COMMENTS ABOUT RIEMANNIAN GEOMETRY, EINSTEIN SPACES, KALUZA-KLEIN, 11-DIMENSIONAL SUPERGRAVITY AND ALL THAT

R. Coquereaux *)
CERN -- Geneva

ABSTRACT

We collect a number of definitions, theorems and properties of Riemannian manifolds as well as a list of references to the mathematical literature which may help the interested physicist. Special attention is devoted to Einstein manifolds (in particular, in seven dimensions). Several comments are made about the possible use (and misuse of this material, and questions are asked by the author about the structure of 11-dimensional supergravity theories.

*) On leave of absence from CPT, CNRS, Marseille.
INTRODUCTION

The recent development of those theories obtained by "dimensional reduction" makes compulsory the use of mathematical results in differential and Riemannian geometry; most of these results can be considered as recent results (being less than 80 years old), and very often do not figure in standard text books. Very often also, it is difficult to extract the relevant information (from a physicist's point of view) out of the mathematical papers. One of the aims of this article is to guide the interested reader (assumed to be a physicist) among the available literature. Many definitions and up-to-date results are given, with special attention being paid to the properties and classification of Einstein spaces. Careful attention is paid to those notions which look similar but are actually logically distinct; it is important to stress that precise words and definitions already exist and that it is very confusing and sometimes misleading to use new words for standard concepts, even worse to give a different meaning to the existing terminology. We therefore collect a number of comments, definitions, counter-examples and theorems of geometry which, we hope, could help the interested physicists. This collection of comments is by no means "complete", in any sense; this would be beyond the scope of such a paper. The paper consists of a list of 33 comments which in principle can be read in any order; however, it is self-consistent in the sense that a notion used in a given comment has always been introduced in a previous comment. The first thirty comments deal mainly with geometrical notions which have been used (or misused) in the recent physical literature (which is not always quoted); these comments can in principle be read by anybody (even by people who have no working familiarity with "dimensional reduction" and related subjects). The last three comments reflect some points of view and questions from the author about the structure of "dimensional reduction" and related topics; these last comments are probably unreadable by people unaware of the current physical literature on the subject.

Comment 1

A topological space is not necessarily a topological manifold. Example: $\infty$ is not a manifold (it contains a cross $\times$).

Comment 2

A given topological manifold may be given several differentiable structures. For example, the number of differentiable structures for spheres is: 1 for $S^p$ ($p < 6$), 28 for $S^7$, 8 for $S^9$, 92 for $S^{11}$, ... [1,2]). This concept has nothing to do with the concepts of linear connection and metric structure; it is more fundamental: given a set $X$, we have to specify, first its topological structure, then its differentiable structure, and finally its Riemannian structure (or a linear connection). At each stage there are groups leaving the structure
invariant; they are called the group of homeomorphisms $T$, the group of diffeomorphisms $D$, the group of isometries $I$. It is clear that $I \subset D \subset T$. We can always construct a lot of Riemannian structures over $X$ for each class of diffeomorphism class of differentiable structures over $X$. There is very often a standard way of realizing a given smoothable manifold (for example, via a coset space representation); the inherited differentiable structure is then called standard; the others, if they exist, are therefore called exotic. It should be stressed that, in all examples of manifolds used until now by physicists (even seven spheres with fancy metrics), the differentiable structure is always the standard one.

Comment 3

Given a Riemannian manifold $X$, i.e., a differentiable manifold and a metric structure ("a shape"), the group of isometries $I$ is a Lie group and it is compact if $X$ is compact$^3$. Notice that the groups $T$ and $D$ introduced in Comment 2 are usually infinite dimensional and moreover are not necessarily Lie groups.

