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EFFECTIVE ACTIONS FROM BLOCK DIAGONALIZATION
IN LATTICE GAUGE THEORIES

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A B S T R A C T

It is shown that the first step in Wilson's real space renormalization program for lattice gauge theories - namely the integration of internal degrees of freedom within one block - can be performed for "block diagonalized" actions, which possess the proper continuum limit, just like the Wilson action. As a result, we obtain, on a lattice with double lattice spacing, an effective action which contains, in addition to the familiar plaquette terms, planar Wilson loops with six and eight links.

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1. - INTRODUCTION

The first generation of Monte Carlo simulations performed during the last three years has revealed the strong but also weak points of QCD on finite (rather small) lattices : asymptotic freedom seems to be visible in the measurements of the string tension ¹⁾ and the mass gap ²⁾ for pure lattice gauge theories. However, the window to asymptotic freedom is still rather narrow in a lattice of order 10^4 and easily closed by a "bad" choice of the action ^{3),4)}. In principle, all kinds of lattice actions having the proper formal continuum limit are equivalent. However, for the practitioner of Monte Carlo simulations on finite lattices, the proper choice of the action will eventually be crucial : e.g., there is a beautiful signal for asymptotic freedom in the measurements of the string tension ^{1),5)} and the mass gap ²⁾ with Wilson's and Manton's actions, but the signal disappears in the corresponding measurements with the mixed fundamental adjoint action ^{3),4)} (with $\beta_A = 1.21$).

In 1981, Symanzik ⁶⁾ has proposed a program for the construction of improved actions, which are contrived to yield a more rapid approach of physical quantities to their continuum limit. The program has been applied first to the ϕ^4 theory and the $O(N)$ non-linear sigma model ⁷⁾. In these models, the improved action contains both nearest neighbour and next-to-nearest neighbour terms. The improved action for the pure Yang-Mills theory has been calculated to lowest order perturbation theory by P. Weisz ⁸⁾, to all leading logarithms by Curci et al. ⁹⁾. Recent Monte Carlo calculations for the $O(3)$ non-linear sigma model ¹⁰⁾ and for the string tension in the pure $SU(2)$ gauge theory ¹¹⁾ confirm the expectation that improved actions open the window to asymptotic freedom much wider.

First attempts for improving the action were already undertaken in 1979 by Wilson ¹²⁾ within his real space renormalization group program. The starting point is the division of a large lattice into blocks ; then one tries to perform the integration over the block internal degrees of freedom. So far, this crucial step could not be done analytically. It is the purpose of this paper to fill this gap. We will prove the existence of a "block diagonalized" lattice action for pure $SU(N)$ gauge theories with the following properties :

- a) the block diagonalized action has the proper continuum limit $a \rightarrow 0$, just as the Wilson action ;
- b) the block internal degrees of freedom within blocks of size $(2a)^4$ can be integrated out analytically.

In doing so, we end up with an effective action defined on a lattice with double lattice spacing $2a$. The effective action contains, in addition to the familiar plaquette terms also planar Wilson loops with six and eight links ; it has a striking similarity to Symanzik's improved action ⁹⁾.

The paper is organized as follows. In Section 2, we rewrite the Wilson action in terms of "double link" and "internal block" variables. These are the appropriate variables for our "block diagonalization procedure" introduced in Ref. 13). In Section 3, we then construct the "block diagonalized" action, which has the desired properties a) and b). In Section 4, we transform the path integral measure to double link - and internal block variables. In Section 5, we perform the integration over the internal block variables. In Section 6, we discuss the properties of the effective action on the sublattice with double lattice spacing and compare it with Symanzik's improved action ⁹⁾.

2. - THE GLUON ACTION, AS SEEN FROM A SUBLATTICE WITH DOUBLE LATTICE SPACING

The basic ideas of our block diagonalization procedure have been described previously in the context of the fermion problem ¹³⁾. To make this paper self-contained, let us briefly review those ingredients which will be important for the treatment of the gluonic part of the action.

A four-dimensional cubic lattice with lattice spacing a can be partitioned into 16 cubic sublattices S_τ , each with lattice spacing $2a$:

$$x = a(n_0, n_1, n_2, n_3) \in S_\tau \quad \tau = (\tau_0, \tau_1, \tau_2, \tau_3) \quad (2.1)$$

if $(-1)^{n_\mu} = \tau_\mu$ for $\mu = 0, 1, 2, 3$.

Therefore, any site on the original lattice can be specified by a point in one preferred sublattice, say $\tilde{x} \in S_{++++} = \tilde{S}$ and the signature vector $\tau = (\tau_0, \tau_1, \tau_2, \tau_3)$ which labels the different sublattices :

$$x = \tilde{x} + \frac{a}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 1-\tau_3) \quad (2.2)$$

τ can be considered as the four-vector of relative co-ordinates within a $(2a)^4$ cubic block $B(\tilde{x})$ with lower left corner $\tilde{x}$. In this section, we want to express the gluonic part of the Wilson action

$$S_g = \sum_p \text{tr } U(p) \quad (2.3)$$

$[U(p)]$ are the usual plaquette variables] in terms of new gauge fields $U_\mu^{(\tau)}(\tilde{x}, \tilde{y})$, which mediate interactions between sites $\tilde{x}, \tilde{y} \in \tilde{S}$ of the sublattice $\tilde{S}$ with double lattice spacing. As in Ref. 13), the starting point is the reformulation of the nearest neighbour Kronecker deltas in terms of the corresponding quantities on the sublattice $\tilde{S}$ with double lattice spacing :

$$\delta_{y, x+\mu} = (T_\mu^{(+)})_{\tau\tau'} \frac{1}{2} (\delta_{\tilde{y}, \tilde{x}} + \delta_{\tilde{y}, \tilde{x}+2\mu}) + (T_\mu^{(-)})_{\tau\tau'} \frac{1}{2} (\delta_{\tilde{y}, \tilde{x}} - \delta_{\tilde{y}, \tilde{x}+2\mu}) \quad (2.4)$$

$$\delta_{y, x-\mu} = (T_\mu^{(+)})_{\tau\tau'} \frac{1}{2} (\delta_{\tilde{y}, \tilde{x}-2\mu} + \delta_{\tilde{y}, \tilde{x}}) + (T_\mu^{(-)})_{\tau\tau'} \frac{1}{2} (\delta_{\tilde{y}, \tilde{x}-2\mu} - \delta_{\tilde{y}, \tilde{x}}) \quad (2.5)$$

x and $\tilde{x}$ are related by Eq. (2.2) and correspondingly :

$$y = \tilde{y} + \frac{a}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 1-\tau_3) \quad (2.6)$$

