{CP^{N-1}}(\text{mean field}) &= \frac{N-1}{N} \left[\frac{1}{2} \left(\beta d + \beta d \sqrt{1 - \frac{2}{\beta d}} - 1 \right) - \ln \beta d - \ln \left(1 + \sqrt{1 - \frac{2}{\beta d}} \right) \right] \\ &- \frac{N-1}{N} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{1}{d} \sum_{\mu} \cos p_{\mu} \right) + \frac{\beta d - 1}{N} - \frac{N-2}{2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{2\beta}{N d^2} \sum_{\mu} \cos p_{\mu} \right) \\ &- \frac{1}{2N} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{2\beta}{d^2} \sum_{\mu} \cos p_{\mu} \right) \xrightarrow{N \rightarrow \infty} f_{CP^{N-1}}(\text{perturbative}) \\ &x = \beta d \left(1 + \sqrt{1 - \frac{2}{\beta d}} \right) \end{aligned} \quad (1.21)$$

We may immediately compare these expressions to our perturbative ones, and find complete agreement, at the order to which the expansion has been performed, between the two classes of results (for all dimensions).

2. VECTOR MODELS IN ONE DIMENSION

It is known³⁾ that the relationship between mean field results and the exact large N result breaks down when $d = 1$. We are therefore led to a more thorough investigation of one-dimensional vector models.

Actually the partition function for one-dimensional $O(N)$ models was found many years ago by Stanley⁴⁾, who realized that in the case of an L link open chain

$$Z_N^{(L)} = \int \prod_{n=1}^{L+1} d\bar{S}_n \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} \delta(\bar{S}_n \bar{S}_{n-1}) \exp N\beta \sum_{n=1}^L \bar{S}_n \bar{S}_{n+1}$$

$$= [Z_N^{(1)}]^L \quad (2.1)$$

where $Z_N^{(1)}$ is the single link partition function

$$Z_N^{(1)} = \int \frac{d\bar{S}_1}{\pi^{N/2}} \frac{d\bar{S}_2}{\pi^{N/2}} \delta(\bar{S}_1 \bar{S}_1 - 1) \delta(\bar{S}_2 \bar{S}_2 - 1) \left[\Gamma(\frac{N}{2}) \right]^2 \exp N\beta \bar{S}_1 \bar{S}_2$$

$$= \frac{\Gamma(\frac{N}{2})}{\left(\frac{N\beta}{2}\right)^{\frac{N}{2}-1}} I_{\frac{N}{2}-1}(N\beta) \quad (2.2)$$

Therefore by taking the infinite volume limit one finds that

$$\mathcal{F}_N = \frac{1}{N} \ln \frac{\Gamma(\frac{N}{2})}{\left(\frac{N\beta}{2}\right)^{\frac{N}{2}-1}} I_{\frac{N}{2}-1}(N\beta) \quad (2.3)$$

The large N limit of this expression is³⁾

$$\mathcal{F}_N = \frac{1}{2} \sqrt{1+4\beta^2} - \frac{1}{2} - \frac{1}{2} \ln \left(\frac{1}{2} + \frac{1}{2} \sqrt{1+4\beta^2} \right) +$$

$$+ \frac{1}{N} \left[\ln \left(\frac{1}{2} + \frac{1}{2} \sqrt{1+4\beta^2} \right) - \frac{1}{2} \ln \sqrt{1+4\beta^2} \right] + O\left(\frac{1}{N^2}\right) \quad (2.4)$$

and we have explicitly checked that not only the leading term³⁾, but also the $(1/N)$ quadratic fluctuations around the saddle point in large N one-dimensional $O(N)$ models agree with Eq. (2.4).

The weak coupling expansion of Eq. (2.2) leads to

$$\frac{\ln Z_N}{V} = \frac{1}{g} + \frac{N-1}{2} \ln g - \frac{(N-1)(N-3)}{8} g + O(g^2) \quad (2.5)$$

and we notice that there is no agreement with our perturbative formulas Eqs. (1.3) except in the Abelian cases $N = 1$ and $N = 2$.

In particular the large N limit, as we have seen, now has a β^{-1} series that does not terminate, in contrast with what happened for $N \rightarrow \infty$ (and mean field) when $d \geq 2$.

The case of one-dimensional CP^{N-1} models has not yet been considered in the literature. We found out that the partition function can be determined exactly in this case also, by applying a method rather similar to that applied by Stanley to $O(N)$ models. It is crucial to observe that when performing the single-link integral

$$\int \frac{dz_2 d\bar{z}_2}{(2\pi i)^N} \Gamma(N) \delta(\bar{z}_2 z_2 - 1) \exp N\beta (\bar{z}_2 z_1)(\bar{z}_1 z_2) \quad (2.6)$$

the result of the z_2 integration does not depend on the orientation of z_1 , but only on its modulus, which is set equal to 1. Therefore, we can recursively integrate an L link open chain

$$\begin{aligned} Z_N^{(L)} &= \int \prod_{n=1}^{L+1} \frac{dz_n d\bar{z}_n}{(2\pi i)^N} \Gamma(N) \delta(\bar{z}_n z_{n-1} - 1) \exp N\beta \sum_{n=1}^L (\bar{z}_n z_{n+1})(z_n \bar{z}_{n+1}) = \\ &= [Z_N^{(1)}]^L \end{aligned} \quad (2.7)$$

and we know that³⁾

$$Z_N^{(1)} = \Gamma(N) \left[\frac{\exp N\beta}{(N\beta)^{N-1}} - \sum_{n=0}^{N-2} \frac{(N\beta)^{n-N+1}}{n!} \right] \quad (2.8)$$

In particular in the large N limit³⁾

$$\ln f_\infty = \beta^{-1} - \frac{N-1}{N} \ln \beta + \frac{1}{2N} \ln 2\pi N + O(e^{-N}) \quad (2.9)$$

and in weak coupling

$$\frac{\ln Z_N}{V} = \frac{1}{g} + (N-1) \ln g + O(e^{-\frac{1}{g}}) \quad (2.10)$$

The series are now truncated both in the β^{-1} and in the $1/N$ expansion.

Again, the exact result is now in disagreement with our previous computation, except in the trivial case $N = 1$. In particular, in the large N limit the perturbative series does not terminate in contrast with the exact result.

We meet here the phenomenon of breakdown of the perturbative series for $d = 1$, which will be confirmed by our subsequent analysis of chiral models.

We observe that, while not converging to the exact result, the perturbative series still agrees with the mean field result, and we interpret this phenomenon in the following way: the perturbative series for these potentially infrared divergent models has the meaning of an analytic continuation from higher dimensions. Therefore it necessarily agrees with mean field results, which have analyticity properties in the (inverse) dimension parameter, while it ceases to converge to the exact result when (for sufficiently low dimensions) one goes out of the domain of convergence of the $1/d$ expansion.

3. PERTURBATION THEORY FOR CHIRAL MODELS

In the same spirit as in the previous section, and for the purpose of completeness, we will study the weak coupling perturbation theory for $U(N)$ and $SU(N)$ chiral models on the lattice.

The usual partition function for these models takes the form

$$Z(\beta) = \int \prod_n dU_n \exp -S \quad (3.1)$$

$$S = -N\beta \sum_{n,\mu} \text{Tr} (U_n U_{n+\mu}^\dagger + U_{n+\mu} U_n^\dagger) \quad \mu = 1, \dots, d \quad (3.2)$$

where U_n denote matrices in $SU(N)$ [or $U(N)$]; dU_n is the Haar measure on this group and, as before, $N\beta = 1/g$.

In order to obtain the weak coupling expansion of these models; we have to expand the matrices U_n around the identity.

$$U_n = 1 + i u_n - \frac{u_n^2}{2} + \dots, \quad u_n = u_n^\dagger \quad (3.3)$$

While the substitution of (3.3) in the action (3.2) leads to the following expansion

$$S(\beta) = -2N\beta \sum_{n,\mu} \text{Tr} \left[1 - \frac{1}{2} (u_n - u_{n+\mu})^2 + \frac{1}{4!} (u_{n+\mu}^4 - 4u_n u_{n+\mu}^3 + 6u_n^2 u_{n+\mu}^2 - 4u_n^3 u_{n+\mu} + u_n^4) + \dots \right] \quad (3.4)$$

the Haar measure becomes an integration over the u_n 's up to a Jacobian which requires further explanation.

Only the $SU(N)$ part of the group is affected by the Jacobian; from the literature⁵⁾, we know that if U belongs to $SU(N)$

$$U \equiv \exp i u = \exp i \lambda^a u_a \quad \lambda^a \equiv \text{fundamental rep.} \quad (3.5)$$

then

$$dU = \frac{2\pi}{\sqrt{N}} K \prod_a du_a \det \frac{\sin \Omega^a u_a}{\Omega^a u_a} \quad \Omega^a \equiv \text{adjoint rep.} \quad (3.6)$$

$$\cong \frac{2\pi}{\sqrt{N}} K \prod_a du_a \exp -\frac{1}{24} \text{Tr} (\Omega^a u_a)^2 - \frac{1}{2880} \text{Tr} (\Omega^a u_a)^4 + \dots$$

$$\text{where } K = \prod_{i=1}^N \frac{\Gamma(i)}{(2\pi)^i}.$$

A relation between traces of matrices in the adjoint and fundamental representations of $SU(N)$ exists due to the identity:

$$\begin{aligned} \text{Tr } U_A &= \text{Tr } U_F \text{Tr } U_F^\dagger - 1 \\ \text{Tr } (\Omega^a u_a)^n &= \sum_{p=0}^n (-1)^p \binom{n}{p} \text{Tr } (\lambda^a u_a)^p \text{Tr } (\lambda^a u_a)^{n-p} \end{aligned} \quad (3.7)$$

$$\begin{aligned} \text{if } U_A &\equiv \exp i \Omega^a u_a \\ U_F &\equiv \exp i \lambda^a u_a \end{aligned}$$

Using these relations, it is easy to obtain the Jacobian (3.6) to the required order. Throughout the paper, we will need only the first two orders that we now write in detail:

$$dU_{SU(N)} \equiv \frac{2\pi}{\sqrt{N}} K \prod_a du_a \exp -\frac{N}{12} \text{Tr } u^2 - \frac{1}{2880} (2N \text{Tr } u^4 + 6 \text{Tr } u^2 \text{Tr } u^2) + \dots \quad (3.8)$$

After a rescaling of the matrices u_n by a factor $\sqrt{2N\beta}$, one easily obtains the expansion of the partition function $Z(\beta)$ in terms of the coupling constant $g (= 1/\beta N)$.

$$\begin{aligned} Z(\beta) &= [C(N)]^V \exp 2N^2 \beta d \quad (2\beta)^{-\frac{\eta(N)V}{2}} Z' \\ Z' &= \int \prod_n da_n \exp -\frac{1}{2} \sum_{n,\mu} \text{Tr } (a_n - a_{n+\mu})^2 \times \\ &\times \left\{ 1 + \frac{1}{N\beta} \left[-\frac{N}{24} \text{Tr } (a_n - \text{Tr } a_n)^2 + \frac{1}{48} \text{Tr } (a_n^4 - 4a_n^3 a_{n+\mu} + 6a_n^2 a_{n+\mu}^2 - 4a_n a_{n+\mu}^3 + a_{n+\mu}^4) \right. \right. \\ &\quad \left. \left. + O\left(\frac{1}{(N\beta)^2}\right) \right] \right\} \end{aligned} \quad (3.9)$$

where $a_n = \sqrt{2N\beta} u_n$

$$C(N) = \begin{cases} \prod_{k=1}^N \frac{\Gamma(k)}{(2\pi)^{N/2}} \frac{N^2}{N^2} = C_0 & \text{for } U(N) \\ (2\pi)^{N/2} C_0 & \text{for } SU(N) \end{cases}, \quad \eta(N) = \begin{cases} N^2 & \text{for } U(N) \\ N^2 - 1 & \text{for } SU(N) \end{cases}$$

and V is the volume of the lattice.