Comment 4

A given differentiable manifold $M$ may also have one or several algebraic structures: it can be a Lie group (multiplication of points in $M$ is defined), it can be also a homogeneous space (there is an action of a group $G$ on $M$, and $M$ as a set, can be realized as $G/H$). It is quite common to find several groups acting transitively on the same differentiable manifold $M$; for example, the standard differentiable sphere $S^7$ can be realized as $SO(8)/SO(7)$, $Spin(7)/G_2$, $U(4)/U(3)$, $SU(4)/SU(3)$, $U(2,H)/U(1,H)$, $U(2,H) \times SU(2)/U(1,H) \times SU(2)$, $U(2,H) \times U(1)/U(1,H) \times U(1)$. Here, $U(n,H)$ denotes $n \times n$ unitary matrices over quaternions, $U(n,H)$ is sometimes called $Sp(n)$, or $Sp(2n)$ or $USp(2n)$; the previous list of groups acting transitively on $S^7$ is exhaustive [Refs. 4 and 5] and references therein.

Comment 5

As any differentiable (paracompact) manifold, a Lie group $G$ may have an infinite number of Riemannian metrics (i.e., shapes). If $G$ is simple, the most "natural" one is the negative of the Killing form, for example on $SU(2) = S^3$ let us choose three left invariant vector fields $\varepsilon_1(\psi, \delta, \chi)$ satisfying $[\varepsilon_1, \varepsilon_2] = 2\varepsilon_3$, etc., and consider the one-parameter family of (inverse) metrics $g^{-1} = 1/r^2[\varepsilon_1 \otimes \varepsilon_1 + \varepsilon_2 \otimes \varepsilon_2 + \varepsilon_3 \otimes \varepsilon_3]$. The isometry group of these metrics is $G \times G$ and for this reason they are called bi-invariant metrics. If we decide to fix the volume of $G$ (which is $2\pi^2 r^3$ in the previous example), then the bi-invariant metric is unique.
If $G$ is not simple however, we can construct several bi-invariant metrics by scaling differently the simple components of the Killing form of $G$. It is clear that a metric such as $g^{-1} = \Lambda_{i j} \varepsilon_i \otimes \varepsilon_j$, where $\Lambda_{i j}$ are constants ($\neq 1$) is at least $G$ invariant (the actual isometry group may be bigger than $G$ but in all cases smaller than $G \times G$); for example, the following metric on $S^3$:

\[ \lambda(e_1 \otimes e_1 + e_2 \otimes e_2) + \mu e_3 \otimes e_3 \] is $SU(2) \times U(1)$ invariant. An exhaustive treatment of left (or right) $G$ invariant metrics on Lie groups $G$ may be found in Ref. 6). If we allow $\Lambda_{i j}$ to depend upon the point where $g^{-1}$ is evaluated [for example, $\Lambda_{i j}(\psi, \theta, \phi)$ in the case of $SU(2)$], the metric will in general not be $G$ invariant, as it may still have an isometry group of the kind $H \times K \subseteq G \times G$. The number of $G$ invariant metrics on $G$ is of course $n(n+1)/2$, with $n = \dim G$.

Comment 6

A torus (like $T^7 = S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1$) is not necessarily flat (think of a tyre with a swelling). However, it is clear that any left invariant metric on a torus (there are $7 \times 9/2$ of them in our example) is flat; it is also an Einstein metric.

Comment 7

Let $G$ be a connected Lie group and $H$ a (closed) Lie subgroup of $G$; if there is a subspace $m$ of Lie $(G)$ so that Lie $(G) = \text{Lie } (H) + m$ such that $[\text{Lie } (H),m] \subseteq m$, then we say that $(G,H)$ is a reductive pair. $m$ can be identified as the tangent space at the origin of the homogeneous space $G/H$. A homogeneous space is called symmetric if, moreover $[m,m] \subseteq \text{Lie } (H)$; it is maybe useful to stress the fact that a homogeneous space is not necessarily symmetric! For example, in Comment 4, among the eight possible realizations of $S^7$ as homogeneous space, only the first one is symmetric. The word "symmetry" possesses, in this context, an algebraic meaning and should not be applied to the underlying differentiable manifold itself. See also comment 9. A complete list of irreducible symmetric spaces may be found for example in Refs. 7) and 8). (Reducible symmetric spaces are of course obtained by making products of irreducible ones.)

