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QUANTUM ORIGIN OF SPONTANEOUS COMPACTIFICATION

Claudio A. Orzalesi

CERN -- Geneva

and

Sezione Teorica, Dipartimento di Fisica dell'Università
43100 Parma, and INFN, Sezione di Milano, Milano, Italy

and

Giovanni Venturi

Dipartimento di Fisica dell'Università and INFN, Bologna
Italy

ABSTRACT

The possible quantum dynamical origin of spin torsion spontaneous compactification is investigated. The Yang-Mills coupling constant is calculated dynamically in a simple model, in terms of a finite cut-off.

A basic ingredient of the modern multidimensional unified theories ¹⁾⁻⁶⁾ is the occurrence of spontaneous compactification ^{4),6)}. The background geometry, which describes "empty space" in $D=4+K$ dimensions, is not trivial but rather corresponds to $M_4^0 \times V^K$, where M_4^0 is $(\text{Minkowski})_4$ and $V^K \simeq G/H$, where G is a non-Abelian compact group and H a subgroup. As is well known, if spontaneous compactification occurs then the zero modes of the metric ^{*}) γ_{MN} on the background $M_4^0 \times V^K$ yield, in four dimensions, one graviton coupled to the Yang-Mills fields of G and possibly to generalized ³⁾ Jordan-Thiry ²⁾ scalar fields.

As explained in Ref. 6), there are two basic mechanisms which can lead to a non-trivial spontaneous compactification :

- (A) a suitable energy momentum condensate in the quantum ground state (q vacuum $|0\rangle$), and/or
- (B) a suitable spin torsion condensate in the q vacuum.

Mechanism (A) can be effective already at the classical level, through suitable q vacuum expectation values of bosonic fields ^{**)}. On the other hand, (B) is typically a quantum mechanism because, in Einstein-Cartan (EC) theories, the spin density leading to torsion is due to local fermion bilinears ⁷⁾. The two mechanisms, (A) and (B), could also operate together : a remarkable example is given by $D=11$ supergravity ⁸⁾, where a phase with a "bosonic" compactification $M_4^A \times V^7$, see Ref. 9), over anti-de Sitter space-time M_4^A , can be followed ¹⁰⁾ by a phase with spin-torsion compactification over $M_4^0 \times V^7$. We recall that mechanism (B) leads to a vanishing cosmological constant ¹¹⁾ and to fermionic zero mass modes in four dimensions ⁶⁾.

By considering the q vacuum expectation values of the EC equations, a further comparison is possible between mechanisms (A) and (B), if one separates the Riemannian Einstein tensor from the piece due to torsion : with (A) one assumes that the torsion contribution vanishes but that the Riemannian Einstein tensor and the Riemannian energy momentum tensor are not trivial in the q vacuum. With (B), the torsion contribution is not vanishing but the

*) Capital Latin indices run from 1 to D ; Greek from 1 to 4 and refer to M_4^0 ; lower-case Latin run from 5 to $D=4+K$ and refer to V^K . "Early" indices are anholonomic, "late" indices are holonomic. We write $X=(x,y)$ for $X \in M_4^0 \times V^K$, $x \in M_4^0$, $y \in V^K$.

**) However, known classical models ⁴⁾ with the flat base space M_4^0 require the fine tuning of a cosmological constant and Yang-Mills fields in D dimensions. Such fields defeat the purpose of a Kaluza-Klein unification ^{1),6)}.

full Einstein-Cartan tensor and the canonical energy-momentum tensor are both vanishing in the q vacuum. Therefore, in approaches based on (A), one is neglecting the possibility of a non-trivial q vacuum spin energy, while with (B) it is assumed that a q vacuum spin energy is crucial to make the background geometry stable *).

The occurrence of non-trivial vacuum expectation values of some components of the spin density is akin to the phenomenon of dynamical symmetry breaking. In fact, the spin-spin contact interaction in the EC theory is similar to the Nambu-Jona Lasinio (N-JL) chiral coupling ¹³⁾, and the functional methods previously used ¹⁴⁾ for the N-JL model are easily generalized ¹²⁾, so as to deal with torsion dynamics in the quantized EC theory.