In Eq. (3.9) the coefficients are computed by Gaussian integration. The diagonalization of the quadratic part is achieved by using the momentum representation of the $a_{n,s}$; it leads to the propagator below

$$a_n^{ij} = \int \frac{d^d p}{(2\pi)^d} e^{i n p} a(p)^{ij} \quad (3.10)$$

$$\begin{aligned} \langle a_{ij}(p) a_{kl}(q) \rangle^{(0)} &= \left(4 \sum_{\mu} \sin^2 p_{\mu} \right)^{-1} \delta(p+q) \delta_{il} \delta_{jk} \quad \text{in } U(N) \\ &= \left(4 \sum_{\mu} \sin^2 p_{\mu} \right)^{-1} \delta(p+q) \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \quad \text{in } SU(N) \end{aligned} \quad (3.11)$$

Use of the standard Wick contractions technique leads to the following result

$$\begin{aligned} \frac{\ln Z(\beta)}{V} &= 2N^2 \beta d - \frac{N^2}{2} \ln 2\beta + \ln C_0 - \frac{N^2}{2} \int \frac{d^d p}{(2\pi)^d} \ln 2 \sum_{\mu} (1 - \cos p_{\mu}) + \frac{N^2 + 1}{32d\beta} + \dots \quad \text{in } U(N) \\ &= 2N^2 \beta d - \frac{N^2}{2} \ln 2\beta + \ln C_0 + \frac{1}{2} \ln 2\pi - \frac{N^2}{2} \int \frac{d^d p}{(2\pi)^d} \ln 2 \sum_{\mu} (1 - \cos p_{\mu}) + \frac{(N^2 - 1)(N^2 - 2)}{32N^2 d \beta} + \dots \\ &\quad \text{in } SU(N) \end{aligned} \quad (3.12a)$$

$$\lim_{N \rightarrow \infty} \ln C_0 = -\frac{3}{4} N^2 - \frac{1}{12} \ln N + \frac{1}{12} - A \quad (3.12b)$$

As in the vector models, the infrared singularities cancel^{1,6)}

For reasons which will become apparent later, it is important to study the reduced version of the chiral model. To this end, let us consider the following partition function $Z_R(\beta)$

$$Z_R(\beta) = \int dU \exp -S_R(\beta) \quad (3.13)$$

$$S_R(\beta) = -N\beta \sum_{\mu=1}^d \text{Tr} (U \gamma_{\mu} U^{\dagger} \gamma_{\mu}^{\dagger} + U \gamma_{\mu}^{\dagger} U^{\dagger} \gamma_{\mu}) \quad (3.14)$$

Here, only one site variable, U , appears in the action, like in the Eguchi-Kawai⁷⁾ scheme of reduction of lattice gauge theory; on the other hand the γ_μ matrices play the role of translation operators [i.e. compare with (3.2)]

$$U_{n+\mu} = \gamma_\mu U_n \gamma_\mu^\dagger \quad (3.15)$$

like the twisting matrices introduced in Refs. 8 and 9.

The model (3.13) has been studied by Bars et al.¹⁰⁾ with γ_μ 's replaced by arbitrary (quenched) diagonal matrices, and by Eguchi and Nakayama¹¹⁾ who used γ_1 and γ_2 as twisting operators in a two-dimensional lattice.

In order to consider the reduced chiral model (3.13) in d dimensions, we will demand that the γ_μ matrices in (3.14) obey the following property.

Given an integer L and an even integer d ; we require

$$(i) \quad \gamma_\mu \in SU(N) \quad N = L^{\frac{d}{2}} \quad (3.16)$$

$$(ii) \quad \forall \mu < \nu \quad \gamma_\mu \gamma_\nu = \exp \frac{2\pi i}{L} \gamma_\nu \gamma_\mu \quad (3.17)$$

The matrices γ_μ thus constrained are straightforward generalizations to even space-time dimensions of the matrices of Refs. 8 and 9 (four-dimensions) and of Ref. 11 (two-dimensions).

A direct consequence of (3.16), (3.17) is that the periodicity of the γ_μ 's is $L = N^{2/d}$. Then, one can define N^2 matrices $G(q)$ by⁸⁾

$$G(q) = \gamma_1^{k_1} \cdots \gamma_d^{k_d} \quad G(q) \in SU(N) \\ k_\mu = 0, 1, \dots, L-1 \quad (3.18)$$

$$\text{where} \quad k_\mu = n_{\mu\nu} q_\nu \quad n_{\mu\nu} = -n_{\nu\mu} = \begin{cases} 1 & \text{if } \mu < \nu \\ 0 & \text{if } \mu = \nu \end{cases}$$

The matrices $G(q)$ satisfy orthogonality relations

$$\text{Tr } G(p) G(q) = N \delta(p+q) \exp \frac{2\pi i}{L} \langle k | l \rangle \quad (3.19)$$

$$k = np \quad l = nq$$

$$\langle k | l \rangle = \sum_{\mu < \nu} k_{\mu} l_{\nu} \quad (3.20)$$

constituting an orthogonal basis for the algebra of $U(N)$.

We will use extensively the basis $G(q)$ in the following. Even if we know little about the existence of the γ_{μ} 's in general; we know they exist for $d = 2$ and 4 , and we can use them formally for other d 's.

The existence of the basis $G(q)$ is very useful for reduced models in general; indeed, expanding any one-site variable a in terms of the $G(q)$'s

$$a = \sum_q a(q) G(q) \quad (3.21)$$

provides a "momentum decomposition" of a , despite the absence of its space-time dependence. In other words: in reduced models where, by definition, the lattice has only one site, the set of group indexes k_{μ} [see Eq. (3.18)] operates in the derivation of the free propagator, as do the momentum variables in an ordinary lattice or continuum field theory.

Before going further, let us also stress that the partition function (3.13) can be interpreted as a single link integral in the adjoint representation of $SU(N)$ and with a specific source depending on the γ_{μ} 's. This point is discussed extensively in Appendix D.

The weak coupling expansion of the reduced model (3.13) is very similar to that of the full model (3.1). We expand U around the identity ($U = e^{iu}$); the rescaling now appears to be

$$u = \frac{a}{2\sqrt{N}\beta} \quad (3.22)$$

The perturbative series of the action is, for the first few orders

$$N\beta \sum_{\mu} \text{Tr} (U \gamma_{\mu} U^{\dagger} \gamma_{\mu}^{\dagger} + U \gamma_{\mu}^{\dagger} U^{\dagger} \gamma_{\mu}) = \quad (3.23)$$

$$= 2N^2 d\beta - \frac{1}{2} \sum_{\mu} \text{Tr} (a^2 - a \gamma_{\mu} a \gamma_{\mu}^{\dagger}) + \frac{1}{4!} N\beta \frac{S_4}{2^3} - \frac{1}{6! (N\beta)^2} \frac{S_6}{2^5} + \dots$$

$$S_4 = \sum_{\mu} \text{Tr} (2a^4 - 4a^3 (\gamma_{\mu} a \gamma_{\mu}^{\dagger} + \gamma_{\mu}^{\dagger} a \gamma_{\mu}) + 6a^2 \gamma_{\mu} a^2 \gamma_{\mu}^{\dagger}) \quad (3.24)$$

$$S_6 = \sum_{\mu} \text{Tr} (2a^6 - 6a^5 (\gamma_{\mu} a \gamma_{\mu}^{\dagger} + \gamma_{\mu}^{\dagger} a \gamma_{\mu}) + 15a^4 (\gamma_{\mu} a^2 \gamma_{\mu}^{\dagger} + \gamma_{\mu}^{\dagger} a^2 \gamma_{\mu}) - 20a^3 \gamma_{\mu} a^3 \gamma_{\mu}^{\dagger}) \quad (3.25)$$

and the partition function, including the correction due to the measure, is equal to

$$Z(\beta) = e^{2N^2 d\beta} (2\beta)^{\frac{1-N^2}{2}} (2\pi)^{\frac{1}{2}C_V} \int da \exp -\frac{1}{2} \sum_{\mu} \text{Tr} (a^2 - a \gamma_{\mu} a \gamma_{\mu}^{\dagger}) \times$$

$$\times \left\{ 1 + \frac{1}{8N\beta} \frac{1}{4!} (S_4 - 4N \text{Tr} a^2) - \frac{1}{32(N\beta)^2} \frac{1}{6!} (S_6 + N \text{Tr} a^4 + 3(\text{Tr} a^2)^2) + \dots \right\} \quad (3.26)$$

The computation of this integral requires the diagonalization of the quadratic term in the exponential; this is achieved by expanding the matrix a in terms of the basis elements $G(q)$ [see Eq. (3.18)].

As we have already stressed, this basis has many interesting properties, the most important ones being listed in Appendix B. Using the propagator (B5) and standard Wick's contractions, the integration of S_4 and S_6 [see (3.24), (3.25)] are straightforward. Here is the result of the one-loop contribution to the free energy in the large N limit

$$F(\beta) = \ln Z(\beta) = 2N^2\beta d - \frac{N^2}{2} \ln 2\beta - \frac{3}{4}N^2 - \frac{1}{12} \ln N + \frac{1}{2} \ln 2\pi + \frac{1}{12} - A \quad (3.27)$$

$$- \frac{N^2}{2} \int \frac{d^d p}{(2\pi)^d} \ln 2 \sum_{\mu} (1 - \cos p_{\mu}) + \frac{N^2}{32\beta d} (1 + O(\frac{1}{N^2})) + O(\beta^{-2})$$

The complete coefficient (including the $1/N$ correction) is quoted in Appendix B, together with a result of the two-loop calculation.

Comparison between the reduced (3.27) and the non-reduced (3.12) free energy exhibits a full agreement in the large N limit (apart from the trivial volume factor V).

4. MEAN FIELD ANALYSIS OF THE CHIRAL MODELS

The application of the mean field technique to chiral models is straightforward. We change constrained variables into unconstrained ones by applying the random field transform¹²⁾

$$Z_N = \int dU_n \exp N\beta \sum_{n,\mu} \text{Tr} (U_n U_{n+\mu}^\dagger + U_{n+\mu} U_n^\dagger) =$$

$$= \int dV_n dA_n \exp N\beta \sum_{n,\mu} \text{Tr} (V_n V_{n+\mu}^\dagger + V_{n+\mu} V_n^\dagger) - N \sum_n \text{Tr} (A_n V_n^\dagger + V_n A_n^\dagger) \times \quad (4.1)$$

$$\times \int dU_n \exp N \sum_n \text{Tr} (A_n U_n^\dagger + U_n A_n^\dagger)$$

The integration over U_n is the now standard single link integral computed by Brower and Nauenberg¹³⁾ and Brezin and Gross¹⁴⁾ in the large N limit, and by Brower, Rossi and Tan¹⁵⁾ for all N . For our purposes we only need its large N weak coupling behaviour:

$$\frac{1}{N} \ln \int dU_n \exp N \sum_n \text{Tr} (A_n U_n^\dagger + U_n A_n^\dagger) =$$

$$= 2 \sum_{i=1}^N \lambda_i^{1/2} - \frac{1}{2N} \sum_{i,j} \ln (\lambda_i + \lambda_j^{1/2}) - \frac{3}{4} N = w(A_n A_n^\dagger) \quad (4.2)$$

where λ_i are the eigenvalues of $A_n A_n^\dagger$.

The application of saddle point techniques, and the ansatz that v_n^{saddle} and A_n^{saddle} are translation-invariant and proportional to the identity, lead to maximizing the effective action

$$-\frac{S}{VN^2} = \beta (2dv^2) - (2av) + \left[2a - \frac{1}{2} \ln 2a - \frac{3}{4} \right] \quad (4.3)$$

The saddle point equations are

$$\begin{aligned} a &= 2\beta dv \\ v &= 1 - \frac{1}{4a} \end{aligned} \quad (4.4)$$

and they are solved by

$$\begin{aligned} \bar{v} &= \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{2\beta d}} \\ \bar{a} &= \beta d \left(1 + \sqrt{1 - \frac{1}{2\beta d}} \right) \end{aligned} \quad (4.5)$$

Direct substitution gives the free energy

$$\frac{F}{VN^2} = \bar{a} - \frac{1}{2} \ln 2\bar{a} - \frac{1}{2} \quad (4.6)$$

and the internal energy per link

$$\frac{1}{2d} \frac{\partial}{\partial \beta} \left(\frac{F}{VN^2} \right) = \bar{v}^2 \quad (4.7)$$

The strong coupling solution is trivial: $v = a = 0$ and there is a critical point at

$$\beta d = \frac{1}{2} \quad \bar{V}_c = \frac{1}{2} \quad \bar{\alpha}_c = \frac{1}{2} \quad (4.8)$$

Mean field therefore predicts a first order transition¹⁶⁻¹⁸).