Comment 8

If a differentiable manifold $S$ can be realized as a homogeneous space $G/H$ (with non-trivial $H$), it is reasonable to look for possible $G$ invariant metrics on $S$. Such metrics are in one-to-one correspondence with $\text{Ad } H$ invariant bilinear symmetric forms on the tangent space at the origin of $S^7$). Finding the dimensionality $d$ of the set of all $G$ invariant metrics on $S$ (this set is a connected manifold) becomes a simple exercise in representation theory; several examples are worked out in Ref. 10) (Par. 4). In the case of $S^7$, this
dimensionality \( d \) is 1, 1, 2, 2, 7, 2, 3 when \( G \) goes through the list 
\( \text{SO}(8), \text{Spin} \, 7, U(n), SU(n), U(2, H), U(2, H) \times SU(2), U(2, H) \times U(1) \) of those groups 
for which \( S \) is a homogeneous space. It should be stressed that, when \( G/H \) 
is symmetric irreducible [taken for example in the list of Ref. 7, p. 288] or 
more generally when \( G/H \) is isotropy-irreducible, i.e., if the representation 
of the (connected component of the) group \( H \) on the tangent space at the origin 
of \( S \) is real irreducible [complete list in Ref. 11] then the number \( d \) is 
one. Very often, people decide to fix the volume of \( S \) to some conventional 
value; in that case the previous number \( d \) is decreased by one unit.

Comment 9

There is another definition of the word "symmetric" which refers, not to the 
'algebraic structure of a given differentiable manifold but to its metric structure'\(^7\). 
However, the confusion is generally harmless: the only isotropy irreducible 
spaces \( G/H \) which are not symmetric from the algebraic point of view (comment 7) 
but symmetric (in the Riemannian sense) for their unique (up to scale) \( G \) 
-invariant metric are spin \((7)/G_2 \) and \( G_2/SU(3) \). For this reason they are identified 
with the algebraically symmetric spaces \( \text{SO}(8)/\text{SO}(7) \) and \( \text{SO}(7)/\text{SO}(6) \).

Comment 10

When a manifold \( S \) can be considered as a homogeneous space \( G/H \), any 
metric on \( S \) induced by a bi-invariant metric on \( G \) is called normal. Such 
metrics are, of course, \( G \) invariant, but are not in general the only ones to 
possess this property (see comment 8). Of course, this notion makes an explicit 
reference to the pair \((G,H)\) and should not be employed without making this 
reference.

Comment 11

For a given metric \( g \) on a manifold \( S \), one should establish a difference 
between a group which is a group of isometries and another one (bigger) which is 
the full group of isometries. In the case of \( S \) with the normal metric inherited 
from \( \text{SO}(8)/\text{SO}(7) \) or from \( \text{Spin}(7)/G_2 \) (it is the same metric), the full group 
of isometry is \( O(8) \).

Comment 12

Let \( M \) be a differentiable manifold and \( N(M) \) the upper bound of the di-
mensions of the full isometry groups of all possible Riemannian metrics on \( M \); 
this number is sometimes called the Hsiang dimension of \( M \)\(^{12} \). When \( M \) is 
the underlying smooth structure of a compact homogeneous space and \( G \) is the 
biggest transitive group on \( M \), it is commonly believed that \( N(M) = \dim G \), 
however, ... it has never been proved\(^{12} \), but nobody knows any counterexample.
Comment 13

In the same way that one usually considers connections in a manifold $P$ (principal fibre bundle) which is a local product $P = (\text{space-time}) \times \text{Lie group}$, one may also consider connections in $G$ considered as a principal fibre bundle over $G/H$; i.e., $G = (G/H) \times \text{Lie (H)}$. These are one-forms valued in the Lie algebra $\text{Lie (H)}$. When $(G,H)$ is a reductive pair (see comment 7), one of these connections is called the canonical connection associated with the reduction: it is obtained by taking the projection on $\text{Lie (H)}$ of the Maurer-Cartan form on $G$.

Let $o$ be the origin of $G/H$, namely the point represented by the coset $H$. Every element $h \in H$ leaves $o$ invariant but "rotates" vectors in the tangent space to $G/H$ at $o$: it induces a linear representation in this vector space (the so-called linear isotropy representation — see comment 8).