In the present paper, such analogies are pursued further, in order to draw some indications on the possible dynamical origin of spin torsion spontaneous compactification. We seek a self-consistent approximation for the spin-spin contact interaction, and we formulate a generalized N-JL consistency condition. In essence, this is an "anti-gap" equation: we require that the lowest bare fermion mass, which inevitably occurs in the Riemannian case ^{15),6)}, is driven down to zero physical mass by the self-energy due to torsion **). In a simple seven-dimensional model, we show that this leads precisely to the parallelizing torsion on S^3 , as advocated in Ref. 6). The consistency of the approach is then checked, as in Refs. 6),11), by showing that $M_4^0 \times S^3$ is indeed a solution of the background EC equations.

As in Ref. 12), we isolate the "Riemannian" gravity effects and we deal with the path integral

$$Z(E^A) = \int \mathcal{D}K \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp i(\mathcal{A}_K + \mathcal{A}_D), \quad (1)$$

where $E^A = E^A_M dx^M$ is an external vielbein (with $A, M = 1, \dots, D$ and with metric $\gamma_{MN} = \eta_{AB} E^A_M E^B_N$), K_{ABC} is the contorsion, and, with $\mathcal{I} = E^1 \wedge \dots \wedge E^D$,

*) As emphasized in Ref. 12), the conventional arguments for studying the stability of background geometries are wholly inadequate when the spin density can develop a non-trivial vacuum expectation value.

**) There is a four-dimensional analogue to this screening of the bare mass: if one introduces a static background torsion along the z axis, then the Fermi sphere becomes depleted along the other two axes ¹⁶⁾. Similarly, a background torsion in the internal space V^K lowers the fermion masses as seen on M_4^0 . The resulting change in the zero-point energy of spinors bears resemblance to the well-known ^{17),18)} Casimir effect.

$$\mathcal{A}_K = -\frac{1}{2\hat{\kappa}} \int \underline{K} \cdot \underline{K} \, \underline{1} \quad , \quad (2)$$

$$\mathcal{A}_D = -\frac{1}{2} \int \bar{\Psi} \{ \nabla_A(\omega), \Gamma^A \} \Psi \, \underline{1} \quad . \quad (3)$$

$\mathcal{A}_K$ and $\mathcal{A}_D$ are the contributions to the EC action due to the contorsion and to a Dirac field Ψ which is "massless" in D dimensions. As usual

$$\nabla_A(\omega) = \nabla_A(\overset{\circ}{\omega}) - \frac{1}{4} K_{ABC} \Gamma^{BC} \quad , \quad (4)$$

where $\overset{\circ}{\omega}_{ABC}$ is the Levi-Civita connection of E^A , and $\hat{\kappa}$ is the D dimensional analogue of Newton's constant κ (we shall later set $\hat{\kappa} = \kappa V$, where V is the volume of V^K).

We consider a simple model in $D=7$, and we choose

$$(E^A) = (e^\alpha, e^a) : \quad e^\alpha_M = e^\alpha_\mu(x) \delta^\mu_M, \quad e^a_M = e^a_m(y) \delta^m_M \quad , \quad (5)$$

where $\alpha, \mu = 1, \dots, 4$ and $a, m = 5, 6, 7$, with co-ordinates $(X^M) = (x^\mu, y^m)$. It will be assumed that $e^a_m(y) dy^m$ is a three-bein on S^3 ; by identifying S^3 with $SU(2)$, we choose e^a to be the left-invariant Maurer-Cartan forms on $SU(2)$. We shall later verify that the choice (5) with $e^\alpha_\mu = \delta^\alpha_\mu$, which corresponds to $M_4^0 \times S^3$, is indeed a solution of the background EC equations, after a suitable background torsion has been generated.

The Cartan field equation gives

$$K_{ABC} = \frac{\hat{\kappa}}{4} \bar{\Psi} \Gamma_{ABC} \Psi = K_{[ABC]} = \frac{1}{2} T_{ABC} \quad , \quad (6)$$

where T_{ABC} is the torsion. In particular, let us consider the "internal" components K_{abc} , $a, b, c = 5, 6, 7$: these are the only components of K_{ABC} which, on the background $e^\alpha_\mu = \delta^\alpha_\mu$, can develop a q vacuum expectation value consistently with the $(\text{Poincaré})_4^0$ -invariance of M_4^0 .