We could also compute the quadratic fluctuations around the mean field saddle point, by performing the Gaussian integral

$$\int d\xi_n d\gamma_n \exp \left\{ \beta \sum_{n\mu} \text{Tr} (\xi_n \xi_{n+\mu}^\dagger + \xi_{n+\mu} \xi_n^\dagger) - \sum_n \text{Tr} (\gamma_n \xi_n^\dagger + \xi_n \gamma_n^\dagger) \right. \\ \left. + \sum_n \text{Tr} \left[\frac{1}{2} \gamma_n \frac{\partial^2 W}{\partial A_n^2} \gamma_n + \gamma_n^\dagger \frac{\partial^2 W}{\partial A_n^\dagger \partial A_n} \gamma_n + \frac{1}{2} \gamma_n^\dagger \frac{\partial^2 W}{\partial A_n^{\dagger 2}} \gamma_n^\dagger \right] \right\} \quad (4.9)$$

After a number of formal manipulations, the effective action appearing in the exponent of the integrand reduces to:

$$\beta \sum_{n\mu} \text{Tr} (\xi_n \xi_{n+\mu}^\dagger + \xi_{n+\mu} \xi_n^\dagger) - \sum_n \text{Tr} (\gamma_n \xi_n^\dagger + \xi_n \gamma_n^\dagger) + \\ + \sum_n \sum_{k,k'} (\gamma_n^\dagger)^{k'k} \frac{\partial W}{\partial \lambda_k} (\gamma_n)^{kk'} + \frac{1}{2} \sum_n \sum_{k,k'} (\gamma_n^{kk} + \gamma_n^{\dagger kk}) \lambda_k^{\frac{1}{2}} \frac{\partial^2 W}{\partial \lambda_k \partial \lambda_{k'}} \lambda_{k'}^{\frac{1}{2}} (\gamma_n^{k'k'} + \gamma_n^{\dagger k'k'}) \\ + \frac{1}{2} \sum_n \sum_{k \neq k'} \left(\gamma_n^{k'k} \lambda_k^{\frac{1}{2}} + \gamma_n^{\dagger k'k} \lambda_{k'}^{\frac{1}{2}} \right) \frac{1}{\lambda_k - \lambda_{k'}} \left(\frac{\partial W}{\partial \lambda_k} - \frac{\partial W}{\partial \lambda_{k'}} \right) \times \\ \times \left(\gamma_n^{kk'} \lambda_{k'}^{\frac{1}{2}} + \gamma_n^{\dagger kk'} \lambda_k^{\frac{1}{2}} \right) \quad (4.10)$$

We can now use the fact that at the saddle point

$$\lambda_k^{\frac{1}{2}} = a \quad \frac{\partial w}{\partial \lambda_k} = \frac{1}{a} - \frac{1}{4a^2} = \frac{1}{2\beta d} \quad (4.11)$$

$$\lambda_k^{\frac{1}{2}} \frac{\partial^2 w}{\partial \lambda_k \partial \lambda_{k'}} \lambda_{k'}^{\frac{1}{2}} = \left[\frac{3}{16a^2} - \frac{1}{2a} \right] \delta_{kk'} + \frac{1}{N} \frac{1}{16a^2} \quad (4.12)$$

$$\frac{1}{\lambda_k - \lambda_{k'}} \left(\frac{\partial w}{\partial \lambda_k} - \frac{\partial w}{\partial \lambda_{k'}} \right) = \left[\frac{3}{16a^4} - \frac{1}{2a^3} \right] \quad (4.13)$$

to diagonalize the mass matrix in Eq. (4.10) and obtain

$$\begin{aligned} \frac{F_{mf}(\text{one loop})}{VN^2} &= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{1}{d} \sum_{\mu} \cos p_{\mu} \right) \\ &- \frac{N^2-1}{2N^2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{\beta}{4a^2} \sum_{\mu} \cos p_{\mu} \right) \\ &- \frac{1}{2N^2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{\beta}{2a^2} \sum_{\mu} \cos p_{\mu} \right) \end{aligned} \quad (4.14)$$

We are in a different situation here with respect to what happened in vector models. Even in the large N limit, the expansion around the mean field does not terminate and we cannot hope to reproduce the whole perturbative series by computing the first correction to mean field. However, the $1/d$ expansion turns out to be also partially an expansion in the coupling (through the dependence of $\bar{\alpha}$ on βd) and therefore the $O(\beta^{-1})$ mean field prediction is testable in perturbation theory and the one-loop correction gives us a prediction for the $O(\beta^{-2})$ perturbative coefficient.

Actually, expansion of Eqs. (4.6) and (4.14) gives

$$\frac{F}{VN^2} = 2\beta d - \frac{1}{2} \ln 2\beta - \frac{3}{4} - \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{1}{d} \sum_r \omega_r p_r \right) 2d \quad (4.15)$$

$$+ \frac{1}{32\beta d} + \frac{1}{256(\beta d)^2} \left(1 + \frac{1}{8d} \right) + O(\beta^{-3})$$

and the $O(\beta^{-1})$ coefficient is in agreement with our perturbative expansion.

It is also possible to compute the one-loop correction to mean field in the strong coupling phase. The effective action is now very simple:

$$\beta \sum_{n\mu} \text{Tr} (\xi_n \xi_{n+\mu}^\dagger + \xi_{n+\mu} \xi_n^\dagger) - \sum_n \text{Tr} (\gamma_n \xi_n^\dagger + \xi_n \gamma_n^\dagger) + \sum_n \text{Tr} (\gamma_n \gamma_n^\dagger) \quad (4.16)$$

and the free energy turns out to be

$$\frac{F_{mf}}{VN^2} = - \int \frac{d^d p}{(2\pi)^d} \ln (1 - 2\beta \sum_r \omega_r p_r) \quad (4.17)$$

with a critical point when $\beta_c d = \frac{1}{2}$ [see Eq. (4.8)]. The strong coupling expansion of Eq. (4.17) starts with $\beta^2 d$, as expected from standard strong coupling results.

5. CHIRAL MODELS IN ONE DIMENSION

As for the vector models, $d = 1$ is anomalous in many respects. We again meet the previously observed phenomena, that is: breakdown of the mean field approximation, agreement of large N perturbation theory in first order with mean field, and therefore breakdown of perturbation theory.

$U(N)$ chiral models in one-dimension are an exactly integrable system¹⁹⁾, thanks to the possibility of performing the change of variables $U_n U_{n+1}^\dagger \rightarrow W_n$ that reduces the partition function to

$$Z_N = \left[\int dW \exp N\beta (\text{Tr} W + \text{Tr} W^\dagger) \right]^L \quad (5.1)$$

In turn, the single link integral can be evaluated exactly (the system is equivalent to two-dimensional lattice YM theory) to give

$$\int dW \exp N\beta (\text{Tr} W + \text{Tr} W^\dagger) = \det (I_{i,j}(2N\beta)) \quad i,j = 1 \dots N \quad (5.2)$$

where I_k are the modified Bessel functions.

The large N limit of the free energy is, in weak coupling²⁰⁾

$$f = \lim_{N \rightarrow \infty} \frac{\ln Z_N}{L N^2} = 2\beta - \frac{1}{2} \ln 2\beta - \frac{3}{4} \quad \beta \geq \frac{1}{2} \quad (5.3)$$

(in strong coupling $f = \beta^2$).

One can also compute weak coupling expansions for f at finite N for both $U(N)$ and $SU(N)$ models⁵⁾

$$f = 2\beta - \frac{1}{2} \ln 2\beta + \frac{1}{16N^2\beta} + \frac{1}{64N^2\beta^2} + O(\beta^{-3})$$

$$\frac{1}{2} \frac{\partial f}{\partial \beta} = 1 - \frac{1}{4\beta} - \frac{1}{32N^2\beta^2} - \frac{1}{64N^2\beta^3} + O(\beta^{-4}) \quad \text{for } U(N) \quad (5.4)$$

$$f = 2\beta - \frac{N^2-1}{N^2} \frac{1}{2} \ln 2\beta - \frac{N^2-1}{N^4} \frac{1}{16\beta} - \frac{(2N^2-5)(N^2-1)}{N^6} \frac{1}{192\beta^2} + O(\beta^{-3})$$

$$\frac{1}{2} \frac{\partial f}{\partial \beta} = 1 - \frac{N^2-1}{N^2} \frac{1}{4\beta} + \frac{N^2-1}{N^4} \frac{1}{32\beta^2} + \frac{(2N^2-5)(N^2-1)}{N^6} \frac{1}{192\beta^3} + O(\beta^{-4}) \quad \text{for } SU(N) \quad (5.5)$$

These exact results disagree with our perturbative results except in the Abelian case $N = 1$, where the two expressions take the same value.

In particular, in the large N limit the exact result implies termination of the perturbative series, while no termination occurs in the perturbative expansion (in analogy with what happened in large N CP^{N-1} models in one-dimension).

6. PERTURBATION THEORY FOR GAUGE MODELS

We will now present our results for the weak coupling expansion of a gauge theory, for instance, the twisted Eguchi-Kawai model recently proposed by González-Arroyo and Okawa^{8,21}). It is a one-plaquette theory "à la Eguchi-Kawai" but with modified boundary conditions. The partition function Z_{TEK} takes the form

$$Z_{\text{TEK}} = \int \prod_{\mu} dU_{\mu} \exp - S_{\text{TEK}} \quad (6.1)$$

$$S_{\text{TEK}} = -N\beta \sum_{\mu \neq \nu} Z_{\mu\nu} U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger} \quad (6.2)$$

Here U_{μ} ($\mu = 1, \dots, d$) denote matrices in the gauge group $SU(N)$ and $Z_{\mu\nu}$ are phases in the centre of the gauge group; these phases (also called twists) are parametrized by integers $n_{\mu\nu}$

$$Z_{\mu\nu} = \exp \frac{2\pi i}{N} n_{\mu\nu} \quad n_{\mu\nu} = -n_{\nu\mu} \quad (6.3)$$

The TEK model was constructed in four dimensions with the special choice

$$n_{\mu\nu} = L \quad \forall \mu < \nu \quad (6.4)$$

It was stressed in Ref. 8, that the action (6.2) can be minimized only if the matrices belong to $SU(L^2)$, a condition which imposes the way of going to the large N limit.

In the following, we will use a generalization of the TEK model for arbitrary dimensions $d \geq 2$; given two integers d and L , d being even, we choose

$$\begin{aligned} n_{\mu\nu} &= L^{\frac{d-2}{2}} & \forall \mu < \nu \\ N &= L^{\frac{d}{2}} \end{aligned} \quad (6.5)$$

which provides a particular way of going to the large N limit for any even d .

The perturbation expansion of the TEK model is realized by expanding the U_μ 's around the configuration γ_μ which minimizes the action S_{TEK} ; in general such γ_μ 's must satisfy

$$\begin{aligned} \gamma_\mu \gamma_\nu &= Z_{\mu\nu}^* \gamma_\nu \gamma_\mu \\ S_{\text{TEK}}(\gamma_\mu) &= -N^2 \beta d(d-1) \end{aligned} \quad (6.6)$$

and the expansion is

$$\begin{aligned} U_\mu &= \gamma_\mu \exp i V_\mu & V_\mu &= V_\mu^\dagger \\ &= \gamma_\mu \left(1 + i V_\mu - \frac{V_\mu^2}{2} + \dots \right) \end{aligned} \quad (6.7)$$

The Eqs. (6.5) have been studied intensively by several authors^{9,22}). Recently, classifications of the solutions were given in four dimensions with all the possible twists including the choice (6.4). Let us stress again that little is known about the solutions of the system (6.6) for general d (different from 2 and 4), but our following results are to be taken formally for arbitrary d .