We therefore have a map $h \in H \rightarrow \tilde{h} \in G \leq (n,R)$ (where $n = \dim G/H$); starting with the canonical connection [valued in $\text{Lie (H)}$] we can then construct the linear canonical connection [valued in $\text{Lie (G \leq (n,R))}$]; this linear connection is, of course, invariant under $G$; however, it usually has non-zero torsion $T$ unless $G/H$ is a symmetric space. Moreover, if $R$ is the curvature, one can prove $VR = VT = 0$ [see Ref. 9].

Comment 14 (see comment 13)

There is only one $G$ invariant torsion-free linear connection with the same geodesics as the linear canonical connection; it is called the natural torsion-free linear connection$^9$. In the case of a symmetric space these two notions coincide, since $T = 0$.

Comment 15

A $G$ invariant metric $g$ on a differentiable manifold $M$ (realized as a homogeneous space $G/H$) is called naturally reductive if $\text{Lie (G)}$ admits a reductive decomposition (see comment 7) in such a way that the Levi-Civita connection associated with $g$ coincides with the natural torsion-free connection associated with the reduction$^9$. To obtain all naturally reductive metrics on a manifold $M$, one has to find all groups $G$ acting transitively on $M$ and choose all metrics on $G/H$ induced by a non-degenerate bi-invariant form (not necessarily positive definite) on $G = \text{Lie G}^{13}$; normal metrics (comment 10) are obtained by such forms which are positive definite.

Comment 16

There is nothing called "squashed spheres" or "squashed manifolds" in the mathematical literature. However, a reasonable definition of the word "squashed" could be the following: let $M$ be a differentiable manifold endowed with a maximally symmetric metric, then fix its volume and consider all other possible metrics (shapes) on $M$; it is therefore natural and intuitive to call "squashed"
all these manifolds by opposition to the first one. With such a definition (which is certainly not necessary), the so-called "squashed seven sphere"\(^{14}\) sometimes used to provide a solution to \(d = 11\) supergravity is not more squashed than infinitely many others, its only virtue is to be an Einstein space.

**Comment 17**

Questions of holonomy. The holonomy group of a connection (nothing to do, of course, with the notion of isometries) can be defined in a purely geometrical way (via parallel transport), but is usually computed by algebraic methods\(^{9,15}\). By "holonomy group of a Riemannian manifold" one means holonomy group of the Levi-Civita connection associated with the metric. It is very often easy to find the holonomy group without any calculations by using a few theorems to be found in Refs. 11, 16 and 17). When the manifold is not simply connected, one has to introduce the notion of restricted holonomy group\(^9\). There is also a weaker notion called "weak holonomy group" [Ref. 18 and references therein]. Notice that the holonomy group of the non-standard homogeneous Einstein metric on \(S^7\) is not \(G_2\), although the group obtained at the end of the calculation in Ref. 14 is \(G_2\) and is indeed what the authors have in mind (it just should not be called "holonomy group of the manifold"). \(G_2\) is not even the weak holonomy group of this non-standard Einstein \(S^7\) (it is the weak holonomy group of the standard \(S^7\)). By the way, there are no known examples of Riemannian manifolds with holonomy group \(G_2\)\(^{18}\). However, if one allows for a non-vanishing torsion, it then becomes possible to find \(G_2\) as holonomy group\(^{19}\).

