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THE CONTINUUM LIMIT OF ONE-DIMENSIONAL NON-LINEAR MODELS

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ABSTRACT

Group invariant expectation values for lattice non-linear vector models $[O(N) \text{ and } CP^{N-1}]$ in one dimension with a general local action may be expressed as explicit functions of the single-link model expectation values. The value of all these averages at the fixed point of the renormalization group can be determined exactly, in such a way that the continuum expectation values are generated. The large N limit and the reconstruction of the fixed point lattice action are discussed.

THE CONTINUUM LIMIT OF ONE-DIMENSIONAL NON-LINEAR MODELS

The basic problem in the lattice formulation of quantum field theory is the recovery of the continuum limit, which is achieved in principle through the determination of a fixed point for the renormalization group of transformations of the lattice theory.

The analytic determination of the fixed point is generally hopeless. However, when special properties of the model (as the presence of a dimensionful parameter and a simple dependence of the expectation values on space geometry) occur, then an explicit evaluation of the Green's functions becomes sometimes possible.

An example was given a few years ago¹⁾ by considering the two-dimensional version of Yang-Mills theory (and the one-dimensional chiral models that are its immediate counterpart).

Recently, the study of one-dimensional non-linear vector models [$O(N)$ and CP^{N-1} models]²⁾ has shown²⁾ that, at least at the level of the partition function, they enjoy a factorization property similar to that of the previously mentioned examples: $Z_L = (Z_1)^L$, where Z_L is the partition function of an L link open chain; this property depends only on the locality of the lattice action (only nearest neighbour interactions are involved) without specification of the explicit form of the Lagrangian.

This observation prompted the search for a fixed point lattice solution for these non-linear models, and such a solution was obtained in the spirit of Ref. 1), which we follow closely in the present exposition of our results.

Section 1 is devoted to a presentation of the models and a brief discussion of the known results. In Section 2 we consider explicitly $O(N)$ vector models in one dimension, showing that when a general local action is assumed, the group invariant Green's functions may be written as a function of the distances among the points involved and the expectation values of powers of a single-link correlation function. All the dependence on the couplings is hidden in these single-link expectation values. In Section 3 we construct explicitly the invariant Green's functions. Our procedure is based on considering the integration of general tensors over L links and solving the recursion equations relating the coefficients of the resulting integrals for different values of L . General results for $O(N)$ invariant integration are thus obtained. As a by-product the L link invariant Green's functions are obtained as a function of the one-link expectation values and L . In Section 4 we show how one can determine the fixed point value of the single-link averages by a renormalization

group prescription, and discuss the properties of the resulting continuum Green's functions. Section 5 is devoted to CP^{N-1} models and their continuum limit. In Section 6 we discuss the large N limit, the reconstruction of the fixed point action and we draw some conclusions.

1. - NON-LINEAR VECTOR MODELS IN ONE DIMENSION

We shall be concerned with two weak coupling expandable, $1/N$ expandable classes of models: $O(N)$ vector models, with a continuum Lagrangian involving N component constrained real vectors

$$\begin{aligned} \mathcal{L}_R &= \frac{1}{2} \frac{N}{g} \partial_\mu s^i \partial_\mu s^i \\ \sum_{i=1}^N s^i s^i &= 1 \end{aligned} \quad (1.1)$$

and CP^{N-1} models involving N component complex vectors and described by the continuum Lagrangian

$$\begin{aligned} \mathcal{L}_C &= \frac{N}{g} \overline{D_\mu z^i} D_\mu z^i \\ \sum_{i=1}^N \overline{z^i} z^i &= 1 \end{aligned} \quad (1.2)$$

where

$$D_\mu z^i = \partial_\mu z^i - \frac{1}{2} (\overline{z}^\ell \partial_\mu z^\ell - z^\ell \partial_\mu \overline{z}^\ell) z^i \quad (1.3)$$

The standard lattice versions of these models are respectively³⁾

$$Z_R = \int \prod_n \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}} d\vec{s}_n \delta(\vec{s}_n \vec{s}_n - 1) \exp N\beta \sum_{n,\mu} \vec{s}_n \cdot \vec{s}_{n+\mu} \quad (1.4)$$

and

$$Z_C = \int \prod_n \frac{\Gamma(N)}{(2\pi i)^N} dz_n d\bar{z}_n \delta(\bar{z}_n z_n - 1) \exp N\beta \sum_{n,\mu} (\bar{z}_{n+\mu} z_n) (\bar{z}_n z_{n+\mu}) \quad (1.5)$$

with the identification $\beta = 1/g$.

It must, however be noticed that, by choosing a proper value for the lattice couplings, the Lagrangians (1.1) and (1.2) may be obtained as the naive continuum limit of the classes of local actions:

$$S_R = -N \sum_{\eta, \mu} \sum_k \beta_k (\bar{S}_n \cdot \bar{S}_{n+\mu})^k = -N \sum_{\eta, \mu} S_R (\bar{S}_n \cdot \bar{S}_{n+\mu}) \quad (1.6)$$

and, respectively

$$S_c = -N \sum_{\eta, \mu} \sum_k \beta_k [(\bar{z}_{n+\mu} z_n)(\bar{z}_n z_{n+\mu})]^k = -N \sum_{\eta, \mu} f_c ((\bar{z}_{n+\mu} z_n)(\bar{z}_n z_{n+\mu})) \quad (1.7)$$

These generalizations are analogous to the possibility of choosing the characters of the plaquette variable in different combinations of representations of the unitary group for gauge and chiral models.

The factorization property present in the one-dimensional case^{2),4)} amounts to the fact that

$$Z_L = \int \prod_{n=1}^{L+1} d\mu(\bar{S}_n) \exp N \sum_{n=1}^{L+1} f_R(\bar{S}_n \cdot \bar{S}_{n+1}) \quad (1.8)$$

can be integrated due to the property that

$$\int d\mu(\bar{S}_1) \exp N f_R(\bar{S}_1 \cdot \bar{S}_2) = Z_1 \quad (1.9)$$

is independent of the orientation of $\vec{S}_2$ because of the invariance of the measure (in principle, the integration depends on the norm of $\vec{S}_2$ but this is fixed to be equal to one by the constraint).

Therefore, the property

$$Z_L = (Z_1)^L \quad (1.10)$$

holds, and the same is trivially true for CP^{N-1} models.