Now, if one expands $K_{abc}(x, y)$ in harmonics over S^3 ,

$$K_{abc}(x, y) = \sum_\lambda K_{abc}^{(\lambda)}(x) D^{(\lambda)}(y) \quad , \quad (7)$$

it is interesting to consider the 0 mode $K_{abc}^{(0)}$, which corresponds to $\lambda = 0$ and, when the left-invariant three-bein e^a is used, to the singlet representation $D^{(0)}$ of $SU(2)$. Notice that $K_{abc} e^a \wedge e^b \wedge e^c$ is a three-form on

$SU(2)$ and that, with the standard $SU(2) \times SU(2)$ invariant metric, only $K_{abc}^{(0)} e^a \wedge e^b \wedge e^c$ is $SU(2) \times SU(2)$ invariant. By a general theorem¹⁹⁾, $K_{abc}^{(0)} \times e^a \wedge e^b \wedge e^c$ is the only harmonic three-form on S^3 : $\Delta^{(3)} K_{abc}^{(0)} = 0$. Therefore, only $K_{abc}^{(0)}$ shall not inherit a mass term by dimensional reduction on M_4^0 , while $K_{abc}^{(\lambda)}$, $\lambda \neq 0$, will correspond to fields on M_4^0 with a mass term induced by the y dependence^{*)}.

Let us deal with $K_{abc}^{(0)}(x)$ and neglect the remaining contorsion contributions appearing in Eqs. (1)-(3). Since $K_{abc} = K_{[abc]}$, we can write

$$K_{abc}^{(0)} = \varepsilon_{abc} \chi(x), \quad (8)$$

where χ is a scalar field on M_4^0 . We are interested in the possibility that χ may develop a q vacuum value,

$$\chi = \hat{\chi} + \chi_0, \quad \chi_0 \equiv \langle 0 | \chi | 0 \rangle \neq 0. \quad (9)$$

Of course, when $e_\mu^\alpha = \delta_\mu^\alpha$, χ_0 must be independent of $x \in M_4^0$, to preserve the (Poincaré)₄ invariance of $|0\rangle$. Notice also that, by (8), the background value

$$\langle K_{abc} \rangle_0 = \varepsilon_{abc} \chi_0, \quad (10)$$

is consistent with the $SU(2) \times SU(2)$ symmetry of the background internal metric on S^3 .

To simplify the discussion as much as possible, we concentrate on the dynamical effects of $K_{abc}^{(0)}$ and we deal with the following amputated action:

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_K + \bar{\mathcal{A}}_D, \quad (11)$$

$$\begin{aligned} \bar{\mathcal{A}}_K &= -\frac{1}{2\hat{\kappa}} \int K_{abc}^{(0)} K^{(0)abc} e^1 \wedge \dots \wedge e^7 \\ &= -\frac{3}{\hat{\kappa}} \int (\hat{\chi} + \chi_0)^2 e^1 \wedge \dots \wedge e^4, \end{aligned} \quad (12)$$

*) Notice that this does not imply that $K_{abc}^{(0)}$ is massless in four dimensions, because it has a bare mass ($\sim m^{-1/2}$) from Eq. (2).

$$\bar{A}_D = - \int \bar{\Psi} (\not{\omega} - \frac{1}{4} \chi_{\varepsilon_{abc}} \Gamma^{abc}) \Psi e^1 \wedge \dots \wedge e^7, \quad (13)$$

where $\hat{\omega}_{ABC}$ is the Levi-Civita connection of (e^α, e^a) and $\hat{\kappa} = \kappa V$, with V the volume of S^3 .