**Comment 18**

Questions of parallelization. Use of "parallelizable manifolds" has been made recently, either in order to introduce new solutions to 11-dimensional supergravity\(^{20}\) or to "solve" the cosmological constant problem\(^{21}\). It should be stressed that there are two distinct notions of parallelism. A differentiable \(n\)-manifold is called "parallelizable" if it is possible to find \(n\) independent vector fields everywhere; (for general properties, the best reference is probably Ref. 22). It is well known that Lie groups are parallelizable and that, among spheres, \(S^1\), \(S^3\) and \(S^7\) are the only ones which have this property\(^{23}\). However, it should not be believed (or written) that these manifolds are the only ones which have these properties; for example, any product of spheres (more than one) where one of them is of odd dimension is parallelizable\(^{24}\); other examples [like \(Sp(n)/SU(2)\)] can be found in Ref. 25); however, the complete classification is not known. If, on such a manifold, one chooses such a set of independent vector fields, one at the same time defines a way of comparing vectors at different points and therefore defines a connection.
(which is flat: $R = 0$, but possesses non-zero torsion); the manifold is then called paralleled. There is another notion, more restrictive, which is the following: a Riemannian manifold (we have chosen a metric) is said to possess a parallelization compatible with its metric if it is parallelizable, and if it is possible to choose a parallelization so that the induced connection is compatible with the metric (autoparallels and geodesics should coincide); in this last case, among compact manifolds, only $S^1$, $S^3$, $S^7$ and Lie groups have these properties\(^{26,27}\). One should notice that the seven sphere used in Ref. 14) to provide a new solution to supergravity should not be called "parallelized" as it is not. The chosen connection is not flat, and the exhibited solution makes use only of a tensor which could be used to parallelize the sphere (it would be the torsion tensor of the flat connection). Finally, notice that the holonomy group of the connection associated with a parallelization of a (parallelizable) manifold is, of course, trivial.

Comment 19

Fibre bundle theory. Working in Kaluza-Klein theories without using the language of fibre bundles is like working in quantum mechanics without using the language of vector spaces ... The first chapters of Ref. 9) provide a good introduction to the subject and more information can be found in Refs. 28) and 29). Roughly speaking, let us say that a principal fibre bundle $(M,K)$ can be thought of as a "local product" of a manifold $M$ (the base) times a Lie group $K$ (the fibre). Starting with a group $K$, one can consider actions of $K$ on other spaces: linear action on a vector space $V$ (representations) or, more generally, non-linear action on another space $S$ (for example, on $K/H$). In the same way, starting with a principal fibre bundle, one can construct associated vector bundles $(M,V)$ or more generally associated bundles $(M,S)$. Given a non-principal bundle $\xi$, one has to identify the structure group $K$ of the bundle and the corresponding principal bundle; $K$ acts "locally" (linearly or not) on $\xi$ and can be "gauged". Besides, there is maybe another group $G$ acting "globally" on $\xi$. Let us give a few basic examples: a group $G$ can be considered as a principal fibre bundle $(G/H,H)$; a given manifold can be sometimes fibrated in several ways, for example, a principal fibre bundle $(M,G)$ with basis $M$ and typical fibre $G$ can also be considered as a principal fibre bundle $(E,H)$ where $H$ is a (closed) subgroup of $G$ and where the new base $E$ is itself an associated bundle $(M,G/H)$. It is traditional in physics to establish a distinction between "active transformations" (which operate on geometrical objects) and "passive transformations" (which leave geometrical objects fixed but operate on their components); this distinction is indeed also crucial in fibre bundle theory and it is very often compulsory to establish a clear distinction between what is "active" and what is "passive", indeed, both groups
of transformations are not necessarily identical. It should be stressed that the structure group of an associated bundle (the one which can be gauged) never acts on this bundle as it is the "passive group". For example, if we assume that we have a global action of a group \( G \) on a fibre bundle \( (M, H, G) \), then the structure group is not \( G \) but \( N/H \) where \( N \) is the normalizer of \( H \) into \( G \) [see Ref. 10] for more details].