The 16×16 matrices $T_\mu^{(\pm)}$ are defined by :

$$T_\mu^{(+)}_{\tau\tau'} = \begin{cases} 1 & \text{for } \tau_\mu \tau'_\mu = -1, \tau_\nu \tau'_\nu = 1 \quad \forall \nu \neq \mu \\ 0 & \text{else} \end{cases} \quad (2.7)$$

$$T_\mu^{(-)}_{\tau\tau'} = \tau_\mu T_\mu^{(+)}_{\tau\tau'}$$

In Ref. 13), we introduced new gauge fields $U_\mu^{(\tau)}(\tilde{x}, \tilde{y})$ which mediate the interactions between the sites on the sublattice $\tilde{x}, \tilde{y} \in \tilde{S}$:

$$U_\mu^{(\tau)}(\tilde{x}, \tilde{x} \pm 2\mu) = \tilde{U}(\tilde{x}, x) U(x, x \pm \mu) \tilde{U}(x \pm \mu, \tilde{x} \pm 2\mu) \quad \text{for } \tau_\mu = \pm 1 \quad (2.8)$$

$$U_\mu^{(\tau)}(\tilde{x}, \tilde{x}) = \tilde{U}(\tilde{x}, x) U(x, x \pm \mu) \tilde{U}(x \pm \mu, \tilde{x}) \quad \text{for } \tau_\mu = \pm 1 \quad (2.9)$$

$$U_\mu^{(\tau)}(\tilde{x}, \tilde{x} \pm 2\mu) = \left(U_\mu^{(\tau)}(\tilde{x} \pm 2\mu, \tilde{x}) \right)^+ \quad \text{for all } \tau \quad (2.10)$$

Here, $\tilde{U}(\tilde{x}, x)$ denotes a product of link variables along a certain path $C(\tilde{x}, x)$ connecting the lower left corner $\tilde{x}$ with the interior site x in one block $B(\tilde{x})$. We used the following convention for $C(\tilde{x}, x)$:

$$\begin{aligned} \tilde{x}, \tilde{x} + \frac{a}{2} (1-\tau_0, 0, 0, 0), \tilde{x} + \frac{a}{2} (1-\tau_0, 1-\tau_1, 0, 0) \\ \tilde{x} + \frac{a}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 0), x = \tilde{x} + \frac{a}{2} (1-\tau_0, 1-\tau_1, 1-\tau_2, 1-\tau_3) \end{aligned} \quad (2.11)$$

The new gauge fields $U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y})$ are ordered products of the original link variables $U(x, x+\mu)$ along a path from $\tilde{x}$ via x to $\tilde{y}$, the path being uniquely specified by τ . From the definition (2.8)-(2.10) one derives the following constraints :

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) = U_{\mu}^{(\tau')}(\tilde{x}, \tilde{x}+2\mu) \quad (2.12)$$

$$U_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}) = \left(U_{\mu}^{(\tau')}(\tilde{x}, \tilde{x}) \right)^+ \quad (2.13)$$

where $\tau'_{\mu} = -\tau_{\mu}$, $\tau'_{\rho} = \tau_{\rho}$ for $\rho \neq \mu$

$$U_{\mu}^{(++++)}(\tilde{x}, \tilde{x}) = 1 \quad \mu = 0, 1, 2, 3 \quad (2.14)$$

$$U_1^{(--++)}(\tilde{x}, \tilde{x}) = U_2^{(--++)}(\tilde{x}, \tilde{x}) = U_2^{(+--+)}(\tilde{x}, \tilde{x}) = U_2^{(-+--)}(\tilde{x}, \tilde{x}) = 1 \quad (2.15)$$

$$U_3^{(\tau)}(\tilde{x}, \tilde{x}) = 1 \quad \text{for all } \tau \quad (2.16)$$

Equations (2.15)-(2.16) result from our special choice (2.11) for the path $C(\tilde{x}, x)$.

Taking into account the constraints (2.12)-(2.16), we are left with 49 independent gauge fields ^{*)}. In the Table, we have listed them in terms of open and closed Wilson lines $W(\mu_1, \mu_2, \dots)$ along the sequence of links $\mu_1, \mu_2, \dots$: four lines are linear, 18 planar, 20 run through three, seven through four dimensions.

The reason for introducing these complicated Wilson lines as new gauge fields instead of the original link variables is that they lead us to the appropriate variables for the block diagonalization procedure : the four linear Wilson lines :

$$U(\tilde{x}, \tilde{x}+2\mu) = U_{\mu}^{(++++)}(\tilde{x}, \tilde{x}+2\mu) = W(\mu, \mu) \quad \mu = 0, 1, 2, 3 \quad (2.17)$$

will provide us with the interaction between the different blocks $B(\tilde{x})$. They are link variables on the sublattice $\tilde{S} = S_{++++}$ with double lattice spacing. They will not be integrated out in the path integral.

^{*)} The number 49 is of course obvious : we have 64 links in the block, with 15 internal gauge degrees of freedom.

The 45 closed Wilson loops :

$$U_{\mu}^{(\tau)}(\vec{x}) = U_{\mu}^{(\tau)}(\vec{x}, \vec{x}) \quad (2.18)$$

$$V_{\mu}^{(\tau)}(\vec{x}) = U_{\mu}^{(\tau)}(\vec{x}, \vec{x}+2\mu) U(\vec{x}+2\mu, \vec{x}) \quad (2.19)$$

define our "internal block variables". They depend on the block internal coordinates τ . Later on, we shall integrate them out in the path integral.

In order to express the gluon action (2.3) in terms of the new link - and internal block variables (2.17)-(2.19), we start from the following representation

$$\int g = \text{TR } H^4 + \text{const.} \quad (2.20)$$

where H is an $N_c L^4 \times N_c L^4$ matrix - N_c being the number of colours, L^4 the number of sites - with matrix elements $^{*})$:

$$H(x, y) \equiv \sum_{\mu} (U(x, x+\mu) \delta_{y, x+\mu} + U(x, x-\mu) \delta_{y, x-\mu}) \quad (2.21)$$

Note that the trace in Eq. (2.20) has to be taken both over the sites x (on the original lattice) and the colour index [which is not written down explicitly in Eq. (2.21)].

Let us define on the sublattice $\tilde{S}$ a corresponding operator $\tilde{H}$:

$$\begin{aligned} \tilde{H}(\vec{x}\tau, \vec{y}\tau') \equiv & V_{\mu}^{(\tau)}(\vec{x}) U(\vec{x}, \vec{x}+2\mu) \frac{1}{2} (T_{\mu}^{(+)} - T_{\mu}^{(-)})_{\tau\tau'} \delta_{\vec{y}, \vec{x}+2\mu} \\ & + U(\vec{x}, \vec{x}-2\mu) V_{\mu}^{(\tau)}(\vec{x}-2\mu) \frac{1}{2} (T_{\mu}^{(+)} + T_{\mu}^{(-)})_{\tau\tau'} \delta_{\vec{y}, \vec{x}-2\mu} \\ & + U_{\mu}^{(\tau)}(\vec{x}) T_{\mu}^{(+)}_{\tau\tau'} \delta_{\vec{y}, \vec{x}} \end{aligned} \quad (2.22)$$