After a rescaling of v_μ

$$V_\mu = \frac{a_\mu}{\sqrt{2N\beta}} \quad (6.8)$$

the action S_{TEK} can be written in powers of g and the first few terms of the series take the form:

$$S_{\text{TEK}} = S_0 - \frac{1}{2} S_2 + \frac{1}{3! 2 \sqrt{2\beta N}} S_3 + \frac{1}{4! 4\beta N} S_4 + \dots \quad (6.9)$$

$$S_0 = \beta d(d-1)N^2 \quad (6.10)$$

$$S_2 = \sum_{\mu, \nu} \text{Tr} (a_\mu^2 - a_\mu \gamma_\nu a_\mu \gamma_\nu^\dagger) + \sum_\mu (a_\mu - \gamma_\mu a_\mu \gamma_\mu^\dagger)^2 \quad (6.11)$$

$$S_3 = \sum_{\mu \neq \nu} \text{Tr} (a_\mu + b_{\mu, \nu}) [a_\nu, b_{\nu, \mu}] + (\mu, \nu) \quad (6.12)$$

$$\text{where } b_{\mu, \nu} = \gamma_\nu^\dagger a_\mu \gamma_\nu$$

$$\begin{aligned} S_4 = \sum_{\mu \neq \nu} \text{Tr} \{ & 2a_\mu^4 + 6a_\mu^2 b_{\mu, \nu}^2 - 4a_\mu^3 (\gamma_\nu^\dagger a_\mu \gamma_\nu + \gamma_\nu a_\mu \gamma_\nu^\dagger) + (\mu, \nu) \\ & + 6(a_\mu^2 + b_{\mu, \nu}^2)(a_\nu^2 + b_{\nu, \mu}^2) + 24a_\mu a_\nu b_{\mu, \nu} b_{\nu, \mu} \\ & - 12(a_\nu b_{\nu, \mu} + b_{\nu, \mu} a_\nu)(a_\mu^2 + b_{\mu, \nu}^2) \} \end{aligned} \quad (6.13)$$

In the computations, we have used the condition

$$\sum_\mu (a_\mu - \gamma_\mu a_\mu \gamma_\mu^\dagger) = 0 \quad (6.14)$$

to fix the gauge. This choice simplifies the quadratic part of S_2 above, but requires the introduction of a Fadeev-Popov determinant Δ_{FP} . In Appendix C, we have written some steps of the calculation of Δ_{FP} , together with some details of the calculation of the $1/\beta N$ coefficient in the TEK model.

Let us stress the important point that, with the gauge fixing (6.14), the quadratic term S_2 in (6.12) is a superposition of the quadratic part of the chiral model for different but independent U_μ matrices; therefore, the calculation proceeds in a similar way, using the $G(q)$ basis to expand the different a_μ . The point is that here the integrands (6.12), (6.13) are more complicated.

Using for each matrix a_μ an expansion in $G(q)$, (See B.3),

$$a_\mu = \sum_{q \neq 0} a_\mu(q) G(q) \quad (6.15)$$

the free propagator of the TEK model takes the form

$$\begin{aligned} \langle a_\mu(q_1) a_\nu(q_2) \rangle &= \delta_{\mu\nu} \delta(q_1 + q_2) e^{\frac{2\pi i}{L} \langle k_1, k_2 \rangle} \\ &\quad 2 \overline{F}(q_1) \\ F(q) &= \sum_{p=1}^d \sin^2 \frac{\pi p}{L} \end{aligned} \quad (6.16)$$

with the definition of Appendix B. (See B.1-B.5). Using the same technique as before, we have obtained the following expansion of the free energy F_{TEK} .

$$\begin{aligned} F_{\text{TEK}} &= \ln Z_{\text{TEK}} = \\ &= \beta d(d-1)N^2 - (d-1) \frac{N^2-1}{2} \ln \beta \\ &+ \frac{1}{64\beta N^2} \left\{ 2(N^2-1)^2 + (6d-8)\mathcal{L} + (d-2)^2 (\mathcal{G} - \mathcal{J}^2) \right. \\ &\quad \left. + 4(d-2)N^2 \mathcal{J} + \mathcal{M} \right\} \end{aligned} \quad (6.17)$$

where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{L}$, $\mathcal{M}$ are the sums over momenta listed below

$$\mathcal{J} = \sum_{p \neq 0} \frac{1}{F(p)} \quad (6.18)$$

$$g = \sum_{p,q \neq 0} \frac{\psi(p,q)}{F(p)F(q)} \quad ; \quad \psi(p,q) = \exp \frac{2\pi i}{L} (\langle k|p \rangle - \langle l|q \rangle) \quad (6.19)$$

$$L = \sum_{p,q \neq 0} \frac{\sum_i \sin^2 \frac{\pi p_i}{L} \sin^2 \frac{\pi q_i}{L} \psi(p,q)}{F(p)F(q)} \quad (6.20)$$

$$M = \sum_{p,q \neq 0} \frac{\sum_i \sin^4 \frac{\pi p_i}{L} + \sin^4 \frac{\pi q_i}{L} - 4(d-1) \sin^2 \frac{\pi p_i}{L} \sin^2 \frac{\pi q_i}{L}}{F(p)F(q)F(p+q)} (1 - \psi(p,q)) \quad (6.21)$$

Let us make a few comments about the result (6.17):

- i) The first two terms of F_{TEK} coincide with their analogue in Wilson theory. This fact was expected and agrees with previous considerations^{8,9,22,23}.
- ii) For $d \geq 4$ [the critical dimension $d = 2$ is treated separately in (v) below] all the sums (6.18-6.21), and consequently the full coefficient, are finite. Moreover we have verified that the structure of the $1/\beta N$ coefficient is quite similar to the one loop calculation in continuum Q.C.D. in $SU(N)$. This result is quoted at the end of Appendix C.
- iii) Let us also notice the importance of being in $SU(N)$ instead of $U(N)$. The same calculation in $U(N)$ would require us to take $G(0) \equiv \mathbb{1}$ in (6.15); then the sums (6.18-21) would also include $p = 0, q = 0$ and would leave us with a hard zero mode problem.

In the present model, taking the large N limit is intimately related to the removal of the infrared cut-off. The two operations act simultaneously.
- iv) In the sums (6.18-21) the presence of the phase $\psi(p,q)$ is due to non-planar diagrams. When going to the large N limit the phases oscillate with a smaller and smaller period and their contribution vanishes. This fact was first observed in Ref. 8.

v) The case $d = 2$ requires special attention because now, the sums $\mathcal{F}$ and $\mathcal{M}$ develop logarithmic divergences when going to the large N limit. However, thanks to the special coefficients these infrared divergences cancel exactly. This is evident for the $\mathcal{F}$ and $\mathcal{F}^2$ terms; for $\mathcal{M}$ we checked carefully that it is also the case.

Up to now, we have not succeeded in finding explicitly the value of $\mathcal{M}$ in (6.21). In the case $d = 2$, we have, however, an argument that this integral vanishes.

Indeed in two dimensions, taking the temporal gauge ($U_0 = 1$) of the Wilson model makes it equivalent to the chiral model in one dimension, for which we know the perturbation expansion. Then, on the basis of gauge invariance of Z (and consequently of F) the comparison of formula (6.17) with $d = 2$ and (3.27) with $d = 1$ clearly indicates that

$$\mathcal{M}(d=2) = 0 \quad \text{when } N \rightarrow \infty \quad (6.22)$$

We must mention that the Wilson model in the temporal gauge has been studied previously by Müller and Rühl²⁴⁾, who also found a discrepancy between perturbation theory with the standard propagator (principal value prescription) and exact two-dimensional results. They have shown that it is possible to redefine the propagator (losing time translation invariance) in such a way that perturbative results reproduce the exact ones. It might be interesting to try and extend their approach to all the models that we have discussed.

7. IMPROVED MEAN FIELD APPROACH FOR REDUCED MODELS

The first attempts^{25,26)} to apply mean field techniques to reduced models had to face the fact that it is not justified to treat each link matrix V_μ as an independent dynamical variable in the action

$$S = -N\beta \sum_{\mu \neq \nu} \text{Tr } U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger Z_{\mu\nu} = -N\beta \sum_{\mu \neq \nu} \text{Tr } V_\mu (\gamma_\mu V_\nu \gamma_\mu^\dagger) (\gamma_\nu V_\mu^\dagger \gamma_\nu^\dagger) V_\nu^\dagger \quad (7.1)$$

because V_μ is always accompanied by its adjoint $V_\mu^\dagger$, and therefore it is more natural to introduce a new dynamical variable

$$K_{ij\ell m}^\mu = V_\mu^{ij} V_\mu^{\dagger \ell m} \quad (7.2)$$

and perform the random field transform and the single link integral in terms of this variable.

In Appendix (D) we shall explicitly obtain the mean field (saddle point) equations in their general form. Here we consider explicitly the original EK model⁷⁾ ($Z_{\mu\nu} = 1$) and make a specific ansatz for the (saddle point) expectation value of the new dynamical variable:

$$C_{ij\ell m}^\mu = \langle V_\mu^{ij} V_\mu^{\dagger \ell m} \rangle = A \delta_{ij} \delta_{\ell m} + \frac{B}{N} \delta_{im} \delta_{je} \quad (7.3)$$

We may then immediately obtain naive mean field equations from Peierls' inequality

$$\langle e^0 \rangle \geq e^{\langle 0 \rangle} \quad (7.4)$$

by defining the measure

$$\prod_\mu (dU_\mu \exp c \operatorname{Tr} U_\mu \operatorname{Tr} U_\mu^\dagger - N^2 \mathcal{J}(c)) \quad (7.5)$$

where

$$\mathcal{J}(c) = \frac{1}{N^2} \ln \int dU_\mu \exp c \operatorname{Tr} U_\mu \operatorname{Tr} U_\mu^\dagger \quad (7.6)$$

and considering the operator

$$O = N\beta \sum_{\mu \neq \nu} \operatorname{Tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger - \sum_\mu c \operatorname{Tr} U_\mu \operatorname{Tr} U_\mu^\dagger \quad (7.7)$$

The resulting inequality takes the form

$$F_N = \frac{1}{N^2} \ln Z_N \geq d \mathcal{F}(c) + \frac{1}{N^2} \left[N \beta \sum_{\mu \neq \nu} \langle U_{\mu}^{ij} U_{\mu}^{+lm} \rangle \langle U_{\nu}^{jl} U_{\nu}^{+mi} \rangle - \sum_{\mu} c \langle \text{Tr} U_{\mu} \text{Tr} U_{\mu}^{\dagger} \rangle \right] \quad (7.8)$$

where

$$\langle U_{\mu}^{ij} U_{\mu}^{+lm} \rangle = \int dU_{\mu} [U_{\mu}^{ij} U_{\mu}^{+lm}] \exp \{ c \text{Tr} U_{\mu} \text{Tr} U_{\mu}^{\dagger} - N^2 \mathcal{F}(c) \} \quad (7.9)$$

It is now easy to derive the relationships

$$A + B = 1$$

$$A = \frac{N^2}{N^2 - 1} \frac{\partial \mathcal{F}(c)}{\partial c} - \frac{1}{N^2 - 1} \quad (7.10)$$

implying

$$\frac{1}{N} \langle U_{\mu}^{ij} U_{\mu}^{+lm} \rangle \langle U_{\nu}^{jl} U_{\nu}^{+mi} \rangle = \frac{N^2}{N^2 - 1} \left[2 \frac{\partial \mathcal{F}}{\partial c} - \left(\frac{\partial \mathcal{F}}{\partial c} \right)^2 - \frac{1}{N^2} \right] \quad (7.11)$$

Substituting in Eq. (7.8), we finally obtain

$$F_N \geq d \left[\mathcal{F}(c) - c \frac{\partial \mathcal{F}}{\partial c} + \gamma \left[2 \frac{\partial \mathcal{F}}{\partial c} - \left(\frac{\partial \mathcal{F}}{\partial c} \right)^2 \right] - \frac{\gamma}{N^2} \right] \quad (7.12)$$

where γ is the effective mean field coupling

$$\gamma = \beta(d-1) \frac{N^2}{N^2 - 1} \quad (7.13)$$

The r.h.s. of Eq. (7.12) is maximized by choosing c to be the solution of the saddle point equation

$$c = 2\gamma \left(1 - \frac{2\gamma}{\gamma c}\right) \quad (7.14)$$

The corresponding mean field value of the free energy is

$$\frac{F_{mf}}{d} = \mathcal{F}(\bar{c}) - \bar{c} + \frac{\bar{c}^2}{4\gamma} + \gamma \left(1 - \frac{1}{N^2}\right) \quad (7.15)$$

In the large N limit, $\mathcal{F}(c)$ can be evaluated exactly. Actually it corresponds to a special case of the mixed action problem considered by many authors^{27,28,29} once we recognize that [for $SU(N)$]

$$\text{Tr}_A U = \text{Tr} U \text{Tr} U^\dagger - 1 \quad (7.16)$$

where Tr_A means the trace taken in the adjoint representation of the group. The resulting expression is

$$\mathcal{F}(c) \xrightarrow{N \rightarrow \infty} \frac{1}{2}c + \frac{1}{2}c\sqrt{1-\frac{1}{c}} - \frac{1}{2}\ln c - \frac{1}{2}\ln\left(1+\sqrt{1-\frac{1}{c}}\right) - \frac{1}{2} \quad (7.17)$$

and it holds for $c \geq 1$, while for $c < 1$ (strong coupling) the large N limit of $\mathcal{F}(c)$ is zero.