We now proceed as in Refs. 6) and 12), with the representation

$$\Gamma^\alpha = \hat{\gamma}^\alpha \otimes 1, \quad \Gamma^a = \gamma^5 \otimes \sigma^a, \quad \hat{\gamma}^\alpha = -i \gamma^5 \gamma^\alpha,$$

and with the effective mass operators ⁶⁾

$$\hat{M}_{op} = i(\sigma^a e_a - \frac{1}{4} \hat{\omega}_{abc} \sigma^{abc}) = i\sigma^a e_a - \frac{3}{4\sqrt{2}L}, \quad (14)$$

$$M_{op} = \hat{M}_{op} - \frac{i}{4} \langle K_{abc} \rangle_0 \sigma^{abc} = \hat{M}_{op} - \frac{3}{2} \chi_0, \quad (15)$$

where L is the length of S^3 as defined in Ref. 6). As in Ref. 6), one can expand $\Psi(x, y)$ into the eigenmodes of M_{op} . We call $\psi_j(x)$, $j=0, \frac{1}{2}, 1, \dots$, the physical Dirac field in four dimensions which corresponds ^{*}) to the harmonic associated to the eigenvalue $\hat{M}_j$ of $|\hat{M}_{op}|$ and M_j of $|M_{op}|$. By performing the integral over S^3 , the action (15) is reduced to

$$\bar{A}_D = - \sum_j \int \bar{\psi}_j (\not{\omega}^{(4)} + M_j - \frac{3}{2} \hat{\chi}) \psi_j e^1 \wedge \dots \wedge e^4, \quad (16)$$

where

$$M_j = \hat{M}_j - \frac{3}{2} \chi_0 \quad (17)$$

and

$$\not{\omega}^{(4)} = \gamma^\alpha (e_\alpha - \frac{1}{4} \hat{\omega}_{[\alpha\beta\gamma]} \gamma^{\alpha\beta\gamma}).$$

Now, the path integral $Z(E^A)$ has been truncated to

^{*}) The spectra of M_{op} are not positive-definite; this simply means that, for negative eigenvalues, ψ_j is related by a Pauli-Gürsey transformation to the corresponding component of Ψ .

$$\bar{Z}(e^\alpha, e^a) = \int \mathcal{D}\hat{\chi} \prod_j \mathcal{D}\bar{\psi}_j \mathcal{D}\psi_j \exp i(\bar{\mathcal{A}}_K + \bar{\mathcal{A}}_D). \quad (18)$$

As in Ref. 12), we proceed by integrating over the fermions, so as to find the effective action for the torsion :

$$\bar{Z}(e^\alpha, e^a) = N(e^\alpha, e^a) \int \mathcal{D}\hat{\chi} \exp i \bar{\mathcal{A}}_K^{\text{eff}}, \quad (19)$$

where

$$\bar{\mathcal{A}}_K^{\text{eff}} = -\frac{3}{\kappa} \int (\hat{\chi} + \chi_0)^2 e^1 \dots e^4 + \sum_{n=1}^{\infty} d_j \mathcal{U}_j^{(n)}, \quad (20)$$

$$\mathcal{U}_j^{(n)} = \frac{i}{n} \text{tr} \left(\frac{3}{2} S_j \hat{\chi} \right)^n, \quad (21)$$

with d_j the multiplicity of the eigenvalue M_j and

$$N(e^\alpha, e^a) = \exp \text{tr} \log \left(- \sum_j d_j S_j^{-1} \right), \quad (22)$$

$$S_j^{-1} = \gamma^\alpha \left(e_\alpha - \frac{1}{4} \omega_{[\alpha\beta\gamma]} \gamma^{\beta\gamma} \right) + M_j. \quad (23)$$

The term $n=1$ in (20) yields a contribution which is linear in $\hat{\chi}$. Since $\hat{\chi}$ has vanishing expectation value, the total contribution to $\bar{\mathcal{A}}_K^{\text{eff}}$ which is linear in $\hat{\chi}$ must vanish, and we obtain the following generalization of the N-JL consistency condition :

$$\chi_0 = i \frac{\kappa}{4} \sum_j d_j \text{tr} S_j. \quad (24)$$

Now, in a "realistic" multidimensional theory, we should : (i) set $e_\mu^\alpha = \delta_\mu^\alpha$, corresponding to the M_4^0 background, and (ii) require that zero mass fermions occur in four dimensions, probably to be identified with leptons²⁰⁾. We therefore demand that the gap $\hat{M}_0$ be completely compensated by the induced torsion mass counter-term $\frac{3}{2}\chi_0$, with zero physical mass for the

lowest mode ($j=0$). This fixes the magnitude of the internal length L in terms of the vacuum expectation value of the contorsion : with the notation of Ref. 6), section 3E, one obtains ^{*})

$$L \overset{\circ}{M}_0 = \frac{3}{4\sqrt{2}} = \frac{3}{2} \chi_0 L. \quad (25)$$