From Eqs. (2.4) and (2.5), and the definitions (2.8), (2.9), (2.18) and (2.19), one verifies that H and $\tilde{H}$ are unitarily equivalent, which of course implies :

$$\text{TR } H^4 = \text{TR } \tilde{H}^4 \quad (2.23)$$

$^{*})$ For some reason, the same matrix H appears in the Wilson term of the fermion action :

$$\bar{\Psi}(x) (H(x, y) - 2\delta_{y, x}) \Psi(y)$$

The evaluation of the trace on the right-hand side yields the gluon action expressed in terms of double link - and internal block variables :

$$S_g = 4 \sum_{\mu \neq \nu} \sum_{\tau, \tau', \tau'', \tau'''} \varepsilon_{\mu\nu}(\tau, \tau', \tau'', \tau''') \sum_{i=1}^3 S_i(\mu, \nu, \tau, \tau', \tau'', \tau''') + \text{const} \quad (2.24)$$

where

$$S_1 = \sum_{\vec{x}} \text{tr} \left\{ V_\mu^{(\tau)}(\vec{x}) U(\vec{x}, \vec{x}+2\mu) V_\nu^{(\tau')}(\vec{x}+2\mu) U(\vec{x}+2\mu, \vec{x}+2\mu+2\nu) \cdot \right. \\ \left. \cdot U(\vec{x}+2\mu+2\nu, \vec{x}+2\nu) V_\mu^{(\tau'')}(\vec{x}+2\nu)^\dagger U(\vec{x}+2\nu, \vec{x}) V_\nu^{(\tau''')}(\vec{x})^\dagger \right\} \quad (2.25)$$

$$S_2 = \sum_{\vec{x}} \text{tr} \left\{ V_\mu^{(\tau)}(\vec{x}) U(\vec{x}, \vec{x}+2\mu) U_\nu^{(\tau')}(\vec{x}+2\mu) U(\vec{x}+2\mu, \vec{x}) V_\mu^{(\tau'')}(\vec{x})^\dagger U_\nu^{(\tau''')}(\vec{x})^\dagger \right. \\ \left. + U_\mu^{(\tau)}(\vec{x}) V_\nu^{(\tau')}(\vec{x}) U(\vec{x}, \vec{x}+2\nu) U_\mu^{(\tau'')}(\vec{x}+2\nu) U(\vec{x}+2\nu, \vec{x}) V_\nu^{(\tau''')}(\vec{x})^\dagger \right\} \quad (2.26)$$

$$S_3 = \sum_{\vec{x}} \text{tr} \left\{ U_\mu^{(\tau)}(\vec{x}) U_\nu^{(\tau')}(\vec{x}) U_\mu^{(\tau'')}(\vec{x}) U_\nu^{(\tau''')}(\vec{x}) \right\} \quad (2.27)$$

and

$$\varepsilon_{\mu\nu}(\tau, \tau', \tau'', \tau''') = 1 \\ \text{if} \quad \begin{cases} \tau_\mu = -, \tau'_\mu = +, \tau''_\mu = +, \tau'''_\mu = - \\ \tau_\nu = -, \tau'_\nu = -, \tau''_\nu = +, \tau'''_\nu = + \\ \tau_\rho = \tau'_\rho = \tau''_\rho = \tau'''_\rho, \rho \neq \mu, \nu \end{cases} \quad (2.28)$$

$\varepsilon_{\mu\nu}(\tau, \tau', \tau'', \tau''') = 0$ in all other cases.

If we put the internal block variables $U_\mu^{(\tau)}, V_\mu^{(\tau)}$ equal to one, we recognize in the first term the plaquette variables on the sublattice $\tilde{S}$, whereas the second and third term become independent on the double link variables $U(\vec{x}, \vec{x}+2\mu)$.

3. - SEPARATION OF INTERNAL BLOCK VARIABLES

As it stands, the gluon action - expressed in terms of our new link and internal block variables - is indeed very complicated and there is no hope of performing the integration over the internal block variables. The situation can be improved, however, from consideration of modified actions with equal continuum limit. Remember that the one-plaquette contribution in the gluon action reproduces the continuum Lagrangian $\text{tr } F_{\mu\nu}(x)F_{\mu\nu}(x)$ within order a^4 . Therefore, any modification of the gluon action, which keeps unaltered all terms within order a^4 is just as legitimate. At this stage it is important to notice that internal block variables are closed (finite) Wilson loops within the block $B(\tilde{x})$. Therefore, in the continuum limit $a \rightarrow 0$, they behave as

$$U_{\mu}^{(\tau)}(\tilde{x}) - 1, V_{\mu}^{(\tau)}(\tilde{x}) - 1 = O(a^2) \quad (3.1)$$

as opposed to open Wilson lines like the link variable

$$U(\tilde{x}, \tilde{x} + 2\mu) - 1 = O(a) \quad (3.2)$$

Our first modification of the gluon action (2.24)-(2.27), therefore, consists in keeping terms at most bilinear in $(V_{\mu}^{(\tau)} - 1)$, $(U_{\mu}^{(\tau)} - 1)$. The contribution S_1 [Eq. (2.25)] to the action, e.g., then acquires the form :

$$\begin{aligned} S_1 = \sum_{\tilde{x}} \text{tr} \{ & P_{\mu\nu}(\tilde{x}, \tilde{x}) + (V_{\mu}^{(\tau)}(\tilde{x}) - 1) P_{\mu\nu}(\tilde{x}, \tilde{x}) + (V_{\nu}^{(\tau)}(\tilde{x} + 2\mu) - 1) P_{\mu\nu}(\tilde{x} + 2\mu, \tilde{x}) \\ & + (V_{\mu}^{(\tau)}(\tilde{x} + 2\nu)^{\dagger} - 1) P_{\mu\nu}(\tilde{x} + 2\nu, \tilde{x}) + (V_{\nu}^{(\tau)}(\tilde{x})^{\dagger} - 1) P_{\mu\nu}(\tilde{x}, \tilde{x}) \\ & + (V_{\mu, \mu}^{(\tau)}(\tilde{x} + 2\mu) - 1) (V_{\nu}^{(\tau)}(\tilde{x} + 2\mu) - 1) P_{\mu\nu}(\tilde{x} + 2\mu, \tilde{x}) \\ & + (V_{\mu, \nu}^{(\tau)}(\tilde{x} + 2\nu) - 1) P_{\mu\nu}(\tilde{x} + 2\nu, \tilde{x}) (V_{\mu}^{(\tau)}(\tilde{x} + 2\nu)^{\dagger} - 1) \\ & + (V_{\mu}^{(\tau)}(\tilde{x}) - 1) P_{\mu\nu}(\tilde{x}, \tilde{x}) (V_{\nu}^{(\tau)}(\tilde{x})^{\dagger} - 1) \\ & + (V_{\nu, \nu}^{(\tau)}(\tilde{x} + 2\mu + 2\nu) - 1) (V_{\mu, \mu}^{(\tau)}(\tilde{x} + 2\nu + 2\mu)^{\dagger} - 1) P_{\mu\nu}(\tilde{x} + 2\mu + 2\nu, \tilde{x}) \\ & + (V_{\nu}^{(\tau)}(\tilde{x} + 2\mu) - 1) P_{\mu\nu}(\tilde{x} + 2\mu, \tilde{x}) (V_{\nu, \mu}^{(\tau)}(\tilde{x} + 2\mu)^{\dagger} - 1) \\ & + (V_{\mu}^{(\tau)}(\tilde{x} + 2\nu)^{\dagger} - 1) (V_{\nu, \nu}^{(\tau)}(\tilde{x} + 2\nu)^{\dagger} - 1) P_{\mu\nu}(\tilde{x} + 2\nu, \tilde{x}) \} \end{aligned} \quad (3.3)$$