In leading orders in the β^{-1} expansion, one recognizes that

$$\mathcal{F}(c) \simeq c - \frac{1}{2} \frac{N^2-1}{N^2} \ln 2c - \frac{3}{4} \quad (7.18)$$

$$\bar{c}^2 \simeq \beta(d-1) = \gamma \frac{N^2-1}{N^2} \quad (7.19)$$

$$\frac{F_{mf}}{d} \simeq \beta(d-1) - \frac{1}{4} \frac{N^2-1}{N^2} \ln 4\beta(d-1) + \frac{1}{4} \frac{N^2-1}{N^2} - \frac{3}{4} \quad (7.20)$$

We therefore recover the factor $1/4$ in front of the logarithmic term that was predicted by the analysis of Bhanot et al.³⁰⁾ and showed the breakdown of the $U(1)^d$ symmetry in the weak coupling phase of the EK model.

This factor could not be obtained in any natural way in the standard mean field approach.

A last comment concerns a curious feature of the mean field equations for the EK model.

Let us observe that, by changing variables to

$$v = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{c}} \quad (7.21)$$

we obtain

$$\frac{F_N}{d} \geq \frac{1}{2} \ln 2 (1-v) - \frac{1}{4} + \gamma [2v^2 - v^4] \quad (7.22)$$

and the saddle point equation

$$2\gamma(1-v^2) = \frac{1}{4v(1-v)} \quad (7.23)$$

Comparing with the (weak coupling, large N) mean field equations for Wilson lattice gauge theory^{25,31)}

$$2\beta(1 + (d-2)v^2) = \frac{1}{4v(1-v)} \quad (7.24)$$

we recognize that

$$\frac{F}{d} \Big|_{EK} = \lim_{d \rightarrow 1} \frac{F}{d-1} \Big|_{Wilson} \quad (7.25)$$

if we identify $\gamma_{EK} = \beta_{Wilson}$, that is, the mean field (large d) limit of EK theory coincides with the analytic continuation to the unphysical value $d = 1$ for standard lattice gauge theory.

8. MEAN FIELD APPROACH FOR THE TWISTED EGUCHI-KAWAI MODEL

The successful approach of the mean field method to the Eguchi-Kawai model encouraged us to apply the same method to the TEK model. Our motivations were strengthened by the knowledge that the TEK model reproduces the first terms of the weak coupling expansion of the Wilson model.

For reasons which will become apparent later, we will start with a modified form of the TEK model.

$$\begin{aligned} Z_{\text{TEK}} &= e^{-N^2 \beta d} \int \prod_{\mu} dU_{\mu} \exp N \beta \sum_{\mu, \nu} U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger} z_{\mu \nu} \\ &= e^{-N^2 \beta d} \int \prod_{\mu} dV_{\mu} \exp N \beta \sum_{\mu, \nu} V_{\mu} \gamma_{\mu} V_{\nu} \gamma_{\mu}^{\dagger} \gamma_{\nu}^{\dagger} V_{\nu}^{\dagger} \end{aligned} \quad (8.1)$$

$$Z_{\text{TEK}} \equiv e^{-N^2 \beta d} \int \prod_{\mu} dV_{\mu} \exp S_T \quad (8.2)$$

The sum in the action is now taken over all couples of indexes μ, ν and the factor in front assures us that nothing has changed.

The approach to the problem is similar to the derivation we performed in the EK model [see formulae (7.3), (7.4), (7.7)], apart from the fact that now, the relevant measure [(7.5), (7.6)] has to be changed into the following one

$$\prod_{\mu} dV_{\mu} \exp \left[\frac{Nc}{2d} \sum_{\mu, \nu} (\text{Tr} V_{\mu} \gamma_{\nu} V_{\mu}^{\dagger} \gamma_{\nu}^{\dagger} + \text{h.c.}) - N^2 \mathcal{J}_T(c) \right] \quad (8.3)$$

with

$$\mathcal{J}_T(c) = \frac{1}{dN^2} \ln \int \prod_{\mu} dV_{\mu} \exp \left[\frac{Nc}{2d} \sum_{\mu, \nu} (\text{Tr} V_{\mu} \gamma_{\nu} V_{\mu}^{\dagger} \gamma_{\nu}^{\dagger} + \text{h.c.}) \right] \quad (8.4)$$

We see that $\mathcal{F}_T(c)$ above is nothing but the free energy of a reduced chiral model (3.13), (3.14), and the interest for these models now becomes clear. The fact that the sum in (8.4) is over all couples μ, ν guarantees the absence of zero modes in the expansion $\mathcal{F}_T(c)$ in the large c region. That is the reason of the change (8.1).

Applying Peierls' inequality with the measure (8.3) to the operator

$$O = S_T - \frac{Nc}{2d} \sum_{\mu, \nu} \text{Tr} (V_\mu \gamma_\nu V_\mu^\dagger \gamma_\nu^\dagger + \text{h.c.}) \quad (8.5)$$

and using the trivial identity

$$c \frac{\partial \mathcal{F}_T}{\partial c} = \frac{1}{dN^2} \frac{Nc}{2d} \left\langle \sum_{\mu, \nu} \text{Tr} (V_\mu \gamma_\nu V_\mu^\dagger \gamma_\nu^\dagger + \text{h.c.}) \right\rangle \quad (8.6)$$

leads to the following inequality:

$$F_{TEk} \geq d \mathcal{F}_T(c) + \frac{1}{N^2} \langle S_T \rangle - dc \frac{\partial \mathcal{F}_T(c)}{\partial c} - \beta d \quad (8.7)$$

It is easy to derive the relationships below, which give us $\langle S_T \rangle$ in function of

$\mathcal{F}_T$:

$$\begin{aligned} \langle \text{Tr} \gamma_\nu V_\mu \gamma_\nu^\dagger V_\mu^\dagger \rangle &= \langle V_\mu^{ij} V_\mu^{\dagger lm} \rangle \gamma_\nu^{+jl} \gamma_\nu^{mi} = \\ &= AN + \left(\frac{1-A}{N} \right) \text{Tr} \gamma_\nu \text{Tr} \gamma_\nu^\dagger = AN \end{aligned} \quad (8.8)$$

$$\sum_{\mu, \nu} \langle \text{Tr} \gamma_\nu V_\mu \gamma_\nu^\dagger V_\mu^\dagger + \text{h.c.} \rangle = 2d^2 AN = 2d^2 N \frac{\partial \mathcal{F}_T}{\partial c} \quad (8.9)$$

So that

$$A = \frac{\partial \mathcal{F}_T}{\partial c} \quad (8.10)$$

and

$$\begin{aligned} \langle S_T \rangle &= N\beta \sum_{\mu, \nu} \langle V_\mu^{ij} V_\mu^{lm} V_\nu^{pq} V_\nu^{ri} \rangle \gamma_\mu^{jp} \gamma_\mu^{qt} \gamma_\nu^{tl} \gamma_\nu^{mr} \\ &= \beta N \sum_{\mu, \nu} \left(A^2 N + \frac{(1-A)^2}{N^2} \mathbb{Z}_{\mu\nu} N \right) \\ &= \beta N^2 A^2 d^2 \left(1 + O\left(\frac{1}{N^2}\right) \right) \end{aligned} \quad (8.11)$$

$$\langle S_T \rangle \xrightarrow{N \rightarrow \infty} N^2 \beta \left(\frac{\partial \mathcal{F}_T}{\partial c} \right)^2 d^2 \quad (8.12)$$

Substituting in Eq. (8.7) finally gives

$$F_{TEK} \geq d \left(\mathcal{F}_T - c \frac{\partial \mathcal{F}_T}{\partial c} + \beta d \left(\frac{\partial \mathcal{F}_T}{\partial c} \right)^2 - \beta \right) \quad (8.13)$$

The new saddle point equation is, obviously

$$c = 2d\beta \frac{\partial \mathcal{F}_T}{\partial c} \quad (8.14)$$

From previous calculations (see Section 3), we know much about the large c behaviour of $\mathcal{F}_T(c)$. For instance, it is the result of Eq. (3.27) with $\beta = c/2d$

$$\begin{aligned}
 \mathcal{H}_T(c) &= c - \frac{N^2-1}{2N^2} \ln c + \frac{1}{16c} \left(\frac{N^2-1}{N^2} \right)^2 + \dots \\
 &\xrightarrow{N \rightarrow \infty} c - \frac{1}{2} \ln c + \frac{1}{16c} + \dots
 \end{aligned}
 \tag{8.15}$$

This provides an approximation for the Eq. (8.14)

$$c = 2\beta d \left(1 - \frac{1}{2c} - \frac{1}{16c^2} + \dots \right) \tag{8.16}$$

which admits among its solutions

$$\bar{c} = 2\beta d - \frac{1}{2} - \frac{5}{32}\beta d + o\left(\frac{1}{(\beta d)^2}\right) \tag{8.17}$$

The mean field approximation of the TEK model associated with the saddle point (8.17) is obtained after a few calculations and appears to be:

$$\begin{aligned}
 F_{\text{TEK}} &\cong d \left[\mathcal{H}_T - c \frac{\partial \mathcal{H}_T}{\partial c} + \beta d \left(\frac{\partial \mathcal{H}_T}{\partial c} \right)^2 - \beta \right]_{c=\bar{c}} \\
 &\cong \beta d(d-1) - \frac{d}{2} \ln \beta + \frac{3}{32}\beta + o(\beta^{-2})
 \end{aligned}
 \tag{8.18}$$

It is pleasant to check that the first two terms agree (in the large d limit) with the original model. For the $1/\beta$ coefficient, we are unable to make a comparison for the reason explained in Section 5.

9. CONVERGENCE OF METHODS IN LARGE d

In conclusion, in order to illustrate the effectiveness of our mean field method, we want to compare three different approaches to the large d (mean field) limit of lattice gauge theories.