Going back to Eqs. (9) and (10), from the result $\langle 0 | K_{abc} | 0 \rangle = \epsilon_{abc} / 2\sqrt{2}L$ and by comparison with Eqs. (3.6b), (3.43) and (3.44) of Ref. 6), we see that the background torsion is precisely the one ^{*}) which leads to vanishing Riemann-Cartan curvature on S^3 . It follows that the background connection has vanishing Einstein-Cartan tensor and that the background geometry is indeed a solution of the EC classical field equations ^{**}).

Next, we combine (25) with the self-consistency condition (24), where now the mass eigenvalues M_j are the same as those calculated in Eq. (3.44) of Ref. 6) :

$$L M_j = c_j, \quad j = 0, \frac{1}{2}, 1, \dots \quad (26)$$

$$\sqrt{2} c_j = J + \frac{1}{2} \pm 1, \quad J = \frac{1}{2}, 1, \dots; \quad j = J \pm \frac{1}{2}; \quad \sqrt{2} c_{1/2} = 3. \quad (27)$$

One obtains, setting $c'_j = c_j d_j$,

$$1 = \frac{\pi}{\sqrt{2}} \sum_{j \neq 0} c'_j \frac{i}{(2\pi)^4} \int \frac{1}{p^2 + M_j^2} d^4 p. \quad (28)$$

In complete analogy with what happens in the N-JL model, the consistency condition (28) is meaningless unless a cut-off $(p_0^2 + p^2 < M^2, j < T)$ is introduced, leading to

^{*}) Since we are dealing with the absolute value of the lowest eigenvalue of M_{op} , there is a sign ambiguity in (25); we arbitrarily chose the positive sign, which corresponds to the (+) Cartan-Schouten connection on $SU(2)$. The alternative choice leads to the (-) connection, and corresponds to the interchange between the left- and right-invariant Maurer-Cartan forms on $SU(2)$, see Ref. 6), Appendix A.

^{**}) The factor $N(e^\alpha, e^a)$, Eq. (22), can be absorbed in the normalization of $\bar{Z}(e^\alpha, e^a)$, although it vanishes when zero mass modes are present. If the path integral were extended also over the vielbein E^A , the factor $N(E^A)/N(e^\alpha, e^a)$ should, however, be included in the integrand.

$$\sum_{0 \neq j < T} c'_j \frac{i}{(2\pi)^4} \int \frac{1}{p^2 + M_j^2} d^4 p = \frac{1}{4\pi^2} \sum_{0 \neq j < T} c'_j \left[-M_j^2 + M_j^2 \log \left(1 + \frac{M_j^2}{M_1^2} \right) \right] \quad (29)$$

The first term ($\sim -T^2 M^2$) is independent of the background radius L and should be absorbed in the definition of $\bar{\Psi}\Psi$, in complete analogy with the procedure commonly adopted¹⁸⁾ in calculations of Casimir energies^{*}). The second term is, however, L dependent and physically significant, hence we rewrite the consistency condition (28) in the following form

$$1 = \frac{\kappa L^{-2}}{4\sqrt{2} \pi^2} \sum_{0 \neq j < T} c_j^3 d_j \log \left(1 + \frac{M_j^2 L^2}{c_j^2} \right) \quad (30)$$

Of course, L cannot yet be computed from (30), unless the cut-off (M^2, T) can be determined by other means. Therefore, the simple model considered here does not quite lead to a numerical prediction for L . However, it should be clear that the mechanisms discussed here are quite general and that a consistency condition of the same kind as Eqs. (24) and (25) should be expected to hold also in other multidimensional theories, where spinors are coupled to EC gravity. If one succeeded in finding a model which is ultra-violet finite^{**)}, the consistency condition should become cut-off independent and L should consequently be completely calculable from the dynamical mechanism of spin-torsion compactification. This possibility is, of course, extremely interesting, because $\sqrt{\kappa}/L$ measures the magnitude of the bare Yang-Mills coupling constant¹⁾⁻⁶⁾ after dimensional reduction to M_4^0 .