Here we have introduced the following notation for the plaquette variable $P_{\mu\nu}(\tilde{y}, \tilde{x})$ on the sublattice $\tilde{S}$: $P_{\mu\nu}(\tilde{y}, \tilde{x})$ is the anti-clockwise ordered product of four link variables in the $\mu\nu$ plane. The loop starts and ends at $\tilde{y}$, which can be either $\tilde{x}$, $\tilde{x}+2\mu$, $\tilde{x}+2\mu+2\nu$ or $\tilde{x}+2\nu$. The second index $\tilde{x}$ refers to the lower left corner of the plaquette (Fig. 1). Therefore, the variables $P_{\mu\nu}(\tilde{x}, \tilde{x})$, $P_{\mu\nu}(\tilde{x}+2\mu, \tilde{x})$, $P_{\mu\nu}(\tilde{x}+2\mu+2\nu, \tilde{x})$, $P_{\mu\nu}(\tilde{x}+2\nu, \tilde{x})$ belong to the same plaquette with lower left corner $\tilde{x}$.

Furthermore, we used the following abbreviation

$$V_{\mu;\nu}^{(\tau)}(\tilde{x}+2\nu) = U(\tilde{x}+2\nu, \tilde{x}) V_{\mu}^{(\tau)}(\tilde{x}) U(\tilde{x}, \tilde{x}+2\nu) \quad (3.4)$$

for the parallel transported closed loop $V_{\mu}^{(\tau)}(\tilde{x})$. The $V_{\mu;\nu}^{(\tau)}(\tilde{x}+2\nu)$ have the same properties as the internal block variables $V_{\mu}^{(\tau)}(\tilde{x})$; they are closed Wilson loops starting and ending at $\tilde{x}+2\nu$ and therefore obey the continuum behaviour (3.1). We could have chosen the $V_{\mu;\mu}^{(\tau)}(\tilde{x}+2\mu)$ as independent block variables as well. From the definition (3.4), one derives immediately the following properties:

$$\text{tr } V_{\mu;\nu}^{(\tau)}(\tilde{x}+2\nu) = \text{tr } V_{\mu}^{(\tau)}(\tilde{x}) \quad (3.5)$$

$$\text{tr } (V_{\mu;\nu}^{(\tau)}(\tilde{x}+2\nu) - 1) P_{\mu\nu}(\tilde{x}+2\nu, \tilde{x}) = \text{tr } (V_{\mu}^{(\tau)}(\tilde{x}) - 1) P_{\mu\nu}(\tilde{x}, \tilde{x}) \quad (3.6)$$

$$V_{\mu;\nu}^{(\tau)}(\tilde{x}+2\nu) - 1 = V_{\mu}^{(\tau)}(\tilde{x}+2\nu) - 1 + O(a^3) \quad (3.7)$$

The last property (3.7) enables us to rewrite the gluon action in a "block diagonalized" form, in which internal block variables of different blocks are decoupled. To illustrate the procedure, let us consider the third term in Eq. (3.3):

$$\begin{aligned} & \sum_{\tilde{x}} \text{tr} \{ (V_{\mu;\mu}^{(\tau)}(\tilde{x}+2\mu) - 1) (V_{\nu}^{(\tau)}(\tilde{x}+2\mu) - 1) P_{\mu\nu}(\tilde{x}+2\mu, \tilde{x}) \} \\ &= \sum_{\tilde{x}} \text{tr} \{ (V_{\mu;\mu}^{(\tau)}(\tilde{x}+2\mu) - 1) (V_{\nu}^{(\tau)}(\tilde{x}+2\mu) - 1) \} + O(a^6) \\ &= \sum_{\tilde{x}} \text{tr} \{ (V_{\mu}^{(\tau)}(\tilde{x}) - 1) (V_{\nu}^{(\tau)}(\tilde{x}) - 1) \} + O(a^6) \end{aligned} \quad (3.8)$$

In the second step of Eq. (3.8), we have used the behaviour of the plaquette variables $P_{\mu\nu}(\tilde{y}, \tilde{x})$ in the limit $a \rightarrow 0$

$$P_{\mu\nu}(\tilde{y}, \tilde{x}) - 1 = O(a^2) \quad (3.9)$$

Due to Eqs. (3.5)-(3.7), the linear terms in $V_{\mu}^{(\tau)}(\tilde{x})-1$ in Eq. (3.3) can be brought in a symmetric form :

$$\begin{aligned} & \sum_{\tilde{x}} \text{tr} \left\{ (V_{\mu}^{(\tau)}(\tilde{x})-1) (P_{\mu\nu}(\tilde{x}, \tilde{x})-1) + (V_{\nu}^{(\tau')}(\tilde{x}+2\mu)-1) (P_{\mu\nu}(\tilde{x}+2\mu, \tilde{x})-1) \right. \\ & \quad \left. + (V_{\mu}^{(\tau'')}(\tilde{x}+2\nu)^{\dagger}-1) (P_{\mu\nu}(\tilde{x}+2\nu, \tilde{x})-1) + (V_{\nu}^{(\tau''')}(\tilde{x})^{\dagger}-1) (P_{\mu\nu}(\tilde{x}, \tilde{x})-1) \right\} \\ &= \frac{1}{4} \sum_{\tilde{x}} \text{tr} \left\{ (V_{\mu}^{(\tau)}(\tilde{x}) + V_{\nu}^{(\tau')}(\tilde{x}) + V_{\mu}^{(\tau'')}(\tilde{x})^{\dagger} + V_{\nu}^{(\tau''')}(\tilde{x})^{\dagger} - 4) \cdot \right. \\ & \quad \left. \cdot (P_{\mu\nu}(\tilde{x}, \tilde{x}) + P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu) + P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu-2\nu) + P_{\mu\nu}(\tilde{x}, \tilde{x}-2\nu) - 4) \right\} \quad (3.10) \end{aligned}$$