As we show in Appendix D, the effective action for large N (reduced, twisted) gauge theories can be cast in the form

$$-S = N\beta \sum_{\mu \neq \nu} A_{\alpha\beta}^{\mu} A_{\gamma\delta}^{\nu} \text{Tr} (T^{\alpha} \gamma_{\nu}^{\dagger} T^{\delta} \gamma_{\mu}^{\dagger} T^{\beta} \gamma_{\nu} T^{\gamma} \gamma_{\mu}) - \sum_{\mu} C_{\mu}^{\alpha\beta} A_{\mu}^{\alpha\beta} + N^2 \sum_{\mu} W(C_{\mu}^{\alpha\beta}) \quad (9.1)$$

where $A_{\alpha\beta}^{\mu}$ and $C_{\mu}^{\alpha\beta}$ are unconstrained variables ($N^2 \times N^2$ matrices) and

$$W(C_{\mu}^{\alpha\beta}) = \frac{1}{N^2} \int dV_{\mu} \exp C_{\mu}^{\alpha\beta} \text{Tr} (T^{\alpha} V_{\mu} T^{\beta} V_{\mu}^{\dagger}) \quad (9.2)$$

We can now make the standard ansatz on the structure of the saddle point solution:

$$A_{\alpha\beta}^{\mu} = A \delta_{\alpha\beta} \quad (9.3)$$

reducing the effective action to the form

$$-\frac{1}{N^2} S(A, C_{\mu}) = \beta d(d-1) A^2 - A \sum_{\mu} \frac{1}{N^2} \text{Tr} C_{\mu} + \sum_{\mu} W(C_{\mu}) \quad (9.4)$$

We consider two possibilities for the form of C_{μ} : a naive one

$$C_{\mu}^{\alpha\beta} = C \delta_{\alpha\beta} \quad (9.5)$$

and a form motivated by the equations of motion following from Eq. (9.1)

$$C_{\mu}^{\alpha\beta} = N^2 \frac{c}{2d} P^{\alpha\beta} \quad (9.6)$$

where $P_{\alpha\beta}$ is a projection operator, $\text{tr } P_{\alpha\beta} = 2d$ and the normalization is chosen in such a way that in both cases $1/N^2 \text{tr } C_{\mu} = c$

The first choice leads to

$$W(c) = J(c) \quad (9.7)$$

as already discussed in Section 7, while the second choice implies that

$$W_T(c) = J_T(c) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \frac{\ln Z_N(\frac{c}{2d})}{V} \quad (9.8)$$

where Z_N is the partition function of (reduced) d-dimensional chiral models with coupling $c/2d$.

In the spirit of the mean field approximation, we can replace $J_T(c)$ with its mean field value computed in Section 4:

$$J_T(c) \cong \bar{a} - \frac{1}{2} \ln 2\bar{a} - \frac{1}{2} \quad (9.9)$$

where $\bar{a}$ is now

$$\bar{a} = \frac{1}{2} c (1 + \sqrt{1 - \frac{1}{c}}) \quad (9.10)$$

By comparing Eq. (9.10) with Eq. (7.17) we recognize that

$$J_T(c) = J(c) \quad (9.11)$$

a highly non trivial result.

We are therefore led to study the saddle point of the effective action

$$-\frac{S}{N}(A, c) = d \left[\gamma A^2 - A c + \mathcal{F}(c) \right] \quad \gamma = \beta(d-1) \quad (9.12)$$

and equations of motion

$$\begin{aligned} 2\gamma A &= c \\ A &= \frac{\partial \mathcal{F}}{\partial c} \end{aligned} \quad (9.13)$$

Inspired by our considerations on the EK model in Section 7, we perform the change of variables

$$v = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{c}} \quad c = \frac{1}{4v(1-v)} \quad (9.14)$$

and, after solving it for A we obtain the effective action

$$-\frac{S_{mf}}{dN^2} = \frac{1}{2} \ln 2(1-v) + \gamma v^4 - \frac{1}{4} \quad (9.15)$$

where we have used the equations of motion

$$8\gamma v^3(1-v) = 1 \quad (9.16)$$

This is exactly the large d limit of the standard Wilson theory, as one would have obtained from usual considerations^{25,31)}.

Therefore, our procedure is fully consistent with standard results, and we conclude that it is the most appropriate way to deal with the mean field approximation in reduced models.

APPENDIX A

AGREEMENT OF EXPANSIONS AND EXACT RESULTS FOR EQUIVALENT MODELS

It is interesting to compare the results for different formulations of the same model, as happens to be the case for some of the $O(N)$, CP^{N-1} and chiral $SU(N)$ and $U(N)$ models.

We check that not only exact results, but also perturbative ones, even when they do not agree with the exact ones, agree for the different formulations of the same model thus confirming the universality of the pathological behaviour.

Abelian models

For Abelian models there is no disagreement between perturbative and exact results. The $O(1)$ vector model is equivalent to the Ising model

$$Z_1^{(1)} = ch \beta \quad (A.1)$$

and

$$\frac{\ln Z_1}{V} = \frac{d}{g} - \ln 2 + O(e^{-\frac{1}{g}}) \quad \text{for all } d \quad (A.2)$$

CP^0 is equivalent to the $SU(1)$ chiral model with coupling $\beta/2$ and trivial

$$Z_1^{(1)} = e^\beta \quad (A.3)$$

$$\frac{\ln Z_1}{V} = \frac{d}{g} \quad \text{for all } d \quad (A.4)$$

$O(2)$ is equivalent to a $U(1)$ chiral model with coupling $\beta/2$ and

$$Z_2^{(1)}(\beta) = I_0(2\beta) = Z_{1c}^{(1)}\left(\frac{\beta}{2}\right) \quad (A.5)$$

$$\begin{aligned} \frac{\ln Z_2(g)}{V} &\cong \frac{d}{g} + \frac{1}{2} \ln g - \frac{1}{2} \ln 2\pi + \frac{1}{8d} g = \frac{2d}{(2g)} + \frac{1}{2} \ln \frac{(2g)}{2} - \frac{1}{2} \ln 2\pi + \frac{1}{16d} (2g) \\ &\cong \frac{\ln Z_{1c}(2g)}{V} \end{aligned} \quad (A.6)$$

Non Abelian models

As we have shown, there is disagreement between perturbative and exact results when $d = 1$. The $O(3)$ vector model is equivalent to the CP model with coupling 3β

$$Z_3^{(1)}(\beta) = e^{-3\beta} Z_{CP}^{(1)}(3\beta) = \frac{\sinh 3\beta}{3\beta} \quad (A.7)$$

$$\begin{aligned} \ln \frac{Z_3(g)}{V} &\cong \frac{d}{g} + \ln g - \ln 2 + \frac{1}{4d} g = \left(\frac{2}{g}\right)d + \ln 2 \left(\frac{g}{2}\right) - \ln 2 + \frac{1}{2d} \left(\frac{g}{2}\right) - \frac{d}{g} \\ &\cong \ln \frac{Z_{CP}^{(1)}\left(\frac{g}{2}\right)}{V} - \frac{d}{g} \end{aligned} \quad (A.8)$$

The $O(4)$ vector model is equivalent to the $SU(2)$ chiral model with coupling $\beta/2$ and

$$Z_4^{(1)}(\beta) = \frac{I_1(4\beta)}{2\beta} = Z_{2c}\left(\frac{\beta}{2}\right) \quad (A.9)$$

$$\begin{aligned} \ln \frac{Z_4(g)}{V} &\cong \frac{d}{g} + \frac{3}{2} \ln 4g - \ln 4\sqrt{2\pi} + \frac{3}{8d} g = \\ &= \frac{4d}{(4g)} + \frac{3}{2} \ln(4g) - \ln 4\sqrt{2\pi} + \frac{3}{32d}(4g) \cong \ln \frac{Z_{2c}(4g)}{V} \end{aligned} \quad (A.10)$$

APPENDIX B

ONE- AND TWO-LOOP CORRECTIONS FOR THE REDUCED CHIRAL MODELS

Here is a list of the useful properties of the matrices $G(q)$ [see definition in (3.18)]

$$G(q_1) \dots G(q_n) = \exp \left(\frac{2\pi i}{L} \sum_{i,j} \langle k_i | k_j \rangle \right) G(q_1 + \dots + q_n)$$

$$k_i = nq_i$$

$$\langle k_i | k_j \rangle = \sum_{\mu < \nu} k_i^\mu k_j^\nu$$
(B.1)

$$\gamma_\mu G(q) \gamma_\mu^\dagger = \exp \left(\frac{2\pi i}{L} q_\mu \right) G(q)$$
(B.2)

$$a = \sum_{q \neq 0} a(q) G(q)$$
(B.3)

$$\sum_\mu \text{tr} (a^2 - \gamma_\mu a \gamma_\mu^\dagger a) = 2 \sum_q a(q) a(-q) \left(\sum_\mu \sin^2 \frac{\pi q_\mu}{L} \right)$$

$$\equiv 2 \sum_q a(q) a(-q) F(q)$$
(B.4)

$$\langle a(q_1) a(q_2) \rangle = \frac{\delta(q_1 + q_2) e^{\frac{2\pi i}{L} \langle k_1 | k_2 \rangle}}{2F(q_1)}$$

$$k_i = nq_i$$
(B.5)

The complete value of the $1/\beta$ coefficient in the formula (3.27) appears to be the following.

$$\frac{1}{32\beta N^2} \left(\left(\frac{N^2-1}{d} \right)^2 + \mathcal{L}_d \right)$$
(B.6)

$$\mathcal{L}_d = \sum_{p,q} \frac{\sum_{\mu} \frac{\sin^2 \pi p_{\mu}}{L} \frac{\sin^2 \pi q_{\mu}}{L}}{F(q) F(p)} \exp \frac{2\pi i}{L} (\langle k|l \rangle - \langle l|k \rangle) \quad (\text{B.7})$$

$k = np \quad l = nq$

It is easy to see that the sum $\mathcal{L}_d$ goes to zero in the large N limit. This sum is associated with a class of non-planar diagrams and the fact that it vanishes was also pointed out in Ref. 8.

The computation of the two-loop correction to the present model is less successful because, even if we keep only the leading order in N, sums appear which we could not evaluate exactly.

For instance, the two-loop contribution to the free energy takes the form

$$\left(\frac{1}{\beta N}\right)^2 \frac{5}{6! 2^9} \left[224 \left(\frac{N^2-1}{d}\right)^2 + 48 \left(\frac{N^2-1}{d}\right) \mathcal{F} - 12 \mathcal{F}^2 + \mathcal{K}_d \right] \quad (\text{B.8})$$

where

$$\mathcal{F} = \sum_{k \neq 0} \frac{1}{F(k)}$$

$$\mathcal{K} = \sum_{\substack{k_1, \dots, k_4 \\ \neq 0}} \frac{\left[\sum_{a=1}^4 \left(2F(k_a) - \frac{3}{2} F(k_a + k_{a+1}) \right) \right]^2}{F(k_1) \dots F(k_4)} \delta \left(\sum_{a=1}^4 k_a \right)$$

Taking the large N limit of the sums, $\mathcal{F}$ and $\mathcal{K}$ become integrals on a domain which includes moments closer and closer to the dangerous region where the denominator vanishes. Despite the fact that we have not succeeded in obtaining a closed form for $\mathcal{F}$ and $\mathcal{K}$ in the large N limit, we have been able to prove that the total coefficient is free of singularities.

For $d > 2$ the sums are separately finite and the problem of singularities does not occur. For $d = 2$, however, $\mathcal{F}$ and $\mathcal{K}$ become infinite but after carefully extracting the singular part of $\mathcal{K}$, we have obtained

$$\chi = 12 \mathfrak{J}^2 - 48 \mathfrak{J} \left(\frac{N^2-1}{d} \right) + O \left(\left(\frac{N^2-1}{d} \right)^2 \right) \quad (\text{B.9})$$

which clearly proves our statement. Notice that the formula (B.9) is valid for any d .