It is possible to compute Z_1 explicitly for the special choice of the actions given in Eqs. (1.4) and (1.5)^{2),4)}. We quote here again the results:

$$\int d\mu(\bar{S}_1) \exp N\beta \vec{S}_1 \cdot \vec{S}_2 = \Gamma\left(\frac{N}{2}\right) \frac{I_{\frac{N-1}{2}}(N\beta)}{\left(\frac{N\beta}{2}\right)^{\frac{N-1}{2}}} \quad (1.11)$$

$$\int d\mu(z_1, \bar{z}_1) \exp N\beta |\bar{z}_1 z_2|^2 = \Gamma(N) \left[\frac{\exp N\beta}{(N\beta)^{N-1}} - \sum_{n=0}^{N-2} \frac{(N\beta)^{n-N+1}}{n!} \right] \quad (1.12)$$

and the lattice single-link correlation functions are immediately obtained by

$$\langle (\bar{S}_1 \cdot \bar{S}_2)^n \rangle = (Z_1)^{-1} \frac{\partial^n}{\partial (N\beta)^n} Z_{1R} \quad (1.13)$$

$$\langle |\bar{Z}_1 \bar{Z}_2|^{2n} \rangle = (Z_1)^{-1} \frac{\partial^n}{\partial (N\beta)^n} Z_{1C} \quad (1.14)$$

We shall come back to these results later. Here, we only want to mention that the leading orders in g of Eqs. (1.13) and (1.14) can be easily computed and, since we shall need them later, we quote them here to be

$$\langle (\bar{S}_1 \cdot \bar{S}_2)^n \rangle \approx 1 - \frac{N-1}{2N} n g \quad (1.15)$$

$$\langle |\bar{Z}_1 \bar{Z}_2|^{2n} \rangle \approx 1 - \frac{N-1}{N} n g \quad (1.16)$$

coinciding (by construction) with the corresponding continuum expressions in the same order of approximation.

2. - INVARIANT LATTICE CORRELATION FUNCTIONS

The most general invariant lattice $2n$ point Green's function for $O(N)$ models takes the form:

$$G(k_1 \dots k_{2n}) = \langle (\bar{S}_{k_1} \cdot \bar{S}_{k_2}) \dots (\bar{S}_{k_{2n-1}} \cdot \bar{S}_{k_{2n}}) \rangle \quad (2.1)$$

where k_i are arbitrary points (in one dimension, in our case). We are going to show that the expectation value Eq. (2.1) may be expressed simply in terms of the distance (number of links) between the points k_i and of the single-link expectation values

$$\langle (\bar{S}_1 \cdot \bar{S}_2)^n \rangle = (Z_1)^{-1} \int (\bar{S}_1 \cdot \bar{S}_2)^n e^{N\beta R(\bar{S}_1 \cdot \bar{S}_2)} d\mu(\bar{S}_1) \quad (2.2)$$

The crucial observation is the following: whenever we deal with the general $O(N)$ invariant one-link integral, the invariance of the measure ensures us that the result may be expressed in terms of a combination of all the n index tensors that we can construct:

$$(Z_1)^{-1} \int s_k^{i_1} \dots s_k^{i_n} e^{N \sum_R (\bar{s}_k \bar{s}_{k+1})} d\mu(s_k) = \sum_{m=0}^n \frac{B_{mn}}{m!} P s_{k+1}^{i_1} \dots s_{k+1}^{i_m} \delta_{i_{m+1} i_{m+2}} \dots \delta_{i_{n-1} i_n} \quad (2.3)$$

where P means a sum over all possible permutations of the indices $(i_1, \dots, i_n)$.

The scalar coefficients B_{mn} can be obtained by solving appropriate recursion relations and they turn out to be linear functions of the single-link averages $\langle (s_k s_{k+1})^m \rangle$ defined as in Eq. (2.2). Therefore, when we consider the expectation value (2.1), we remember that in one dimension the sites k_i are naturally ordered along the line and we can imagine performing explicitly the integrations in

$$G(k_1 \dots k_{2n}) = Z_1^{-L} \int (\bar{s}_{k_1} \bar{s}_{k_1}) \dots (\bar{s}_{k_{2n-1}} \bar{s}_{k_{2n}}) e^{N \sum_{i=1}^L \sum_R (\bar{s}_i \bar{s}_{i+1})} d\mu(\bar{s}_1) \dots d\mu(\bar{s}_L) \quad (2.4)$$

starting from s_1 and using the factorization property that we have discussed in the previous section.

Every integration involving a given spin $\vec{s}_k$ with free indices as in Eq. (2.3) "moves" the free indices to its neighbour $\vec{s}_{k+1}$ as we have previously discussed, until all indices (since we deal with invariant functions) are contracted between nearest neighbours and the full Green's function is expressed as a combination of single-link averages.

The only other parameter involved is obviously the "distance", (i.e., the number of links) between the sites k_j involved in the definition of the correlation function we are considering. This number, or set of numbers, will appear as an exponent for the combination of single-link averages involved. This completes the proof of our statement. Later we shall explicitly see how this mechanism works.

It is almost needless mentioning that we could use the result of the previous section, Eq. (1.13), and thus obtain an expression for an arbitrary Green's function for the lattice version of the model corresponding to the standard Lagrangian. This is not, however, the spirit of our approach that will instead lead to a determination of $\langle (\vec{s}_1 \vec{s}_2)^n \rangle$ for the fixed point version of the theory.

3. - RECURSION EQUATIONS FOR $O(N)$ LINK INTEGRALS

Let us now consider the general $O(N)$ invariant L link integral, generalizing Eq. (2.3)

$$\begin{aligned} z_1^{-L} \int s_1^{j_1} \dots s_1^{j_n} e^{N \sum_{i=1}^L f_R(\vec{s}_i; \vec{s}_{i+1})} d\mu(s_1) \dots d\mu(s_L) &\equiv \langle s_1^{j_1} \dots s_1^{j_n} \rangle_L \\ &= \sum_{m=0}^n \frac{B_{mn}(L)}{m!} p_{s_{L+1}^{j_1} \dots s_{L+1}^{j_m}} \delta_{j_{m+1} j_{m+2}} \dots \delta_{j_{n-1} j_n} \end{aligned} \quad (3.1)$$

By considering separately the effect of the L^{th} integration and making use of Eq. (2.3) itself, we easily obtain:

$$B_{kn}(L) = \sum_{m \leq k} B_{mn}(L-1) B_{km} \quad (3.2)$$

where it is clear from the definitions that

$$B_{kn}(1) = B_{kn} \quad (3.3)$$

and only the values of k such that $n-k$ is even are allowed in the definition of $B_{kn}(L)$.

Equation (3.2) is a system of ordinary finite-difference linear equations and therefore its general solution including the boundary condition Eq. (3.3) may be expressed in the form

$$B_{kn}(L) = \sum_{\ell=k}^n c_{k\ell} [B_{\ell\ell}]^L c_{\ell n}^{-1} \quad (3.4)$$

where we have used the fact that B_{kn} is an upper triangular "chess-board" matrix and its eigenvalues therefore coincide with its diagonal values $B_{\ell\ell}$. The coefficients $c_{k\ell}$ are determined by the equations

$$B_{kn} = \sum_{\ell=k}^n c_{k\ell} B_{\ell\ell} c_{\ell n}^{-1} \quad (3.5)$$

Let us notice that $c_{k\ell}$ also form an upper triangular chess-board matrix (that is only k such that $\ell-k$ is even correspond to non-zero values) and we can choose the normalization

$$c_{\ell\ell} = 1 \quad (3.6)$$

thus fixing all the residual free parameters.