Our final result for the block diagonalized gluon action can be written as follows :

$$S_g = 16 \sum_{\tilde{x}} \sum_{\mu \neq \nu} \text{tr} P_{\mu\nu}(\tilde{x}, \tilde{x}) + 4 \sum_{\tilde{x}} S_g(\tilde{x}) + \text{const.} \quad (3.11)$$

where $S_g(\tilde{x})$ is the action of the block $B(\tilde{x})$ which we split up for later convenience into three parts :

$$S_g(\tilde{x}) = \sum_{\mu \neq \nu} \sum_{\tau, \tau', \tau'', \tau'''} \mathcal{E}_{\mu\nu}(\tau, \tau', \tau'', \tau''') \sum_{i=1}^3 \mathcal{S}_i(\tilde{x}, \mu, \nu, \tau, \tau', \tau'', \tau''') \quad (3.12)$$

where

$$\begin{aligned} \mathcal{S}_1 &= \frac{1}{4} \text{tr} \left\{ (V_{\mu}^{(\tau)}(\tilde{x}) + V_{\nu}^{(\tau')}(\tilde{x}) + V_{\mu}^{(\tau'')}(\tilde{x})^{\dagger} + V_{\nu}^{(\tau''')}(\tilde{x})^{\dagger} - 4) \cdot \right. \\ & \quad \left. \cdot (P_{\mu\nu}(\tilde{x}, \tilde{x}) + P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu) + P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu-2\nu) + P_{\mu\nu}(\tilde{x}, \tilde{x}-2\nu) - 4) \right\} \quad (3.13) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_2 &= 2 \text{tr} \left\{ V_{\mu}^{(\tau)}(\tilde{x}) + V_{\nu}^{(\tau')}(\tilde{x}) + V_{\mu}^{(\tau'')}(\tilde{x})^{\dagger} + V_{\nu}^{(\tau''')}(\tilde{x})^{\dagger} - 4 \right. \\ & \quad \left. + U_{\mu}^{(\tau)}(\tilde{x}) + U_{\nu}^{(\tau')}(\tilde{x}) + U_{\mu}^{(\tau'')}(\tilde{x}) + U_{\nu}^{(\tau''')}(\tilde{x}) - 4 \right\} \quad (3.14) \end{aligned}$$

$$\begin{aligned}
 s_3 = & \frac{1}{4} \left\{ \left(V_\mu^{(\tau)}(\bar{x}) + V_\mu^{(\tau'')}(\bar{x})^\dagger + U_\mu^{(\tau)}(\bar{x}) + U_\mu^{(\tau'')}(\bar{x}) - 4 \right) \cdot \right. \\
 & \cdot \left(V_\nu^{(\tau')}(\bar{x}) + V_\nu^{(\tau''')}(\bar{x})^\dagger + U_\nu^{(\tau')}(\bar{x}) + U_\nu^{(\tau''')}(\bar{x}) - 4 \right) \\
 & + 2 \left(V_\mu^{(\tau)}(\bar{x}) - 1 \right) \left(V_\mu^{(\tau')}(\bar{x})^\dagger - 1 \right) + 2 \left(V_\nu^{(\tau')}(\bar{x}) - 1 \right) \left(V_\nu^{(\tau''')}(\bar{x})^\dagger - 1 \right) \\
 & \left. + 2 \left(U_\mu^{(\tau)}(\bar{x}) - 1 \right) \left(U_\mu^{(\tau'')}(\bar{x}) - 1 \right) + 2 \left(U_\nu^{(\tau')}(\bar{x}) - 1 \right) \left(U_\nu^{(\tau''')}(\bar{x}) - 1 \right) \right\} \quad (3.15)
 \end{aligned}$$

The first part of s_1 shows how the "external" plaquette variables $P_{\mu\nu}(\bar{x}, \bar{y})$ couple to the internal block variables $V_\mu^{(\tau)}(\bar{x})$, $U_\mu^{(\tau)}(\bar{x})$, which will be integrated out later. Notice that internal block variables from a given block $B(\bar{x})$ couple to all plaquette variables $P_{\mu\nu}(\bar{x}, \bar{y})$ ($\bar{y} = \bar{x}, \bar{x}-2\mu, \bar{x}-2\mu-2\nu, \bar{x}-2\nu$), which have $\bar{x}$ as a common corner point (Fig. 2).

The second and third terms do not depend on the "external" plaquette variables. The contribution s_3 contains the couplings among the internal block variables within one block.

Finally, we want to emphasize that the block diagonalized action is gauge invariant and reproduces for $a \rightarrow 0$ the continuum action just as the original Wilson action.

4. - TRANSFORMATION OF THE INTEGRATION MEASURE TO DOUBLE LINK - AND INTERNAL BLOCK VARIABLES

In the last section, we achieved a block diagonalization of the gluon action in the following sense : internal block variables are no longer coupled among neighbouring blocks. We are left only with the couplings of the 45 independent internal block variables within one block. Note, however, that the internal block variables of different blocks might still be coupled via the integration measure in the path integral. We will show in this section that

this is not the case. Indeed, we will prove that the change from the original link variables $U(x, x+\mu)$ to the double link - and internal block variables, is accompanied by a Jacobian equal to one :

$$\prod_{x,\mu} \int dU(x, x+\mu) \dots = \prod_{\tilde{x}, \mu, \tau} \int dU(\tilde{x}, \tilde{x}+2\mu) dV_{\mu}^{(\tau)}(\tilde{x}) dU_{\mu}^{(\tau)}(\tilde{x}) \dots \quad (4.1)$$

Due to Eq. (2.19), the right-hand side is equal to

$$\prod_{\tilde{x}, \mu, \tau} \int dU_{\mu}^{(\tau)}(\tilde{x}, \tilde{x}+2\mu) dU_{\mu}^{(\tau)}(\tilde{x}, \tilde{x})$$

The gauge variables $U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y})$, $\tilde{y} = \tilde{x}+2\mu, \tilde{x}$ are expressed in terms of open and closed Wilson lines in Tables a to d.

For the proof of Eq. (4.1), we make use of the invariance of the Haar measure under left and right multiplication with $SU(N_c)$ matrices on the one hand and gauge invariance on the other hand. We proceed in four steps, transforming the original link variable $U(x, x+\mu)$ lying

- a) within the $\tilde{S}$ links (these are the links on the sublattice $\tilde{S}$ with double lattice spacing) ;
 - b) inside the $\tilde{S}$ plaquettes ;
 - c) inside the $\tilde{S}-(2a)^3$ cubes ;
 - d) inside the $\tilde{S}-(2a)^4$ cubes
- into the new gauge fields $U_{\mu}^{(\tau)}(\tilde{x}, \tilde{y})$:

- a) We fix the gauge for the link variables :

$$U(\tilde{x}+\mu, \tilde{x}+2\mu) = 1 \quad \mu = 0, 1, 2, 3 \quad (4.2a)$$

Then we can substitute the original link variables $U(\tilde{x}, \tilde{x}+\mu)$ by the new ones :

$$U_{\mu}^{(++++)}(\tilde{x}, \tilde{x}+2\mu) = W(\mu, \mu) = U(\tilde{x}, \tilde{x}+\mu) \quad (4.2b)$$

- b) Next, let us consider the gauge variables associated with Wilson lines on the $\tilde{S}$ plaquette in the (0,1) plane (cf. Table b) :

$$\begin{aligned} U_0^{(--++)}(\tilde{x}, \tilde{x}+2\hat{0}) &= W(\hat{0}, \hat{1}, \hat{0}, -\hat{1}) \\ U_1^{(--++)}(\tilde{x}, \tilde{x}+2\hat{1}) &= W(\hat{0}, \hat{1}, \hat{1}, -\hat{0}) \\ U_0^{(---+)}(\tilde{x}, \tilde{x}) &= W(\hat{0}, \hat{1}, -\hat{0}, -\hat{1}) \end{aligned} \quad (4.3)$$