APPENDIX C

FADEEV-POPOV DETERMINANT IN THE BACKGROUND GAUGE OF TEK

Using the relation

$$f(a) = \sum_{\mu} (a_{\mu} - \gamma_{\mu} a_{\mu} \gamma_{\mu}^{\dagger}) = 0 \quad (C.1)$$

as a gauge-fixing term requires the introduction of a Fadeev-Popov term in the action (6.2) of the model. To this end, the knowledge of the gauge transformation of a_{μ} is necessary

$$\begin{aligned} U_{\mu} &= \gamma_{\mu} e^{i a_{\mu}} \rightarrow \\ U'_{\mu}(\phi) &= e^{i\phi} \gamma_{\mu} e^{i a_{\mu}} e^{-i\phi} = \gamma_{\mu} (\gamma_{\mu}^{\dagger} e^{i\phi} \gamma_{\mu} e^{i a_{\mu}} e^{-i\phi}) \\ &= \gamma_{\mu} e^{i a'_{\mu}(\phi)} \end{aligned} \quad (C.2)$$

Using the Baker-Hausdorff formula and keeping only the part of a'_{μ} linear in ϕ leads, for the first few terms, to the following expansion

$$a'_{\mu}(\phi) = a_{\mu} + \frac{i}{2} [\phi + \gamma_{\mu} \phi \gamma_{\mu}^{\dagger}, a_{\mu}] - \frac{1}{12} [[\phi - \gamma_{\mu} \phi \gamma_{\mu}^{\dagger}, a_{\mu}], a_{\mu}] + \dots \quad (C.3)$$

The Fadeev-Popov determinant can now be written as an effective part of the action, using the usual trick of anticommuting variables C^a ; λ_a is a basis of $SU(N)$

$$c = C^a \lambda_a \quad a = 1 \dots N^2 - 1 \quad (C.4)$$

$$\begin{aligned}
 \Delta_{FP} &= \det \frac{\delta f(a'(\phi))}{\delta \phi} \\
 &= \int dc d\bar{c} \exp - \text{Tr } \bar{c} f(a'(c)) \\
 &= \int dc d\bar{c} \exp - S_{FP}
 \end{aligned} \tag{C.5}$$

$$\begin{aligned}
 S_{FP} &= \text{Tr} \sum_{\mu} [\bar{c} - \gamma_{\mu} \bar{c} \gamma_{\mu}^+] [c - \gamma_{\mu} c \gamma_{\mu}^+] \\
 &\quad + \frac{i}{2} [\bar{c} - \gamma_{\mu} \bar{c} \gamma_{\mu}^+] [c + \gamma_{\mu} c \gamma_{\mu}^+, a_{\mu}] \\
 &\quad - \frac{i}{12} [\bar{c} - \gamma_{\mu} \bar{c} \gamma_{\mu}^+] [[c - \gamma_{\mu} c \gamma_{\mu}^+, a_{\mu}], a_{\mu}]
 \end{aligned} \tag{C.6}$$

Let us remark that the third term in S_{FP} is specific to the lattice; it is absent in the continuum theory.

We now present an intermediate step of the calculation of the $1/\beta N$ coefficient given in (6.17). To give support to the formulae, we write them in front of the Feynman diagram to which they correspond and we give their analogue for continuum QCD in d dimensions and $SU(N)$. This comparison is the object of the Table.

The definitions of F , $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{L}$ are in the text and

$$Q \equiv \sum_{p,q} \frac{\sum_{\mu} \sin^2 \frac{\pi q_{\mu}}{L} \sin^2 \frac{\pi p_{\mu}}{L}}{F(p)F(q)} = \frac{(N^2 - 1)^2}{d} \tag{C.7}$$

Table 1

Diagrams	TEK integrals	QCD - d dim. - SU(N) to be multiplied by $g^2(N^2 - 1)$
$-\frac{d}{96} \gamma$	—
$-\frac{2}{96} \gamma$	—
$\frac{1}{64} \left[(d-d^2)(\gamma^2 - \gamma) + \frac{14}{3} N^2 (d-1) \gamma \right. \\ \left. + 2(d-1)Q + 2(d-3)L - 4(N^2-1) \right]$	$\frac{d-d^2}{4} \gamma_c^2$ ($\gamma_c = \int \frac{d^d p}{p^2}$)
$\frac{1}{64} \sum_{k,m} \frac{\sum_{\mu \neq \nu} \cos^2 \frac{\pi(m+k)}{L} \sin^2 \frac{\pi(m-k)}{L}}{F(k) F(m) F(k+m)} (1-\psi)$	$\frac{d-1}{2} \int dk dm \frac{m^2 + k^2 + mk}{m^2 k^2 (m+k)^2}$
$-\frac{1}{32} \sum_{k,m} \frac{\sum_{\mu} \sin \frac{\pi k}{L} \cos \frac{\pi k m}{L} \sin \frac{\pi m}{L} \cos \frac{\pi m}{L}}{F(k) F(m) F(k-m)} (1-\psi)$	$-\frac{1}{2} \int dk dm \frac{m \cdot k}{m^2 k^2 (m-k)^2}$
SUM	see (6.17)	$(d/2 - 1) \gamma_c^2$

APPENDIX D

MEAN FIELD EQUATIONS FOR REDUCED MODELS

We start from the action described in Eq. (7.1)

$$S = -N\beta \sum_{\mu \neq \nu} \text{Tr} V_{\mu} (\gamma_{\mu} V_{\nu} \gamma_{\mu}^{\dagger}) (\gamma_{\nu} V_{\mu}^{\dagger} \gamma_{\nu}^{\dagger}) V_{\nu}^{\dagger} \quad (\text{D.1})$$

We want to change variables to $K_{ij\ell m}^{\mu} = v_{\mu}^{ij} v_{\mu}^{+\ell m}$ and to perform a random field transform

$$S = -N\beta \sum_{\mu \neq \nu} K_{ij\ell m}^{\mu} K_{pqri}^{\nu} (\gamma_r)^{jp} (\gamma_r^{\dagger})^{iq} (\gamma_{\nu})^{ke} (\gamma_{\nu}^{\dagger})^{mr} \quad (\text{D.2})$$

Actually we find it more convenient to redefine our variables by introducing a complete set of Hermitian normalized generators of $U(N)$: T_{ij}^{α} , so that

$$T_{ij}^{\alpha} T_{ji}^{\beta} = \delta_{\alpha\beta} \quad (\text{D.3})$$

$$\sum_{\alpha} T_{ij}^{\alpha} T_{\ell m}^{\alpha} = \delta_{im} \delta_{je} \quad (\text{D.4})$$

and defining

$$K_{\alpha\beta}^{\mu} = K_{ij\ell m}^{\mu} T_{mi}^{\alpha} T_{je}^{\beta} \quad K_{ij\ell m}^{\mu} = K_{\alpha\beta}^{\mu} T_{im}^{\alpha} T_{\ell j}^{\beta} \quad (\text{D.5})$$

we may immediately recognize that

$$K_{\alpha\beta}^{\mu} = \text{Tr} (T^{\alpha} V_{\mu} T^{\beta} V_{\mu}^{\dagger}) \quad (\text{D.6})$$

are the elements of the adjoint representation of $U(N)$. The action takes the form

$$S = -N\beta \sum_{\mu \neq \nu} K_{\alpha\beta}^{\mu} K_{\gamma\delta}^{\nu} \text{Tr} (T^{\alpha} \gamma_{\nu}^{\dagger} T^{\gamma} \gamma_{\mu}^{\dagger} \gamma_{\nu} T^{\beta} \gamma_{\mu} T^{\delta}) \quad (\text{D.7})$$

and after performing a random field transform, the expression for the partition function reduces to

$$\begin{aligned} Z_N = \int dA_{\mu}^{\alpha\beta} dC_{\mu}^{\alpha\beta} \exp \{ N\beta \sum_{\mu \neq \nu} V_{\alpha\beta}^{\mu} V_{\gamma\delta}^{\nu} \text{Tr} (T^{\alpha} \gamma_{\nu}^{\dagger} T^{\gamma} \gamma_{\mu}^{\dagger} \gamma_{\nu} T^{\beta} \gamma_{\mu} T^{\delta}) \\ - \sum_{\mu} C_{\mu}^{\alpha\beta} V_{\mu}^{\alpha\beta} + N^2 \sum_{\mu} w(C_{\mu}^{\alpha\beta}) \} \end{aligned} \quad (\text{D.8})$$

where

$$w(C_{\mu}^{\alpha\beta}) = \frac{1}{N^2} \ln \int dV_{\mu} \exp C_{\mu}^{\alpha\beta} K_{\mu}^{\alpha\beta} \quad (\text{D.9})$$

Let us notice that it is a property of the adjoint representation for $U(N)$ that

$$\begin{aligned} K_{\mu}^{00} &= 1 \\ K_{\mu}^{0\alpha} &= K_{\mu}^{\beta 0} = 0 \end{aligned} \quad (\text{D.10})$$

Therefore, the integration in C_{μ}^{00} and $C_{\mu}^{\alpha 0}$, $C_{\mu}^{0\beta}$ is trivial and leads to

$$\begin{aligned} A_{\mu}^{00} &= 1 \\ A_{\mu}^{0\alpha} &= A_{\mu}^{\beta 0} = 0 \end{aligned} \quad (\text{D.11})$$

The partition function can therefore be represented as an integral over $SU(N)$ indices

$$Z_N = \int dA_\mu^{\alpha\beta} dC_\mu^{\alpha\beta} \exp \left\{ \beta \sum_{\mu \neq \nu} Z_{\mu\nu} + \beta \sum_{\mu \neq \nu} A_{\alpha\beta}^\mu \text{Tr} (T^\alpha \gamma_\nu^\dagger \gamma_\mu^\dagger \gamma_\nu T^\beta \gamma_\mu + \right. \\ \left. + T^\alpha \gamma_\mu^\dagger T^\beta \gamma_\nu^\dagger \gamma_\mu \gamma_\nu) + N\beta \sum_{\mu \neq \nu} A_{\alpha\beta}^\mu A_{\gamma\delta}^\nu \text{Tr} (T^\alpha \gamma_\nu^\dagger T^\delta \gamma_\mu^\dagger \gamma_\nu T^\beta \gamma_\mu T^\delta) \right. \\ \left. + N^2 \sum_\mu w(C_\mu^{\alpha\beta}) - \sum_\mu C_\mu^{\alpha\beta} A_\mu^{\alpha\beta} \right\} \quad (D.12)$$

and the saddle-point equations of motion are

$$\beta \sum_{\nu \neq \mu} \text{Tr} (T^\alpha \gamma_\nu^\dagger \gamma_\mu^\dagger \gamma_\nu T^\beta \gamma_\mu + T^\alpha \gamma_\mu^\dagger T^\beta \gamma_\nu^\dagger \gamma_\mu \gamma_\nu) \\ + N\beta \sum_{\nu \neq \mu} A_{\gamma\delta}^\nu \text{Tr} (T^\alpha \gamma_\nu^\dagger T^\delta \gamma_\mu^\dagger \gamma_\nu T^\beta \gamma_\mu T^\delta + T^\delta \gamma_\mu^\dagger T^\beta \gamma_\nu^\dagger \gamma_\mu T^\delta \gamma_\nu T^\alpha) = \\ = C_\mu^{\alpha\beta} \quad (D.13)$$

$$A_\mu^{\alpha\beta} = N^2 \frac{\delta}{\delta C_\mu^{\alpha\beta}} w(C_\mu) \quad (D.14)$$

In order to solve the equations of motion we make an Ansatz

$$A_\mu^{\alpha\beta} = A \delta_{\alpha\beta} \quad (D.15)$$

[equivalent to the ansatz Eq. (7.3)], and determine a consistent ansatz for $C_\mu^{\alpha\beta}$.

As a consequence of Eq. (D.15), Eqs. (D.13) and (D.14) reduce to

$$\beta(1-A) \sum_{\nu \neq \mu} \text{Tr} (T^\alpha \gamma_\nu^\dagger \gamma_\mu^\dagger \gamma_\nu T^\beta \gamma_\mu + T^\alpha \gamma_\mu^\dagger T^\beta \gamma_\nu^\dagger \gamma_\mu \gamma_\nu) + \\ + N\beta A \sum_{\nu \neq \mu} \text{Tr} (T^\alpha \gamma_\nu^\dagger) \text{Tr} (\gamma_\nu T^\beta) + \text{Tr} (T^\beta \gamma_\nu^\dagger) \text{Tr} (\gamma_\nu T^\alpha) = C_\mu^{\alpha\beta} \quad (D.16)$$

$$A \delta_{\alpha\beta} = N^2 \frac{\delta}{\delta c_{\mu}^{\alpha\beta}} w(c) \quad (D.17)$$

Let us now consider the special case $Z_{\mu\nu} = 1$, when $\gamma_{\mu} = 1$. Equation (D.16) reduces further to

$$2\beta(1-A)(d-1) \delta_{\alpha\beta} = c_{\mu}^{\alpha\beta} = c \delta^{\alpha\beta} \quad (D.18)$$

implying

$$w(c) = J(c) - \frac{1}{N^2} c \quad (D.19)$$

because of Eq. (7.16). The effective action takes the form

$$d(d-1) \left[\beta + 2A(N^2-1) + A^2(N^2-1) \right] - (N^2-1)cA + [N^2 J(c) - c] \quad (D.20)$$

and one can easily recognize that this becomes exactly the free energy quoted in the text [Eq. (7.12)], when one makes use of Eq. (D.17) in the form

$$(N^2-1)A = N^2 \frac{\partial J}{\partial c} - 1 \quad (D.21)$$

The twisted case is, in principle, more intriguing. However, one is interested only in the large N limit of these equations, and therefore one may observe that the first term in the l.h.s of Eq. (D.16) is now depressed by two powers of $1/N$ with respect to the second term, and it may therefore be omitted.