One can explicitly construct C_{kl} in terms of B_{kn} , obtaining

$$C_{kl} = \sum_{t=0}^{l-k-1} \sum_{\{k < n_1 < \dots < n_t < l\}} \frac{B_{kn_1} B_{n_1 n_2} \dots B_{n_t l}}{(B_{ll} - B_{kk})(B_{ll} - B_{n_1 n_1}) \dots (B_{ll} - B_{n_t n_t})} \quad (3.7)$$

but, as we shall immediately see, there is a more natural way to explicitly compute the coefficients C_{kl} . Let us indeed consider the form assumed by Eq. (3.1) after substitution of Eq. (3.4):

$$\begin{aligned} \langle s_1^{j_1} \dots s_l^{j_l} \rangle_L &= \sum_{m=0}^n \frac{1}{m!} \sum_{\ell=m}^n C_{m\ell} [B_{\ell\ell}]^L C_{\ell n}^{-1} \mathbb{P} s_{L+1}^{j_1} \dots s_{L+1}^{j_m} \delta_{j_{m+1} j_{m+2}} \dots \delta_{j_{n-1} j_n} \\ &= \sum_{\ell=0}^n [B_{\ell\ell}]^L C_{\ell n}^{-1} \left(\sum_{m=0}^{\ell} \frac{1}{m!} C_{m\ell} \mathbb{P} s_{L+1}^{j_1} \dots s_{L+1}^{j_m} \delta_{j_{m+1} j_{m+2}} \dots \delta_{j_{n-1} j_n} \right) \end{aligned} \quad (3.8)$$

thus introducing a natural decomposition in irreducible tensors. Indeed, when we consider the combination

$$T_{j_1 \dots j_\ell}(s) = \sum_{m=0}^{\ell} \frac{1}{m!} C_{m\ell} \mathbb{P} s^{j_1} \dots s^{j_m} \delta_{j_{m+1} j_{m+2}} \dots \delta_{j_{\ell-1} j_\ell} \quad (3.9)$$

we easily find that the use of Eq. (3.8) leads us to

$$Z_1^{-L} \int T_{j_1 \dots j_\ell}(s_1) e^{N \sum_{i=1}^L \int_R (\bar{s}_i \bar{s}_{i+1})} d\mu(s_i) = [B_{\ell\ell}]^L T_{j_1 \dots j_\ell}(s_{L+1}) \quad (3.10)$$

It is possible to reconstruct $T_{j_1, \dots, j_\ell}$ for arbitrary ℓ by making use of a few simple properties. The definition of $T_{j_1, \dots, j_\ell}$ implies it is a completely symmetric tensor, while its irreducibility requires it to be traceless. It is clear that if we can find a unique tensor $T_{j_1, \dots, j_\ell}$ that is completely symmetric and traceless, it will a priori satisfy an equation like Eq. (3.10). Indeed both symmetry and tracelessness are preserved by the integration, and therefore for each link integration the result must have the same tensorial structure as the integrand.

Actually the condition:

$$\delta_{j_1 j_2} T_{j_1 \dots j_\ell} = 0 \quad (3.11)$$

completely determines the coefficients $C_{m\ell}$. We must only observe that

$$\begin{aligned}
 & \delta_{j_1 j_2} P s^{j_1} \dots s^{j_m} \delta_{j_{m+1} j_{m+2}} \dots \delta_{j_{l-1} j_l} = \\
 & = (l-m)(l+m+N-2) P s^{j_3} \dots s^{j_{m+2}} \delta_{j_{m+3} j_{m+4}} \dots \delta_{j_{l-1} j_l} + \\
 & + (m-1)m P s^{j_3} \dots s^{j_m} \delta_{j_{m+1} j_{m+2}} \dots \delta_{j_{l-1} j_l}
 \end{aligned} \tag{3.12}$$

From Eqs. (3.9), (3.11) and (3.12) the following recursion equation follows:

$$(l-m)(l+m+N-2) C_{ml} + C_{m+2,l} = 0 \tag{3.13}$$

and its solution [with boundary condition Eq. (3.6)] is simply

$$C_{ml} = \frac{(-1)^{\frac{l-m}{2}}}{2^{l-m} \frac{l-m}{2}!} \frac{\Gamma(\frac{N}{2} + \frac{l}{2} + \frac{m}{2} - 1)}{\Gamma(\frac{N}{2} + l - 1)} \tag{3.14}$$

A list of the first few irreducible tensors is presented in Appendix A. It is also possible to compute explicitly the inverse matrix:

$$C_{ml}^{-1} = \frac{1}{2^{l-m} \frac{l-m}{2}!} \frac{\Gamma(\frac{N}{2} + m)}{\Gamma(\frac{N}{2} + \frac{l}{2} + \frac{m}{2})} \tag{3.15}$$

Another property that we shall need soon concerns the sums of coefficients C_{mp} . One can show that

$$\sum_{m=0}^l \frac{C_{ml}}{m!} \equiv I_l(1) = \frac{1}{l!} \frac{(N+l-3)!}{(N+2l-4)!! (N-3)!!} \tag{3.16}$$

A sketch of the proofs of Eqs. (3.15) and (3.16) is presented in Appendix B.

In order to determine completely Eq. (3.1), we still must evaluate B_{ll} in terms of single-link averages. For this purpose we define a new class of invariant irreducible Green's functions

$$\begin{aligned}
 I_l^L(\bar{s}_1 \bar{s}_{L+1}) &= \frac{1}{l!} \langle T_{j_1 \dots j_l}(\bar{s}_1) s_{L+1}^{j_1} \dots s_{L+l}^{j_l} \rangle \equiv \\
 &\equiv \sum_{m=0}^l \frac{C_{ml}}{m!} \langle (\bar{s}_1 \bar{s}_{L+1})^m \rangle
 \end{aligned} \tag{3.17}$$

From Eq. (3.9) we obtain

$$I_\ell^L(\bar{S}_1 \bar{S}_{L+1}) = [B_{\ell\ell}]^L I_\ell(1) \quad (3.18)$$

and in particular

$$B_{\ell\ell} = \frac{I_\ell(\bar{S}_1 \bar{S}_2)}{I_\ell(1)} = \frac{\sum_{m=0}^L \frac{C_{m\ell}}{m!} \langle (\bar{S}_1 \bar{S}_2)^m \rangle}{\sum_{m=0}^L \frac{C_{m\ell}}{m!}} \quad (3.19)$$

To complete our analysis, we can now go back to Eq. (3.1) and recognize that

$$\begin{aligned} \langle \frac{(\bar{S}_1 \bar{S}_{L+1})^n}{n!} \rangle &= \sum_{k=0}^n \frac{B_{kn}(L)}{k!} = \sum_{\ell=0}^n (B_{\ell\ell})^L C_{\ell n}^{-1} \left(\sum_{k=0}^L \frac{1}{k!} C_{k\ell} \right) \\ &= \sum_{\ell=0}^n \left[\frac{I_\ell(\bar{S}_1 \bar{S}_2)}{I_\ell(1)} \right]^L I_\ell(1) C_{\ell n}^{-1} \end{aligned} \quad (3.20)$$

and in particular

$$\langle \frac{(\bar{S}_1 \bar{S}_2)^n}{n!} \rangle = \sum_{\ell=0}^n I_\ell(\bar{S}_1 \bar{S}_2) C_{\ell n}^{-1} \quad (3.21)$$

that could have been obtained directly from inversion of Eq. (3.17).