Notice that the first and the last link variables in all W 's of Eq. (4.3) were already transformed in step a). As can be seen from Fig. 3 [and Eq. (4.3)], all three paths coincide up to the centre $(x+\hat{0}+\hat{1})$ of the plaquette ; then they branch into three different directions $\hat{0}, \hat{1}, -\hat{0}$. The corresponding old link variables are expressed as products of the new gauge variables (4.2b) and (4.3), after fixing the gauge for the second link ($\hat{1}$)

$$u(\vec{x}+\hat{0}, \vec{x}+\hat{0}+\hat{1}) = 1$$

which is common to all three Wilson lines. The corresponding procedure is followed for the remaining plaquettes in the $(0,2), (0,3), (1,2), (1,3)$ and $(2,3)$ planes.

c) There are five gauge variables associated with the Wilson lines in the $(2a)^3$ cube with axis $(\hat{0}, \hat{1}, \hat{2})$:

$$\begin{aligned} u_0^{(+--+)}(\vec{x}, \vec{x}+2\hat{0}) &= W(\hat{0}, \hat{1}, \hat{2}, \hat{0}, -\hat{2}, -\hat{1}) \\ u_1^{(-+-+)}(\vec{x}, \vec{x}+2\hat{1}) &= W(\hat{0}, \hat{1}, \hat{2}, \hat{1}, -\hat{2}, -\hat{0}) \\ u_2^{(--++)}(\vec{x}, \vec{x}+2\hat{2}) &= W(\hat{0}, \hat{1}, \hat{2}, \hat{2}, -\hat{1}, -\hat{0}) \\ u_0^{(+--+)}(\vec{x}, \vec{x})^+ &= W(\hat{0}, \hat{1}, \hat{2}, -\hat{0}, -\hat{2}, -\hat{1}) \\ u_1^{(-+-+)}(\vec{x}, \vec{x})^+ &= W(\hat{0}, \hat{1}, \hat{2}, -\hat{1}, -\hat{2}, -\hat{0}) \end{aligned} \quad (4.4)$$

Notice that the first two and the last two links are lying on the $\tilde{S}$ plaquettes. The corresponding link variables were already transformed in steps a) and b). All the five paths coincide on the first three links up to the centre of the cube $(x+\hat{0}+\hat{1}+\hat{2})$, then they branch into five different directions $\hat{0}, \hat{1}, \hat{2}, -\hat{0}, -\hat{1}$. The corresponding old link variables are expressed in terms of the new gauge variables (4.2)-(4.4) after fixing the gauge for the third link ($\hat{2}$) :

$$u(\vec{x}+\hat{0}+\hat{1}, \vec{x}+\hat{0}+\hat{1}+\hat{2}) = 1$$

which is common to all five Wilson lines in Eq. (4.4). The same procedure is performed for the $(2a)^3$ cubes with axis $(\hat{0}, \hat{1}, \hat{3}), (\hat{0}, \hat{2}, \hat{3}), (\hat{1}, \hat{2}, \hat{3})$.

d) The remaining seven gauge variables correspond to paths in the $(2a)^4$ block $B(\tilde{x})$. As can be seen from Table d, the first three and the last three links in each W are lying in the $(2a)^3$ cubes treated in steps a)-c). The first four links are common to all seven paths and lead to the centre $(\tilde{x} + \hat{0} + \hat{1} + \hat{2} + \hat{3})$ of the block $B(\tilde{x})$. Then there is a branching into seven different directions $\hat{0}, \hat{1}, \hat{2}, \hat{3}, -\hat{0}, -\hat{1}, -\hat{2}$. The corresponding link variables are expressed in terms of the new gauge variables after fixing the gauge for the fourth link $(\hat{3})$:

$$U(\tilde{x} + \hat{0} + \hat{1} + \hat{2}, \tilde{x} + \hat{0} + \hat{1} + \hat{2} + \hat{3}) = 1$$

which is common to all Wilson lines in Table d. This completes our proof of Eq. (4.1).

Now we are prepared to perform the integration over the 45 internal block variables $V_{\mu}^{(\tau)}(\tilde{x}), U_{\mu}^{(\tau)}(\tilde{x})$.

5. - INTEGRATION OVER INTERNAL BLOCK VARIABLES

Due to the results obtained in Sections 3 and 4, we are now in the position to rewrite the path integral for the partition function in terms of double link and internal block variables :

$$Z = \prod_{\tilde{x}, \mu} \int dU(\tilde{x}, \tilde{x} + 2\mu) \exp\{16\beta \sum_{\tilde{x}, \mu\nu} \text{tr} P_{\mu\nu}(\tilde{x}, \tilde{x})\} \cdot \prod_{\tilde{x}} I\{P_{\mu\nu}(\tilde{x}, \tilde{y})\} \quad (5.1)$$

$$I\{P_{\mu\nu}(\tilde{x}, \tilde{y})\} = \prod_{\mu, \tau} \int dV_{\mu}^{(\tau)}(\tilde{x}) dU_{\mu}^{(\tau)}(\tilde{x}) \exp\{4\beta S_g(\tilde{x})\} \quad (5.2)$$

The block action $S_g(\tilde{x})$ is given in Eqs. (3.12)-(3.15). Remember that there is no coupling between block variables of different blocks ; we can therefore perform the integration of block variables for each block separately, as indicated in Eq. (5.2). The double link variables $U(\tilde{x}, \tilde{x} + 2\mu)$ enter via the plaquette variables $P_{\mu\nu}(\tilde{x}, \tilde{y})$ as "external" fields into the block action.

We will evaluate the block integrals (5.2) perturbatively. In the limit $\beta \rightarrow \infty$ the support of $V_{\mu}^{(\tau)}(\tilde{x}), U_{\mu}^{(\tau)}(\tilde{x})$ shrinks to the unit matrix, so that we can expand the group elements

$$V_{\mu}^{(\tau)}(\vec{x}) = \exp\{i(2a)^2 G_{\mu}^{(\tau)}\}, \quad U_{\mu}^{(\tau)}(\vec{x}) = \exp\{i(2a)^2 H_{\mu}^{(\tau)}\} \quad (5.3)$$

in terms of a power series in the Hermitian and traceless generators $(2a)^2 G_{\mu}^{(\tau)}$, $(2a)^2 H_{\mu}^{(\tau)}$. Inserting these expansions into Eqs. (3.13)-(3.15), and keeping all terms of order a^4 , we arrive at a surprisingly simple form of the block action :

$$S_1 = -(2a)^4 \text{tr} \left\{ (G_{\mu}^{(\tau)} + G_{\nu}^{(\tau')} - G_{\mu}^{(\tau'')} - G_{\nu}^{(\tau''')}) \bar{F}_{\mu\nu} \right\} \quad (5.4)$$

$$\begin{aligned} S_2 + S_3 = & -\frac{1}{2} (2a)^4 \text{tr} \left\{ (G_{\mu}^{(\tau)} - G_{\mu}^{(\tau'')} + G_{\nu}^{(\tau')} - G_{\nu}^{(\tau''')})^2 + (H_{\mu}^{(\tau)} + H_{\mu}^{(\tau'')} + H_{\nu}^{(\tau')} + H_{\nu}^{(\tau''')})^2 \right. \\ & \left. + (G_{\mu}^{(\tau)} - G_{\mu}^{(\tau'')} + H_{\nu}^{(\tau')} + H_{\nu}^{(\tau''')})^2 + (H_{\mu}^{(\tau)} + H_{\mu}^{(\tau'')} + G_{\nu}^{(\tau')} - G_{\nu}^{(\tau''')})^2 \right\} \quad (5.5) \end{aligned}$$