It should also be observed that the term $n=2$ in Eq. (21) leads to a kinetic term and to a mass counter-term for the internal torsion field $K_{abc}^{(0)}$. Therefore, this field can in principle acquire a physical mass much lower than the bare mass ($\sim \kappa^{-\frac{1}{2}}$).

Other possible physical effects of the spin-spin contact interaction shall be discussed elsewhere, particularly in connection with the proposal made in Ref. 20).

*) One could also modify our approach and keep the leading divergence of Eq. (29), as in the N-JL model. The "wrong" sign ($\sim -M^2$) might be avoided by exploiting the lack of positivity of M_{op} . We feel that the advocated subtraction is, however, physically better motivated.

**) It is possible that a multidimensional theory with local supersymmetry could be finite.

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REFERENCES

- 1) T. Kaluza, Z.Sitzber.Preuss.Akad.Wiss. 966 (1921);
O. Klein, Z-Phys. 37 (1925) 895;
A. Einstein and P. Bergmann, Ann.Math. 39 (1938) 683.
- 2) P. Jordan, Astronom.Nachr. 276 (1948) 193;
Y.R. Thiry, Compt.Rend. 226 (1948) 216.
- 3) Y.M. Cho and P.G.O. Freund, Phys.Rev. D12 (1975) 1711;
C.A. Orzalesi, Fortschr.Phys. 29 (1981) 413, and references therein.
- 4) E. Cremmer and J. Scherk, Nuclear Phys. B108 (1976) 409;
J.F. Luciani, ibid.B135 (1978) 111.
- 5) E. Witten, Nuclear Phys. B186 (1981) 412;
A. Salam and J. Strathdee, Ann.Phys. 141 (1982) 316.
- 6) C. Destri, C.A. Orzalesi and P. Rossi, Ann.Phys. 147 (1983) 321.
- 7) A. Trautman, Sympos.Mathem. 12 (1973) 139;
F.W. Hehl, P. Von der Heyde, G.D. Kerlick and J.M. Nester, Revs.Mod.
Phys. 48 (1975) 393.
- 8) E. Cremmer, B. Julia and J. Scherk, Phys.Letters 76B (1978) 409;
E. Cremmer and B. Julia, Nuclear Phys. B159 (1979) 141.
- 9) P.G.O. Freund and M.A. Rubin, Phys.Letters 94B (1980) 233;
M.J. Duff and C.N. Pope, in Supersymmetry and Supergravity, Ed. by
S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen, World Scientific
(1982);
B. Biran, F. Englert, B. de Wit and H. Nicolai, Phys.Letters 124B (1983)
45.
- 10) M.J. Duff and C.A. Orzalesi, Phys.Letters 122B (1983) 37.
- 11) C.A. Orzalesi and M. Pauri, Phys.Letters 107B (1981) 186;
C.A. Orzalesi, in Proceedings of Conf. on Differential Geometric Methods
in Theor.Physics, Trieste 1981, Ed. by G. Denardo and H.D. Doebner,
World Scientific (1983).
- 12) C.A. Orzalesi, preprint CERN.TH-3647, to be published.

- 13) Y. Nambu and G. Jona-Lasinio, Phys.Rev. 122 (1961) 345.
- 14) T. Eguchi and H. Sugawara, Phys.Rev. D10 (1974) 4257;
T. Kugo, Progr.Theor.Phys. 55 (1976) 2032;
K. Kikkawa, ibid 56 (1976) 974.
- 15) A. Lichnerowicz, Compt.Rend.Acad.Sci.Paris 257 (1963) 7.
- 16) G.D. Kerlick, Phys.Rev. D12 (1975) 3004;
M. Soffel, B. Müller and W. Greiner, Phys.Letters 70A (1979) 167.
- 17) H.B.G. Casimir, Proc.Kon.Ned.Akad.Wet. 51 (1948) 793.
- 18) B.S. De Witt, Phys.Reports 19 (1975) 295.
- 19) A. Lichnerowicz, Géométrie des Groupes de Transformation, Dunod (1958).
- 20) S. Bergia, C.A. Orzalesi and G. Venturi, Phys.Letters 123B (1983) 205.