The first few invariant Green's functions are presented in Appendix A. A few comments on the results expressed by Eqs. (3.20) and (3.21) are worth being mentioned. Besides the obvious consistency of the results, that can be checked also by observing that when $L = 0$

$$\lim_{L \rightarrow 0} \langle (\bar{S}_1 \bar{S}_{L+1})^n \rangle = 1 \quad \text{for all } N \quad (3.22)$$

we also notice that if we assume the extreme weak coupling condition

$$\langle (\bar{S}_1 \bar{S}_2)^n \rangle = 1 \quad (3.23)$$

then automatically $B_{nn} = 1$ and

$$\langle (\bar{S}_1 \bar{S}_{L+1})^n \rangle = 1 \quad (3.24)$$

as expected when all spins are aligned and the state is totally ordered.

Another interesting limit is achieved when considering $N = 1$. This corresponds to the one-dimensional Ising model. From Eq. (3.17) it follows that in this case

$$I_0(1) = I_1(1) = 1 \quad I_\ell(1) = 0 \quad \ell \geq 2 \quad (3.25)$$

and therefore

$$\langle (\bar{S}_1 \bar{S}_{L+1})^n \rangle = \begin{cases} 1 & \text{when } n \text{ is even} \\ [\langle \bar{S}_1 \bar{S}_2 \rangle]^L & \text{when } n \text{ is odd} \end{cases} \quad (3.26)$$

as one expected from independent considerations.

Finally, let us consider the limit $N \rightarrow \infty$. In this limit one easily recognizes that

$$C_{m\ell} = C_{m\ell}^{-1} = \delta_{m\ell} \quad (3.27)$$

and therefore

$$B_{\ell\ell} = \langle (\bar{S}_1 \bar{S}_2)^\ell \rangle \quad (3.28)$$

$$\langle (\bar{S}_1 \bar{S}_{L+1})^n \rangle = (B_{nn})^L = \langle (\bar{S}_1 \bar{S}_2)^n \rangle^L \quad (3.29)$$

We can easily understand this result if we recall that due to large N factorization (dominance of the functional integral by a saddle point)

$$\langle (\bar{S}_1 \bar{S}_{L+1})^n \rangle \xrightarrow{N \rightarrow \infty} (\langle \bar{S}_1 \bar{S}_{L+1} \rangle)^n \quad (3.30)$$

and in particular

$$\langle (\bar{S}_1 \bar{S}_2)^n \rangle \rightarrow \langle \bar{S}_1 \bar{S}_2 \rangle^n \quad (3.31)$$

therefore

$$\langle (\bar{S}_1 \bar{S}_{L+1})^n \rangle \xrightarrow{N \rightarrow \infty} [\langle \bar{S}_1 \bar{S}_2 \rangle^L]^n = \langle (\bar{S}_1 \bar{S}_2)^n \rangle^L \quad (3.32)$$

4. - FIXED POINT VALUE OF THE SINGLE-LINK AVERAGES

Now comes the crucial point in our derivation: we have to find the fixed point value of $\langle (\bar{s}_1 \bar{s}_2)^n \rangle$. Recalling that in one dimension the coupling g has the physical dimensions of an inverse length, and that all Green's functions depend on co-ordinates only through the distance between sites, we find that our renormalization group prescription amounts to:

$$\langle (\bar{s}_1 \bar{s}_{L+1})^n \rangle \left(\{ \langle (\bar{s}_1 \bar{s}_2)^m \rangle(g) \}, L \right) \rightarrow \langle (\bar{s}_1 \bar{s}_{L+1})^n \rangle(gL) \quad (4.1)$$

and therefore we must choose $\langle (\bar{s}_1 \bar{s}_2)^m \rangle(g)$ as a function of g in such a way that Eq. (4.1) is satisfied.

Actually, when considering Eq. (3.21), it appears that it is more natural to apply the renormalization group prescription to the irreducible Green's functions:

$$\frac{I_\ell^L(\bar{s}_1 \bar{s}_{L+1})}{I_\ell(1)} = \left(\frac{I_\ell(\bar{s}_1 \bar{s}_2)}{I_\ell(1)} \right)^L \quad (4.2)$$

thus obtaining the dependence

$$\frac{I_\ell(\bar{s}_1 \bar{s}_2)}{I_\ell(1)} = e^{-k_\ell g} \quad (4.3)$$

The free numerical parameters k_ℓ are fixed by the request that g agrees with the continuum coupling, or in other words the weak coupling expansion of $I_\ell(\bar{s}_1 \bar{s}_2)/I_\ell(1)$ must agree with what is predicted starting from Eq. (1.15).

Substituting Eq. (1.15) in Eq. (3.20) we find that

$$\frac{I_\ell(\bar{s}_1 \bar{s}_2)}{I_\ell(1)} \cong 1 - \frac{N-1}{2N} g \frac{\sum_{m=0}^{\ell} \frac{C_{m\ell} m}{m!}}{\sum_{m=0}^{\ell} \frac{C_{m\ell}}{m!}} = 1 - \frac{N-1}{2N} g \ell \left(\frac{N+\ell-2}{N-1} \right) \quad (4.4)$$

The proof of Eq. (4.4) is sketched in Appendix B. As a consequence

$$k_\ell = \ell \frac{N+\ell-2}{2N} \quad (4.5)$$

and we can write down a closed form expression for the fixed point values of the one-link averages:

$$\langle (\bar{s}_1 \bar{s}_2)^n \rangle = \sum_{\ell=0}^{\infty} \frac{I_\ell(1)}{n!} e^{-g \ell \frac{N+\ell-2}{2N}} c_{\ell n}^{-1} \quad (4.6)$$

and we can construct the fixed point invariant Green's functions,

$$\langle (\bar{S}_1 \bar{S}_{1+1})^n \rangle = \sum_{\ell=0}^{\infty} n! I_{\ell}(1) e^{-g L \ell \frac{N+\ell-2}{2N}} C_{\ell n}^{-1} \quad (4.7)$$

Equation (4.7) is now naturally interpreted as the continuum value of the invariant Green's functions for the $O(N)$ model, through the obvious substitutions

$$\langle (\bar{S}(x) \bar{S}(y))^n \rangle = \sum_{\ell=0}^{\infty} n! I_{\ell}(1) e^{-g |x-y| \ell \frac{N+\ell-2}{2N}} C_{\ell n}^{-1} \quad (4.8)$$

and

$$I_{\ell}(\bar{S}(x) \bar{S}(y)) = I_{\ell}(1) e^{-g |x-y| \ell \frac{N+\ell-2}{2N}} \quad (4.9)$$