In Eq. (5.4), we have introduced the abbreviation

$$\bar{F}_{\mu\nu} = -i(2a)^{-2} \frac{1}{4} \left(P_{\mu\nu}(\vec{x}, \vec{x}) + P_{\mu\nu}(\vec{x}, \vec{x}-2\mu) + P_{\mu\nu}(\vec{x}, \vec{x}-2\mu-2\nu) + P_{\mu\nu}(\vec{x}, \vec{x}-2\nu) \right) \quad (5.6)$$

Remember that we have fixed the 15 internal gauge degrees of freedom in each block $B(\vec{x})$, when we passed in the last section from the original link variables to the internal block variables. Therefore, in the large β limit, we can substitute the Haar measure $dV_{\mu}^{(\tau)} dU_{\mu}^{(\tau)}$ by the measure for the traceless Hermitian matrices $G_{\mu}^{(\tau)}$, $H_{\mu}^{(\tau)}$. Inserting Eqs. (5.4), (5.5) and (3.12) for the block action, we realize that the integrals (5.2) are gaussian :

$$\int \{ F_i \} = \frac{45}{\pi} \int dX_i \exp \{ 4\beta S_g(\vec{x}) \} \quad (5.7)$$

where

$$S_g(\vec{x}) = -(2a)^4 \text{tr} \left(\sum_{j=1}^{28} F_j X_j + \frac{1}{2} \sum_{i,j=1}^{45} X_j C_{ji} X_i \right) \quad (5.8)$$

Here we have collected the 45 independent block variables $G_{\mu}^{(\tau)}$, $H_{\mu}^{(\tau)}$ in one vector X_i , $i = 1, \dots, 45$. The $G_{\mu}^{(\tau)}$ are identified with the first 28 components (X_i , $i = 1, \dots, 28$), the $H_{\mu}^{(\tau)}$ with the remaining 17 components ($X_i = 29, \dots, 45$). The F_i ($i = 1, \dots, 28$) are related in a similar way to the external plaquette

variables (5.6). Notice from Eq. (5.4) that they couple only to the $G_{\mu}^{(\tau)}$, i.e., to the first 28 components X_i , $i = 1, \dots, 28$. The C_{ij} are the matrix elements of the symmetric and positive 45×45 "block internal coupling matrix", which can be determined from Eqs. (3.12) and (5.5). The leading behaviour of the integral (5.7) in the large β limit is given by the absolute maximum of $S_g(X)$ [e.g. (5.8)]. Differentiating with respect to X_i we get, for the position of the maximum

$$X_i = - \sum_{j=1}^{28} (C^{-1})_{ij} F_j \quad i=1, \dots, 45 \quad (5.9)$$

which yields

$$\text{Max } S_g(X) = \frac{1}{2} (2a)^4 \sum_{i,j=1}^{28} (C^{-1})_{ij} \text{tr}(F_i F_j) \quad (5.10)$$

We have calculated the inverse C^{-1} of the block internal coupling matrix numerically and find for the block integral (5.2) in the large β limit

$$\log I \{ P_{\mu\nu}(X, \tilde{X}) \} = 4\beta (2a)^4 \frac{3}{2} \sum_{\mu \neq \nu} \text{tr}(\overline{F}_{\mu\nu}^2) \quad (5.11)$$

Consequences for the effective action on the sublattice $\tilde{S}$ with double lattice spacing will be discussed in the next section.

6. - THE EFFECTIVE ACTION

Having performed the integration over the block internal degrees of freedom, it is now straightforward to read, from Eqs. (5.1), (5.6) and (5.11), the resulting effective action :

$$\begin{aligned} S_{\text{eff}} = & 2 \cdot 11 \cdot \sum_{\mu\nu, \tilde{X}} \left[\text{tr } P_{\mu\nu}(\tilde{X}, \tilde{X}) \right. \\ & - \frac{3}{4} \frac{1}{11} \left\{ \text{tr } P_{\mu\nu}(\tilde{X}, \tilde{X}) P_{\mu\nu}(\tilde{X}, \tilde{X}-2\mu) + \text{tr } P_{\mu\nu}(\tilde{X}, \tilde{X}) P_{\mu\nu}(\tilde{X}, \tilde{X}-2\nu) \right\} \\ & \left. - \frac{3}{8} \frac{1}{11} \left\{ \text{tr } P_{\mu\nu}(\tilde{X}, \tilde{X}) P_{\mu\nu}(\tilde{X}, \tilde{X}-2\mu-2\nu) + \text{tr } P_{\mu\nu}(\tilde{X}, \tilde{X}-2\mu) P_{\mu\nu}(\tilde{X}, \tilde{X}-2\nu) \right\} \right] \quad (6.1) \end{aligned}$$

According to the definition of our plaquette variables $P_{\mu\nu}(\tilde{x}, \tilde{y})$ (Section 3), the new terms in the action (6.1) represent planar Wilson loops with six and eight links shown in Fig. 4. The similarity between our effective action and Symanzik's improved action for pure gauge theories⁹⁾ is indeed striking. The relative weight between the familiar plaquette term and the rectangular Wilson loops is, in our case, $-\frac{3}{4}\frac{1}{11} = -0.068$, in Symanzik's improved action -0.05 . The planar eight link loops, which enter here with a relative weight $-\frac{3}{8}\frac{1}{11}$, do not appear in Symanzik's action, since loops with more than six links have been discarded from the beginning.

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$$\begin{aligned} U_0^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{o}) & U_1^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{1}, \hat{1}) \\ U_2^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{2}, \hat{2}) & U_3^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{3}, \hat{3}) \end{aligned}$$