Comment 20

It has been said that the choice of a metric \( g \) on a smooth manifold \( M \) determines its shape; however, two distinct metrics may lead to the same "shape". To be more precise, we need to define the action of the group \( D \) of diffeomorphisms of \( M \) on the manifold \( M \) of metrics. A diffeomorphism \( \phi \) of \( M \) is in particular an active transformation: \( P \in M \Rightarrow Q = \phi(P) \in M \); the tangent map (Jacobian matrix) denoted by \( \phi_* \) acts on vectors of the tangent space \( T_P \) at \( P \) and gives vectors of the tangent space \( T_Q \) at \( Q \): \( V \in T_P \Rightarrow \phi_* V = \phi_* (V) \in T_Q \). Given a metric \( g_1 \), one knows how to compute the scalar product \( g_1 (V_1, V_2) \) of two vectors \( V_1 \) and \( V_2 \) at \( P \) and also the scalar product \( g_1 q (W_1, W_2) \) of their image \( W_1 \) and \( W_2 \) at \( Q \). One can now define a new metric \( g_2 \) by defining the value of \( g_2 \) at \( P \) as follows: \( g_2 (V_1, V_2) \equiv g_1 q (W_1, W_2) \). One says that \( g_2 \) is the pull-back of \( g_1 \) by \( \phi \) and usually writes \( g_2 = \phi^* g_1 \). If two distinct metrics \( g_1 \) and \( g_2 \) are such that there exists such a diffeomorphism carrying \( g_1 \) to \( g_2 \) via pull-back, the "shapes" determined by \( g_1 \) and \( g_2 \) should be identified (indeed, in that case, a change in the metric can be compensated by a change of the co-ordinate system). Mathematically, one is therefore led to consider the space of Riemannian structures \( R \) (the space of "shapes") which is defined as the coset space \( R \cong M/D \). This space \( R \) is the physically relevant object and is at the root of what we mean by the "equivalence principle" (it is the Souriau "hyperspace" [30, 31]). The full group of isometries \( I \) of a given metric \( g \) is the set of all \( \phi \in D \) such that \( \phi^* g = g \) (see also comments 2 and 11); two distinct metrics \( g_1 \) and \( g_2 \) are also called isometric if there is a \( \phi \in D \) such that \( g_2 = \phi^* g_1 \), i.e., if they are in the same orbit under the action of \( D \), meaning that they lead to the same Riemannian structure. Two metrics of the same orbit (under the \( D \) action) will have conjugate stabilizers (i.e., conjugate isometry groups) but more generally we will decompose the infinite dimensional manifold of Riemannian metrics \( M \) into strata: two metrics are in the same stratum if (by definition) they have conjugate isometry groups. A given stratum \( M(I) \) is therefore a collection of orbits of the same type and can be labelled by \( (I) \in J \), where \( J \) is the set of conjugacy classes of those subgroups of \( D \) which are isometry groups for at least one metric on \( M \). A given stratum \( M(I) \) can itself be considered as a fibre bundle above \( R(I) \).
the space of Riemannian structures whose isometry group is \( I \) (up to conjugacy). One is therefore led to a stratification of the space \( R \) of Riemannian structures \( R = \bigcup R(I) \). [See Ref. 32 for details]. Notice that, although defined as a coset space, \( R \) is usually not a manifold; besides, it is important to notice that \( R(I) \) is finite dimensional if and only if the group \( I \) is transitive over \( M \). The space \( R \) plays, in Riemannian geometry, a role analogous to the role played in Yang-Mills theory by the set of connections modulo gauge transformations. Let us finish this comment with a few examples: in the case of \( SU(2) \), we saw in comment 5 that the manifold of left invariant metrics (meaning: at least left invariant) is of dimension \( 3 \times 4/2 = 6 \); however, it is enough to know three numbers \( \lambda_1, \lambda_2, \lambda_3 \) in order to precise the Riemannian structure (it is always possible in this case to choose three vectors \( \{e_i\} \) satisfying \( [e_1, e_2] = 2 \varepsilon_3 \), etc., and to characterize a given Riemannian structure by the (inverse) metric \( g^{-1} = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3 \). One needs actually only two numbers if the volume is fixed. In the same way, the manifold of \( U(2,4) \) (\( = USp(4) \)) invariant metrics on \( S^7 \) is of dimension 7 (see comment 8) but four numbers only are needed to specify the Riemannian structure (only three if the volume is fixed). It is well known that the left invariant metrics on \( G \) with given volume can be parametrized by \( SL(n, \mathbb{R})/SO(n, \mathbb{R}) \) where \( n = \dim G \); if \( G \) is simple, two such metrics are isometric if and only if they are equivalent by an element of \( \text{Aut}^+(G) \subset SL(n, \mathbb{R}) \), \( \text{Aut}^+(G) \) being the automorphisms of \( G = \text{Lie}(G) \) with determinant 1]; the space of non-isometric left invariant metrics - i.e., the space of Riemannian structures on \( G \) which are at least \( G \) invariant - is therefore parametrized by \( \text{Aut}^+(G) \setminus SL(n, \mathbb{R})/SO(n, \mathbb{R}) \); it needs not be a manifold\(^{13} \).