It is easy to recognize the quantum mechanical meaning of Eqs. (4.8) and (4.9) by recalling that the path integral formulation of quantum mechanics allows for the interpretation

$$Z^{-1} \int \mathcal{D}s(t) e^{-\int_x^y H(t) dt} \psi_{\ell}(s(x)) = e^{-E_{\ell} |y-x|} \psi_{\ell}(s(y)) \quad (4.10)$$

where $\psi_{\ell}(s)$ are Schrödinger wave functions solving the associated quantum mechanical equation

$$H \psi_{\ell}(s) = E_{\ell} \psi_{\ell}(s) \quad (4.11)$$

Therefore, by comparing Eq. (4.10) with the continuum limit of Eq. (3.10), we recognize that the operators $T_{j_1, \dots, j_{\ell}}(s)$ are exactly the Schrödinger wave functions associated with the energy eigenvalues

$$E_{\ell} = \frac{g}{2N} \ell(N+\ell-2) \quad (4.12)$$

In particular, by choosing a parametrization over the N sphere:

$$\bar{S}_1 \bar{S}_2 = \cos \theta \quad (4.13)$$

and expressing Schrödinger's equation Eq. (4.11) in this co-ordinate, we must be able to prove that

$$H I_{\ell}(\cos \theta) = \frac{g}{2N} \ell(N+\ell-2) I_{\ell}(\cos \theta) \quad (4.14)$$

where

$$H = \frac{N}{2g} \left(\frac{\partial \vec{S}}{\partial t} \right)^2 = \frac{g}{2N} \vec{P}^2 \rightarrow -\frac{g}{2N} \left[\frac{\partial^2}{\partial \theta^2} + (N-2) \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} \right] \quad (4.15)$$

It is immediate to recognize that

$$-\left[\frac{\partial^2}{\partial \theta^2} + (N-2) \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} \right] P_\ell(\cos \theta) = \ell(N+\ell-2) P_\ell(\cos \theta) \quad (4.16)$$

admits solutions that are polynomials in $\cos \theta$ (generalized Legendre polynomials)

$$P_\ell(\cos \theta) = \sum_{m=0}^{\ell} \frac{C_{m\ell}}{m!} (\cos \theta)^m \quad (4.17)$$

whose coefficients obey the recursion equation

$$C_{m+2,\ell} + (\ell-m)(N+\ell+m-2)C_{m\ell} = 0 \quad (4.18)$$

exactly the same as in Eq. (3.13).

5. - THE CP^{N-1} MODELS

The analysis given in Section 3 can be repeated with minor modifications in the case of the CP^{N-1} models. We recognize that

$$\begin{aligned} & Z_{1c}^{-L} \int z_1^{i_1} \dots z_n^{i_n} \bar{z}_1^{j_1} \dots \bar{z}_n^{j_n} e^{N \sum_{i=1}^L f(|\bar{z}_i z_{i+1}|^2)} d\mu(z_1, \bar{z}_1) \dots d\mu(z_L, \bar{z}_L) = \\ & = \sum_{m=0}^n \frac{B_{mn}(L)}{(m!)^2} P_i P_j z_1^{i_1} \dots z_n^{i_n} \bar{z}_1^{j_1} \dots \bar{z}_n^{j_n} \delta_{i_{m+1} j_{m+1}} \dots \delta_{i_n j_n} \end{aligned} \quad (5.1)$$

and Eqs. (3.4)-(3.7) formally still hold, with the specification that indices now run over all integer values.

We can define irreducible tensors

$$T_{i_1 \dots i_n j_1 \dots j_n} = \sum_{k=0}^n \frac{C_{kn}}{(k!)^2} P_i P_j z_1^{i_1} \dots z_n^{i_n} \bar{z}_1^{j_1} \dots \bar{z}_n^{j_n} \delta_{i_{k+1} j_{k+1}} \dots \delta_{i_n j_n} \quad (5.2)$$

and impose the tracelessness condition

$$\delta_{i_n j_n} T_{i_1 \dots i_n j_1 \dots j_n} = 0 \quad (5.3)$$

implying the recursion relations

$$(n-k)(n+k+N-1)C_{kn} + C_{k+1,n} = 0 \quad (5.4)$$

that are solved by

$$C_{kn} = \frac{(-1)^{n-k}}{(n-k)!} \frac{\Gamma(N+n+k-1)}{\Gamma(N+2n-1)} \quad (5.5)$$

which inverse is (see Appendix B):

$$C_{kn}^{-1} = \frac{1}{(n-k)!} \frac{\Gamma(N+2k)}{\Gamma(N+n+k)} \quad (5.6)$$

and the following property holds (see Appendix B):

$$\sum_{k=0}^n \frac{C_{kn}}{(k!)^2} = \frac{1}{(n!)^2} \frac{[(N+n-2)!]^2}{(N+2n-2)!(N-2)!} \quad (5.7)$$

The first few irreducible tensors are presented in Appendix A. We can define irreducible Green's functions

$$\begin{aligned} I_n^L(|\bar{z}_1, \bar{z}_{L+1}|^2) &= \frac{1}{(n!)^2} \langle T_{i_1 \dots i_n j_1 \dots j_n}(\bar{z}_1) \bar{z}_{L+1}^{i_1} \dots \bar{z}_{L+1}^{i_n} z_{L+1}^{j_1} \dots z_{L+1}^{j_n} \rangle \equiv \\ &\equiv \sum_{k=0}^n \frac{C_{kn}}{(k!)^2} \langle |\bar{z}_1, \bar{z}_{L+1}|^{2n} \rangle \end{aligned} \quad (5.8)$$

and obtain

$$B_{ee} = \frac{I_e(|\bar{z}_1, \bar{z}_2|^2)}{I_e(1)} = \frac{\sum_{k=0}^n \frac{C_{kn}}{(k!)^2} \langle |\bar{z}_1, \bar{z}_2|^{2n} \rangle}{\sum_{k=0}^n \frac{C_{kn}}{(k!)^2}} \quad (5.9)$$

$$\frac{\langle |\bar{z}_1, \bar{z}_2|^{2n} \rangle}{(n!)^2} = \sum_{k=0}^n I_e(|\bar{z}_1, \bar{z}_2|^2) C_{kn}^{-1} \quad (5.10)$$

In particular,

$$B_{11} = \frac{N \langle |\bar{z}_1 z_2|^2 \rangle - 1}{N-1} \quad (5.11a)$$

$$B_{22} = \frac{2 - 4(N+1) \langle |\bar{z}_1 z_2|^2 \rangle + (N+1)(N+2) \langle |\bar{z}_1 z_2|^4 \rangle}{N(N-1)} \quad (5.11b)$$

The renormalization group prescription now amounts to

$$B_{\ell\ell} = e^{-k'_\ell g} \quad (5.12)$$

and computation shows (see Appendix B) that

$$k'_\ell = \ell \frac{N+\ell-1}{N} \quad (5.13)$$

Therefore, the continuum solution of the CP^{N-1} models in one dimension is

$$\langle |\bar{z}(x) z(y)|^{2n} \rangle = \sum_{\ell=0}^n (n!)^2 I_\ell(1) e^{-g|x-y|^\ell} \frac{N+\ell-1}{N} C_{\ell n}^{-1} \quad (5.14)$$

6. - DISCUSSION AND CONCLUSIONS

The fact that we are able to reconstruct the continuum Green's functions for one-dimensional non-linear vector models leads us naturally to the presumption that we have indeed found the fixed point lattice action for these models and that we should be able to reconstruct it (at least implicitly) from our results.