(a)

$$\begin{aligned} U_0^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{1}, \hat{o}, -\hat{1}) & U_0^{(-+-+)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{2}, \hat{o}, -\hat{2}) \\ U_1^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{o}, \hat{1}, \hat{1}, -\hat{o}) & U_2^{(-+-+)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{o}, \hat{2}, \hat{2}, -\hat{o}) \\ U_0^{(----)}(\tilde{x}, \tilde{x}) &= W(\hat{o}, \hat{1}, -\hat{o}, -\hat{1}) & U_0^{(-+-+)}(\tilde{x}, \tilde{x}) &= W(\hat{o}, \hat{2}, -\hat{o}, -\hat{2}) \end{aligned}$$

$$\begin{aligned} U_0^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{3}, \hat{o}, -\hat{3}) & U_1^{(+---)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{1}, \hat{2}, \hat{1}, -\hat{2}) \\ U_3^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{o}, \hat{3}, \hat{3}, -\hat{o}) & U_2^{(+---)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{1}, \hat{2}, \hat{2}, -\hat{1}) \\ U_0^{(----)}(\tilde{x}, \tilde{x}) &= W(\hat{o}, \hat{3}, -\hat{o}, -\hat{3}) & U_1^{(+---)}(\tilde{x}, \tilde{x}) &= W(\hat{1}, \hat{2}, -\hat{1}, -\hat{2}) \end{aligned}$$

$$\begin{aligned} U_1^{(+---)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{1}, \hat{3}, \hat{1}, -\hat{3}) & U_2^{(+---)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{2}, \hat{3}, \hat{2}, -\hat{3}) \\ U_3^{(+---)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{1}, \hat{3}, \hat{3}, -\hat{1}) & U_3^{(+---)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{2}, \hat{3}, \hat{3}, -\hat{2}) \\ U_1^{(+---)}(\tilde{x}, \tilde{x}) &= W(\hat{1}, \hat{3}, -\hat{1}, -\hat{3}) & U_2^{(+---)}(\tilde{x}, \tilde{x}) &= W(\hat{2}, \hat{3}, -\hat{2}, -\hat{3}) \end{aligned}$$

(b)

$$\begin{aligned}
U_0^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{o}-\hat{2}, -\hat{1}) & U_0^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{1}, \hat{3}, \hat{o}-\hat{3}, -\hat{1}) \\
U_1^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{1}, -\hat{2}, -\hat{o}) & U_1^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{o}, \hat{1}, \hat{3}, \hat{1}, -\hat{3}, -\hat{o}) \\
U_2^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{2}, -\hat{1}, -\hat{o}) & U_3^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{o}, \hat{1}, \hat{3}, \hat{3}, -\hat{1}, -\hat{o}) \\
U_0^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{o}, \hat{1}, \hat{2}, -\hat{o}, -\hat{2}, -\hat{1}) & U_0^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{o}, \hat{1}, \hat{3}, -\hat{o}, -\hat{3}, -\hat{1}) \\
U_1^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{o}, \hat{1}, \hat{2}, -\hat{1}, -\hat{2}, -\hat{o}) & U_1^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{o}, \hat{1}, \hat{3}, -\hat{1}, -\hat{3}, -\hat{o}) \\
U_0^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{2}, \hat{3}, \hat{o}, -\hat{3}, -\hat{2}) & U_1^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{1}, \hat{2}, \hat{3}, \hat{1}, -\hat{3}, -\hat{2}) \\
U_2^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{o}, \hat{2}, \hat{3}, \hat{2}, -\hat{3}, -\hat{o}) & U_2^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{1}, \hat{2}, \hat{3}, \hat{2}, -\hat{3}, -\hat{1}) \\
U_3^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{o}, \hat{2}, \hat{3}, \hat{3}, -\hat{2}, -\hat{o}) & U_3^{(++++)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{1}, \hat{2}, \hat{3}, \hat{3}, -\hat{2}, -\hat{1}) \\
U_0^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{o}, \hat{2}, \hat{3}, -\hat{o}, -\hat{3}, -\hat{2}) & U_1^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{1}, \hat{2}, \hat{3}, -\hat{1}, -\hat{3}, -\hat{2}) \\
U_2^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{o}, \hat{2}, \hat{3}, -\hat{2}, -\hat{3}, -\hat{o}) & U_2^{(++++)}(\tilde{x}, \tilde{x})^+ &= W(\hat{1}, \hat{2}, \hat{3}, -\hat{2}, -\hat{3}, -\hat{1})
\end{aligned}$$

(c)

$$\begin{aligned}
U_0^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{o}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, \hat{o}, -\hat{3}, -\hat{2}, -\hat{1}) \\
U_1^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{1}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, \hat{1}, -\hat{3}, -\hat{2}, -\hat{o}) \\
U_2^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{2}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, \hat{2}, -\hat{3}, -\hat{1}, -\hat{o}) \\
U_3^{(----)}(\tilde{x}, \tilde{x}+2 \cdot \hat{3}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, \hat{3}, -\hat{2}, -\hat{1}, -\hat{o}) \\
U_0^{(----)}(\tilde{x}, \tilde{x}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, -\hat{o}, -\hat{3}, -\hat{2}, -\hat{1}) \\
U_1^{(----)}(\tilde{x}, \tilde{x}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, -\hat{1}, -\hat{3}, -\hat{2}, -\hat{o}) \\
U_2^{(----)}(\tilde{x}, \tilde{x}) &= W(\hat{o}, \hat{1}, \hat{2}, \hat{3}, -\hat{2}, -\hat{3}, -\hat{1}, -\hat{o})
\end{aligned}$$

(d)

TABLE - The new gauge fields $U_\mu^{(\tau)}(\tilde{x}, \tilde{y})$ [Eqs. (2.8)-(2.10)] are products of the old link variables $U(x, x+\mu)$ along a path ϕ starting at $\tilde{x}$ ending at $\tilde{y}$. The sequence of links along this path ϕ is indicated in the brackets.

- (a) linear paths ϕ ;
- (b) planar paths ϕ ;
- (c) paths ϕ in three dimensions ;
- (d) paths ϕ in four dimensions.

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FIGURE CAPTIONS

- Fig. 1 The plaquette variable $P_{\mu\nu}(\tilde{y}, \tilde{x})$ is defined as the anti-clockwise ordered product of four link variables in the $\mu\nu$ plane. The loop starts and ends at $\tilde{y}$, which can be either $\tilde{x}$, $\tilde{x}+2\mu$, $\tilde{x}+2\mu+2\nu$, or $\tilde{x}+2\nu$. The second index $\tilde{x}$ always refers to the lower left corner of the plaquette.
- Fig. 2 The definition of the neighbouring plaquette variables $P_{\mu\nu}(\tilde{x}, \tilde{x})$, $P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu)$, $P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu-2\nu)$ and $P_{\mu\nu}(\tilde{x}, \tilde{x}-2\nu)$.
- Fig. 3 The linear and planar paths which enter in Eqs. (4.2b) and (4.3). The links for which the gauge is fixed are marked by "1".
- Fig. 4 The planar six and eight link Wilson loops appearing in the effective action (6.1). The numbers indicate the relative weight to the plaquette term in the effective action.

Fig. 1

Fig. 2

Fig. 3

$$-\frac{3}{4} \frac{1}{11} \text{tr } P_{\mu\nu}(\tilde{x}, \tilde{x}) P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu)$$

$$-\frac{3}{4} \frac{1}{11} \text{tr } P_{\mu\nu}(\tilde{x}, \tilde{x}) P_{\mu\nu}(\tilde{x}, \tilde{x}-2\nu)$$

$$-\frac{3}{8} \frac{1}{11} \text{tr } P_{\mu\nu}(\tilde{x}, \tilde{x}) P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu-2\nu)$$

$$-\frac{3}{8} \frac{1}{11} \text{tr } P_{\mu\nu}(\tilde{x}, \tilde{x}-2\mu) P_{\mu\nu}(\tilde{x}, \tilde{x}-2\nu)$$

Fig. 4