Moreover the matrix

$$\frac{1}{N} \sum_{\nu \neq \mu} \text{Tr}(\Gamma_{\nu}^{\alpha} \delta_{\nu}^{\dagger}) \text{Tr}(\Gamma_{\nu}^{\beta} T^{\beta}) + \text{Tr}(\Gamma_{\nu}^{\beta} \delta_{\nu}^{\dagger}) \text{Tr}(\delta_{\nu} T^{\alpha}) = P_{\alpha\beta} \quad (D.22)$$

is a projection operator, that is

$$P_{\alpha\beta} P_{\beta\gamma} = P_{\alpha\gamma} \quad \text{and} \quad P_{\alpha\alpha} = 2(d-1) \quad (\text{D.23})$$

These considerations justify the Ansatz for the mean field of the TEK model introduced in the text and lead to an investigation of group integrals in the form

$$\int dU \exp N^2 c P_{\alpha\beta} K_{\alpha\beta} = \int dU \exp N c \sum_v \text{Tr}(\gamma_v U \gamma_v^\dagger U^\dagger + \gamma_v^\dagger U \gamma_v U^\dagger) \quad (\text{D.24})$$

From Eq. (24), we recognize that the single-link integral for twisted reduced gauge theory is expressed in terms of the partition function for (twisted reduced) chiral models.

In the next Appendix we shall discuss some properties of these group integrals.

APPENDIX E

ON GROUP INTEGRALS

As we have seen in the previous section, in order to solve the chiral models in the large N limit and in order to perform a mean field expansion of large N TEK models, it is crucial to evaluate the group integral defined in Eq. (D.24), which is in turn only a special case of the integral Eq. (D.9)

$$Z = \int dV \exp C_{\alpha\beta} K_{\alpha\beta} \quad (E.1)$$

where $K_{\alpha\beta} = \text{Tr}(T^\alpha V T^\beta V^\dagger)$ is the adjoint representation of V and $C_{\alpha\beta}$ is a real matrix. More generally, we could start from

$$\int dV \exp A_{\beta\alpha}^+ K_{\alpha\beta} + A_{\alpha\beta} K_{\beta\alpha}^+ \quad (E.2)$$

but $K_{\beta\alpha}^+ = K_{\alpha\beta}^* = K_{\alpha\beta}$ and $A_{\beta\alpha}^+ = A_{\alpha\beta}^*$. Therefore, by defining the real matrix

$$C_{\alpha\beta} = A_{\alpha\beta} + A_{\alpha\beta}^*$$

we reduce to the previous case.

$K_{\alpha\beta}$, being an element of the adjoint representation of $SU(N)$ is also an element of the fundamental representation of $O(N^2-1)$. Indeed, if the integration is extended to the whole group $O(N^2-1)$, it is possible to obtain its large N limit explicitly. Restriction to the subgroup $SU(N)$, however, leaves us with an apparently intractable problem.

Let us first discuss the $O(N)$ integration. Every real matrix can be decomposed into the form

$$C = H O \quad (E.3)$$

where H is a real symmetric (Hermitian) matrix, and O is an orthogonal matrix

$$\begin{aligned} H &= \sqrt{CC^t} \\ O &= H^{-1}C \qquad O^t = C^t H^{-1} \end{aligned} \tag{E.4}$$

H can be diagonalized by an orthogonal transformation

$$H = Q D Q^t \tag{E.5}$$

and therefore changing variables to

$$K \rightarrow O^t Q K' Q^t \tag{E.6}$$

and exploiting the invariance of the $O(N^2-1)$ measure, one shows that the integral depends only on the eigenvalues λ_i of the matrix CC^+ .

We now apply the standard technique based on writing down an appropriate Schwinger-Dyson equation for Z . The orthogonality condition takes the form

$$\frac{\delta}{\delta C_{\alpha\beta}} \frac{\delta}{\delta C_{\gamma\beta}} Z = \delta_{\alpha\gamma} Z \tag{E.7}$$

and since Z depends only on λ_i , one obtains the differential equation

$$\lambda_i \frac{\partial^2 Z}{\partial \lambda_i^2} + \frac{N}{2} \frac{\partial Z}{\partial \lambda_i} + \frac{1}{2} \sum_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \left(\frac{\partial Z}{\partial \lambda_j} - \frac{\partial Z}{\partial \lambda_i} \right) = \frac{1}{4} Z \tag{E.8}$$

This equation is similar to that derived by Brezin and Gross¹⁴⁾ for $U(N)$: the only difference is in the numerical coefficients. However, while the equation for $U(N)$ could be explicitly integrated for all N , in this case we found solutions only for $N = 1, 2$ and in the large N limit

O(1): it is very easy to solve the ordinary differential equation leading to

$$Z_1 = e^{\lambda^{\frac{1}{2}}} \quad (E.9)$$

O(2): with some effort, one recognizes that the equation is solved (with proper boundary conditions) by

$$Z_2 = I_0(\lambda_1^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}}) \quad (E.10)$$

Another soluble case occurs when the source matrix $C_{\alpha\beta}$ is a projector (only one eigenvalue is non zero). Then the equation reduces to

$$\lambda \frac{\partial^2 Z}{\partial \lambda^2} + \frac{N}{2} \frac{\partial Z}{\partial \lambda} = \frac{1}{4} Z \quad (E.11)$$

and the solution is

$$Z = \Gamma\left(\frac{N}{2}\right) \frac{I_{\frac{N-1}{2}}(\lambda^{\frac{1}{2}})}{\left(\frac{1}{2}\right)^{\frac{N-1}{2}}} \quad (E.12)$$

Equations (E.9) and (E.10) can be checked by direct integration; Equations (E.12) can also be checked for $N = 1, 2, 3$.

Finally, let us consider the large N limit and rescale the eigenvalues according to $\lambda_i \rightarrow N^2 \lambda_i$. We can write an equation for $W = 1/N \ln Z$; when $N \rightarrow \infty$ this equation reduces to

$$\lambda_i \left(\frac{\partial W}{\partial \lambda_i} \right)^2 + \frac{1}{2} \left(\frac{\partial W}{\partial \lambda_i} \right) + \frac{1}{2N} \sum_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \left(\frac{\partial W}{\partial \lambda_j} - \frac{\partial W}{\partial \lambda_i} \right) = \frac{1}{4} \quad (E.13)$$

It is trivial to check that this equation is solved by

$$W = \frac{1}{2} \bar{W} \quad (\text{E.14})$$

where $\bar{W}$ is the solution found by Brower and Nauenberg¹³⁾, and Gross and Brezin¹⁴⁾ for the group $U(N)$. The large N solution can be checked explicitly at least in the case when c is proportional to the identity matrix:

$$\begin{aligned} \int dO \exp c \text{Tr } O &= \sum_m \frac{c^m}{m!} \int dO (\text{Tr } O)^m \\ &= \sum_{m \text{ even}} \frac{c^m}{m!} (m-1)!! = \sum_n \frac{c^{2n}}{2n!!} = \exp \frac{c^2}{2} \end{aligned} \quad (\text{E.15})$$

We would like to re-express Eq. (E.8) in terms of the eigenvalues c_i of the matrix H :

$$\left[\frac{\partial^2}{\partial c_i^2} + \sum_{j \neq i} \frac{1}{c_j^2 - c_i^2} \left(c_j \frac{\partial}{\partial c_j} - c_i \frac{\partial}{\partial c_i} \right) \right] Z = Z \quad (\text{E.16})$$

H would be agreeable to have a solution to this equation for finite values of N . Unfortunately in the case $N = 3$ [corresponding to the integral Eq. (E.1) for $SU(2)$ because of the $SU(2)$ - $SO(3)$ isomorphism] we only know how to solve it when c is proportional to the identity

$$\int dO \exp c \text{Tr } O = \exp c \left[I_0(2c) - I_1(2c) \right] \quad (\text{E.17})$$

Let us now go back to the case when the integration is restricted to $SU(N)$. We have computed the first few cumulants for the strong coupling expansion:

$$\int dV K_{\alpha\beta} = 0 \quad (\text{E.18})$$

$$\int dV \quad K_{\alpha\beta} K_{\delta\delta} = \frac{1}{N^2-1} \delta_{\alpha\delta} \delta_{\beta\delta} \quad (E.19)$$

$$\int dV \quad K_{\alpha\beta} K_{\delta\delta} K_{\epsilon\epsilon} = \frac{1}{2N(N^2-1)} f_{\alpha\delta\epsilon} f_{\beta\delta\epsilon} + \frac{N}{2(N^2-1)(N^2-4)} d_{\alpha\delta\epsilon} d_{\beta\delta\epsilon} \quad (E.20)$$

where

$$T_\alpha T_\beta = \frac{1}{N} \delta_{\alpha\beta} + \frac{i}{2} f_{[\alpha\beta\gamma]} T_\gamma + \frac{1}{2} d_{\{\alpha\beta\gamma\}} T_\gamma \quad (E.21)$$

It is immediately obvious that there are many more invariants one can construct for the $SU(N)$ integration than there were for $O(N^2-1)$. Actually the result of performing the $O(N^2-1)$ integration with the integrand of Eq. (E.20) would have been zero for $N > 2$. It is therefore, very hard to compute higher cumulants and get significant information.

It is possibly worth observing that in the special case when $C_{\alpha\beta} = N^2 c P_{\alpha\beta}$, explicit computation shows that

$$\int dV \quad (P_{\alpha\beta} K_{\alpha\beta})^3 = 0 \quad (E.22)$$

and more generally, the strong coupling expansion reduces in this case to a series containing only even powers of C .

This is not, however, an obvious property of Eq. (E.20) and its generalizations. It rather relates to the fact that the trace of an odd number of γ_μ , γ_μ^+ matrices is always zero.

Therefore, we may expect even notable simplifications in the aforementioned case (after all, the only one that has physical interest); but we are not yet able to predict the full consequences of these simplifications.

Finally we would like to present at least one explicitly integrable case, corresponding to $d = 1$ chiral models in the large N limit. Going back to Eq. (D.24) we study the special case

$$\begin{aligned} Z &= \int dV \exp Nc \operatorname{Tr} (\gamma V \gamma^\dagger V^\dagger + \gamma^\dagger V \gamma V^\dagger) = \\ &= \int dU \exp \{ Nc \operatorname{Tr} U + \operatorname{Tr} U^\dagger \} \delta(U, \gamma V \gamma^\dagger V^\dagger) dV \\ &= \int dU \exp \{ Nc \operatorname{Tr} U + \operatorname{Tr} U^\dagger \} \sum_{\{r,s\}} \frac{1}{d_{\{r,s\}}} \chi_{\{r,s\}}^*(U) \chi_{\{r,s\}}(\gamma) \chi_{\{r,s\}}(\gamma^\dagger) \end{aligned}$$

where the sum is over all irreducible representations of $SU(N)$, and $\chi_{\{r\}}$ are the corresponding characters. However, because of the property $\operatorname{Tr} \gamma^n = 0$, $1 < n < N$ in the large N limit, only the trivial representation gives a contribution and therefore

$$Z = \int dU \exp Nc \operatorname{Tr} U + \operatorname{Tr} U^\dagger$$

This integral is well known; its large N limit has been computed many times and is known to correspond to the large N limit of one-dimensional chiral models, as expected (actually the correspondence holds for all values of N).

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