First of all, it is clear from our procedure that this action will be local on the lattice. Then it is clear that, for arbitrary N , we can write down the following (defining) equation for the fixed point action $f(\bar{s}_1, \bar{s}_2, g)$:

$$Z_1^{-1} \int (\bar{s}_1, \bar{s}_2)^n e^{N f(\bar{s}_1, \bar{s}_2, g)} d\mu(s_1) = \langle (\bar{s}_1 \bar{s}_2)^n \rangle \quad (6.1)$$

where $\langle (\bar{s}_1 \bar{s}_2)^n \rangle_c$ is extracted from Eq. (4.6).

Furthermore, we can use the invariance of the measure to parametrize the integration in terms of the relative angle $\cos \theta = \vec{s}_1 \vec{s}_2$:

$$\frac{\int_{-\pi/2}^{\pi/2} \omega^n \theta e^{Nf(\omega\theta, g)} \sin^{N-2} \theta d\theta}{\int_{-\pi/2}^{\pi/2} e^{Nf(\omega\theta, g)} \sin^{N-2} \theta d\theta} = \langle (\bar{s}_1 \bar{s}_2)^n \rangle \quad (6.2)$$

which allows us to determine

$$\begin{aligned} z^{-1} e^{Nf(\omega\theta, g)} \sin^{N-3} \theta &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip\omega\theta} \sum_{\ell=0}^{\infty} e^{-g^{\ell} \frac{N+\ell-2}{2N}} \sum_{n=\ell}^{\infty} (ip)^n I_{\ell}(1) C_{ln}^{-1} \\ &= \sum_{\ell=0}^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{N-1}{2})} \frac{(N+\ell-3)!(N+2\ell-2)}{\ell!} e^{-g^{\ell} \frac{N+\ell-2}{2N}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip\omega\theta} \frac{(i)^{\ell} J_{N+\ell-1}(p)}{(2p)^{\frac{N}{2}-1}} \\ &= \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})} \sum_{\ell=0}^{\infty} \frac{2^{\ell} \Gamma(\frac{N}{2}+\ell)}{\Gamma(\frac{N}{2})} e^{-g^{\ell} \frac{N+\ell-2}{2N}} I_{\ell}(\omega\theta) \sin^{N-3} \theta \end{aligned} \quad (6.3)$$

In particular, when $N \rightarrow \infty$ as we have seen

$$\langle (\bar{s}_1 \bar{s}_2)^n \rangle \rightarrow e^{-\frac{n}{2}g} \quad (6.4)$$

and therefore

$$\lim_{N \rightarrow \infty} z^{-1} e^{Nf(\omega\theta, g)} \sin^{N-3} \theta = \delta(e^{-\frac{g}{2}} - \omega\theta) \quad (6.5)$$

which shows that the large N saddle point (master field) of this model can be expressed through the equation

$$(\bar{s}_1 \bar{s}_2) = e^{-\frac{g}{2}} \quad (6.6)$$

A similar analysis can obviously be extended to CP^{N-1} models. The derivation of the explicit formulas for the fixed point action of the two models is presented in Appendix C.

What are the possible implications and uses of these results ? We do not need to stress the formal interest of having analytically solved yet another class of quantum mechanical systems with a systematic lattice renormalization group approach. Equation (4.8) in particular, is the complete solution

of the quantum mechanical problem corresponding to the free movement of a massless particle constrained to live on the surface of the N sphere: Schrödinger wave functions emerge naturally as eigenvectors of the lattice transfer matrix.

Besides that, we believe that the fixed point action defined through the procedure described by Eq. (6.3), (or some approximation to it), can be used as a better starting point in order to improve the lattice action even in higher dimensions. This action is the vector model correspondent of the heat-kernel type of action for lattice gauge theories⁵⁾. It is reasonable to expect better properties of convergence to the continuum from an action that is exact at least in some low dimensionality, even if it is known that in higher dimensions a purely local lattice action will always fail to reproduce the exact (asymptotically free) behaviour of the continuum theory⁶⁾.

APPENDIX A

Here, we present explicitly some formulas which appeared in a compact form throughout the text. The first few irreducible tensors which are relevant in the study of the $O(N)$ models are

$$\begin{aligned}
 T_{(0)} &= 1 \\
 T_{j_1} &= S_{j_1} \\
 T_{j_1 j_2} &= S_{j_1} S_{j_2} - \frac{1}{N} \delta_{j_1 j_2} \\
 T_{j_1 j_2 j_3} &= S_{j_1} S_{j_2} S_{j_3} - \frac{1}{N+2} [S_{j_1} \delta_{j_2 j_3} + S_{j_2} \delta_{j_1 j_3} + S_{j_3} \delta_{j_1 j_2}] \\
 T_{j_1 j_2 j_3 j_4} &= S_{j_1} S_{j_2} S_{j_3} S_{j_4} - \frac{1}{N+4} [S_{j_1} S_{j_2} \delta_{j_3 j_4} + S_{j_3} S_{j_4} \delta_{j_1 j_2} \\
 &\quad + S_{j_1} S_{j_3} \delta_{j_2 j_4} + S_{j_2} S_{j_4} \delta_{j_1 j_3} + S_{j_1} S_{j_4} \delta_{j_2 j_3} + S_{j_2} S_{j_3} \delta_{j_1 j_4}] \\
 &\quad + \frac{1}{(N+2)(N+4)} [\delta_{j_1 j_2} \delta_{j_3 j_4} + \delta_{j_1 j_3} \delta_{j_2 j_4} + \delta_{j_1 j_4} \delta_{j_2 j_3}]
 \end{aligned} \tag{A.1}$$

The first few invariant Green's functions are

$$\begin{aligned}
 \langle \bar{S}_1 \bar{S}_{L+1} \rangle &= (B_{11})^L \\
 \langle (\bar{S}_1 \bar{S}_{L+1})^2 \rangle &= \frac{1}{N} + \frac{N-1}{N} (B_{22})^L \\
 \langle (\bar{S}_1 \bar{S}_{L+1})^3 \rangle &= \frac{3}{N+2} (B_{11})^L + \frac{N-1}{N+2} (B_{33})^L \\
 \langle (\bar{S}_1 \bar{S}_{L+1})^4 \rangle &= \frac{3}{N(N+2)} + \frac{6(N-1)}{N(N+4)} (B_{22})^L + \frac{N^2-1}{(N+2)(N+4)} (B_{44})^L
 \end{aligned} \tag{A.2}$$

where

$$\begin{aligned}
 B_{11} &= \langle \bar{S}_1 \bar{S}_2 \rangle \\
 B_{22} &= \frac{N \langle (\bar{S}_1 \bar{S}_2)^2 \rangle - 1}{N-1} \\
 B_{33} &= \frac{(N+2) \langle (\bar{S}_1 \bar{S}_2)^3 \rangle - 3 \langle \bar{S}_1 \bar{S}_2 \rangle}{N-1} \\
 B_{44} &= \frac{3 - 6(N+2) \langle (\bar{S}_1 \bar{S}_2)^2 \rangle + (N+2)(N+4) \langle (\bar{S}_1 \bar{S}_2)^4 \rangle}{N^2-1}
 \end{aligned} \tag{A.3}$$

The first few irreducible tensors appearing in the CP^{N-1} models are

$$T_{(0)} = 1$$

$$T_{i,j_1} = z_{i_1} \bar{z}_{j_1} - \frac{1}{N} \delta_{i_1 j_1}$$

$$\begin{aligned} T_{i_1 i_2 j_1 j_2} = & z_{i_1} z_{i_2} \bar{z}_{j_1} \bar{z}_{j_2} - \frac{1}{N+2} [z_{i_1} \bar{z}_{j_1} \delta_{i_2 j_2} + z_{i_1} \bar{z}_{j_2} \delta_{i_2 j_1} \\ & + z_{i_2} \bar{z}_{j_1} \delta_{i_1 j_2} + z_{i_2} \bar{z}_{j_2} \delta_{i_1 j_1}] \\ & + \frac{1}{(N+2)(N+4)} [\delta_{i_1 j_1} \delta_{i_2 j_2} + \delta_{i_1 j_2} \delta_{i_2 j_1}] \end{aligned} \quad (A.4)$$

APPENDIX B

Throughout the text, we have given several formulas without proof. This appendix is devoted to the presentation of arguments which will make the proofs easier. They are straightforward but not always obvious.

First of all, let us mention two identities among the symmetric and traceless tensors introduced in the text

$$s_{j_1} T_{j_1 \dots j_\ell}(s) = \frac{N+\ell-3}{N+2\ell-4} T_{j_2 \dots j_\ell}(s) \quad \text{for } O(N) \quad (B.1)$$

$$\bar{z}_{i_1} z_{j_1} T_{i_1 \dots i_\ell j_1 \dots j_\ell}(\bar{z}, z) = \frac{(N+\ell-2)^2}{(N+2\ell-2)(N+2\ell-3)} T_{i_2 \dots i_\ell j_2 \dots j_\ell}(\bar{z}, z) \quad \text{for } CP^{N-1} \quad (B.2)$$

$$\frac{\partial}{\partial s_{j_1}} T_{j_1 \dots j_\ell}(s) = \frac{(N+\ell-2)(N+\ell-3)}{(N+2\ell-4)} T_{j_2 \dots j_\ell}(s) \quad \text{for } O(N) \quad (B.3)$$

$$z_{j_1} \frac{\partial}{\partial \bar{z}_{i_1}} T_{i_1 \dots i_\ell j_1 \dots j_\ell}(\bar{z}, z) = \frac{(N+\ell-2)^2(N+\ell-1)}{(N+2\ell-2)(N+2\ell-3)} T_{i_2 \dots i_\ell j_2 \dots j_\ell}(\bar{z}, z) \quad \text{for } CP^{N-1} \quad (B.4)$$

Furthermore, if we denote by $t_{i_1, \dots, i_\ell}^{\ell, m}$ the elementary symmetric tensor (not traceless) with degree n in the spin variable, which enters in the definition of $T_{i_1, \dots, i_\ell}$, we get the following identity

$$\begin{aligned} \frac{\partial}{\partial s_{j_1}} t_{j_1 \dots j_\ell}^{\ell, m} &= m(N+\ell-m-1) t_{j_2 \dots j_\ell}^{\ell-1, m-1} - m(\ell-m) t_{j_2 \dots j_\ell}^{\ell-1, m+1} \\ z_{j_1} \frac{\partial}{\partial \bar{z}_{i_1}} t_{i_1 \dots i_\ell j_1 \dots j_\ell}^{\ell, m} &= m^2(N+\ell-m-1) t_{i_2 \dots i_\ell j_2 \dots j_\ell}^{\ell-1, m-1} + m(\ell-m)(N+\ell-2m-2) t_{i_2 \dots i_\ell j_2 \dots j_\ell}^{\ell-1, m} \\ &\quad - m(\ell-m)(\ell-m-1) t_{i_2 \dots i_\ell j_2 \dots j_\ell}^{\ell-1, m+1} \end{aligned} \quad (B.5)$$

It is easy to understand the identities (B.1)-(B.4) thanks to the unique character of the T tensors. The computation of the coefficients of proportionality is straightforward. However, to establish (B.3), [respectively, (B.4)] one has to remember that the spin variables s_i (respectively, z_i) are subject to the constraint $|s|^2 = 1$ (respectively $|z|^2 = 1$) which requires the following formulas of derivation

$$\frac{\partial}{\partial s_i} s_j = \delta_{ij} - s_i s_j \quad (B.6)$$

$$\frac{\partial}{\partial \bar{z}_i} z_j = \delta_{ij} - \bar{z}_i z_j \quad (B.7)$$

The identities (B.1) and (B.2) furnish a proof by induction for the summation formulas (3.16) and (5.7) respectively. For instance, in the $O(N)$ case, we have

$$\begin{aligned} I_\ell(1) &= \frac{1}{\ell!} s_{j_1} \dots s_{j_\ell} T_{j_1 \dots j_\ell} \\ &= \frac{1}{\ell!} \frac{N+\ell-3}{N+2\ell-4} s_{j_2} \dots s_{j_\ell} T_{j_2 \dots j_\ell} = \frac{N+\ell-3}{\ell(N+2\ell-4)} I_{\ell-1}(1) \end{aligned} \quad (B.8)$$

The demonstration of the summation formula in (4.4) requires the use of the identities (B.3) and (B.5). Contracting (B.3) with $s_{j_2}, \dots, s_{j_\ell}$ and using (3.16), (B.5) leads to the following

$$\ell!(N+\ell-2) I_\ell(1) = (\ell-1)! \left[\sum_{m=0}^{\ell} m(N+\ell-m-1) \frac{C_{m\ell}}{m!} - \sum_{m=0}^{\ell} m(\ell-m) \frac{C_{m\ell}}{m!} \right] \quad (B.9)$$

and after some elementary algebra it can be shown that

$$\sum_{m=0}^{\ell} m \frac{C_{m\ell}}{m!} = I_\ell(1) \frac{\ell(N+\ell-2)}{N-1} \quad (B.10)$$

which clearly proves (4.4).

For (5.9), the analogous formula in CP^{N-1} models, the proof works in the same way using (B.4).

Now, let us consider the inversion formula for the matrix C_{pm} , (3.15) and (5.6) respectively, for $O(N)$ and CP^{N-1} models. We have found it more convenient to first prove it for CP^{N-1} . Indeed, once (5.6) is established, (3.15) can also be derived, once we remark that

$$C_{2m, 2\ell}^{O(2N)} = \frac{1}{2^{2\ell-2m}} C_{m\ell}^{CP^{N-1}} \quad (B.11)$$

$$C_{m+\ell, \ell}^{O(N)} = C_{m\ell}^{O(N+2)} \quad (B.12)$$

To prove (3.15), we have to show that

$$\sum_{n=k}^{\ell} C_{kn} C_{n\ell}^{-1} = \sum_{n=k}^{\ell} \frac{(-1)^{n-k}}{(n-k)!(\ell-n)!} \frac{(N+n+k-2)!(N+2n-1)!}{(N+n+\ell-1)!} = \delta_{k\ell} \quad (B.13)$$

with the new variables d, m, q defined by

$$\begin{aligned} d &= \ell - k \\ m &= n - k \\ q &= N + 2k - 1 \end{aligned} \quad (B.14)$$

Equation (B.13) can be written as

$$\frac{1}{d!} \sum_{m=0}^d (-1)^m \binom{d}{m} \frac{(q+m-1)! (q+2m)}{(q+d+m)!} = \delta_{d0} \quad (B.15)$$

Multiplying both sides of Eq. (B.15) by $(x/2)^{2d}$, summing over d and exchanging the two summation symbols leads to the identity (B.16) below which we have found in Ref. 7) [see Eq. (8.512-2)]

$$\left(\frac{x}{2}\right)^{-q} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (q+m-1)! (q+2m) I_{q+2m}(x) = 1 \quad (B.16)$$

where

$$I_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+q}}{(q+n)! n!}$$

is the Bessel function of imaginary argument.

This terminates the proof.

APPENDIX C

The reconstruction of the fixed point action can be achieved in a very straightforward way that does not involve taking inverse Fourier transforms as in Section 6.

We simply notice that, following our definitions and Eqs. (3.17), (3.18) and (4.3)

$$\int_{-\pi/2}^{\pi/2} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})} \sin^{N-2} \theta d\theta e^{Nf(\omega\theta, g)} I_\ell(\omega\theta) = e^{-g\ell \frac{N+\ell-2}{2N}} I_\ell(1) \quad (C.1)$$

We now observe that $I_\ell(\cos \theta)$ are orthogonal polynomials with respect to the measure

$$\frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})} \sin^{N-2} \theta d\theta \quad (C.2)$$

[they are related by a multiplicative factor to the Gegenbauer polynomials $C_\ell^{N/2-1}(\cos \theta)$] and their normalization is given by

$$\int_{-\pi/2}^{\pi/2} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})} \sin^{N-2} \theta d\theta I_\ell(\omega\theta) I_{\ell'}(\omega\theta) = \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})} \frac{2\pi \Gamma(N+\ell-2)}{2^{N+\ell-2} \ell! \Gamma(\frac{N}{2}+\ell) \Gamma(\frac{N}{2}+\ell-1)} \delta_{\ell\ell'} \quad (C.3)$$

Therefore, it is clear that the fixed point action must have the structure

$$e^{Nf(\omega\theta, g)} = \sum_{\ell=0}^{\infty} K_\ell I_\ell(\omega\theta) e^{-g\ell \frac{N+\ell-2}{2N}} \quad (C.4)$$

and the coefficients K_ℓ are determined by the condition

$$K_\ell \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})} \frac{2\pi \Gamma(N+\ell-2)}{2^{N+\ell-2} \ell! \Gamma(\frac{N}{2}+\ell) \Gamma(\frac{N}{2}+\ell-1)} = I_\ell(1) = \frac{\Gamma(N+\ell-2) \Gamma(\frac{1}{2})}{2^{N+\ell-3} \ell! \Gamma(\frac{N-1}{2}) \Gamma(\frac{N}{2}+\ell-1)} \quad (C.5)$$

$$K_\ell = 2^\ell \frac{\Gamma(\frac{N}{2}+\ell)}{\Gamma(\frac{N}{2})} \quad (C.6)$$

As a by-product of our analysis let us consider the Abelian case $O(2)$. It is easy to show that the limit $N \rightarrow 2$ of the polynomials $I_\ell(\cos \theta)$ is defined and yields:

$$I_\ell(\cos \theta) \rightarrow \frac{2 \cos \ell \theta}{2^\ell \ell!} \quad \ell \neq 0 \quad (C.7)$$

Therefore, the fixed point action is

$$e^{2f(\cos \theta, g)} = 1 + \sum_{\ell=1}^{\infty} 2 \cos \ell \theta e^{-g \frac{\ell^2}{4}} = \sum_{\ell=-\infty}^{\infty} e^{-g \frac{\ell^2}{4}} e^{i \ell \theta} \quad (C.8)$$

that is exactly the Villain action.

The same procedure can be successfully applied to CP^{N-1} models. The crucial observation that allows us to perform the corresponding analysis is the fact that the natural variable in CP^{N-1} models is best parametrized in the form

$$|\bar{z}_1 z_2|^2 = \frac{1 + \cos \theta}{2} = \cos^2 \frac{\theta}{2} \quad (C.9)$$

After this change of variables one recognizes that the polynomials $I_\ell(|\bar{z}_1 z_2|^2)$ are related (up to a normalization) to the Jacobi polynomials $P_\ell^{(N-2, 0)}(\cos \theta)$. In particular, it appears that when $N = 2$:

$$P_\ell^{(0, 0)}(\cos \theta) = P_\ell(\cos \theta) = C_\ell^{\frac{1}{2}}(\cos \theta) \quad (C.10)$$

reflecting the equivalence between CP^1 and $O(3)$ models [and motivating our parametrization Eq. (C.9)].

The polynomials $I_\ell(\frac{1+\cos \theta}{2})$ are orthogonal with respect to the normalized measure

$$\frac{N-1}{2^{N-1}} (1 - \cos \theta)^{N-2} d(\cos \theta) \quad (C.11)$$

and their normalization is given by

$$\int \frac{N-1}{2^{N-1}} (1 - \cos \theta)^{N-2} d(\cos \theta) I_\ell(\frac{1+\cos \theta}{2}) I_{\ell'}(\frac{1+\cos \theta}{2}) = \frac{N-1}{N+2\ell-1} \frac{[(N+\ell-2)!]^2}{[(N+2\ell-2)!] (\ell!)^2} \delta_{\ell \ell'} \quad (C.12)$$

by repeating the previous argument we find therefore

$$e^{N f_c(\omega\theta, g)} = \sum_{l=0}^{\infty} \frac{(N+2l-1)!}{(N-1)!} e^{-g^l \frac{N+l-1}{N}} I_l\left(\frac{1+\omega\theta}{2}\right) \quad (C.13)$$

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