Comment 21

When studying a Riemannian manifold \((M, g)\), one usually considers the Riemann tensor \( R \) (curvature), the Ricci tensor \( \text{Ric} \) and the scalar curvature \( \tau \); general relations can be found in all textbooks [for example in Refs. 33 and 34]. Let us remember only that \( \langle \text{Ric} \rangle_\nu = R_{\nu \alpha \nu}^\alpha \) and that \( \tau = \langle \text{Ric} \rangle_\nu = \text{Tr}(\text{Ric}) \). Two important notions appear frequently in the mathematical literature: the sectional curvature function \( K(\ldots) \) and the Ricci curvature function \( \rho(\cdot) \); the first one is a real valued function which, at the point \( P \), measures the curvature of the two-surface generated by geodesics with tangent vectors lying in the vector space generated by orthonormal vectors \( u \) and \( v \). Explicitly \( K(u, v) = v^\flat u^\flat \alpha \gamma R_{\alpha \beta u v} \). One proves easily that \( K \) is independent of the choice of (orthogonal) \( u \) and \( v \) in a given two-dimensional subspace. Let \( \{u, u_2, \ldots, u_n\} \) be an orthonormal basis at the point \( P \), then the Ricci curvature function is \( \rho(u) = \sum_{i=1}^n K(u, u_i) = \text{Ric}(u, u) \). Many details can be found, especially in Ref. 35].
Comment 22

It should not be believed that a compact Riemannian manifold always has a positive scalar curvature, ..., the easiest counter-example is provided by the 3 sphere $S^3 = SU(2)$ endowed with the following $SU(2) \times U(1)$ invariant metric $\sigma_1 = \epsilon_1 \otimes \epsilon_1 + \epsilon_2 \otimes \epsilon_2 + \epsilon_3 \otimes \epsilon_3 / t^2$, (with $[\epsilon_1, \epsilon_2] = 2 \epsilon_3$). The associated scalar curvature is $\tau = 2(4 - t^2)$; it is constant on $S^3$ but is zero when $t^2 = 4$ and becomes negative when $t^2 > 4$. (The calculation is easy but the result can be found in Ref. 36) - it can also be obtained by using a general formula, Ref. 10) par. 4.3, applied to the case of $S^3$ fibred in circles over $S^2$. $S^3$ is then considered as $U(2)/U(1)$ and $c = 0, k = 1, l = 2$. Actually, any compact Riemannian manifold admits a metric with constant negative scalar curvature.

Comment 23

It is a classical result [Ref. 9 vol. 1, Ref. 32 vol. 2] that the sectional curvature function determines completely the curvature (the Riemann tensor) of a Riemannian manifold. A manifold is called a manifold of constant curvature if the sectional curvature function is a constant function $K_\theta$ all over the manifold. It should be stressed that a homogeneous space $G/H$ endowed with a $G$ invariant metric (or a Lie group $G$ with a left invariant metric) is usually not a manifold of constant curvature: the individual components of the Riemann tensor are the same at all points of the manifold but they are not all equal (the manifold is not isotropic in general). Let us mention also the obvious implications: $[\text{constant sectional curvature}] \Rightarrow [\text{constant Ricci curvature}] \Rightarrow [\text{constant scalar curvature}]$ which do not go in the other direction! Spaces of constant sectional curvature are classified in Ref. 7). This is basically an algebraic problem, since all such manifolds have the standard simply-connected examples as their universal covering space (for example, those of positive curvature all locally isometric to the standard sphere with its standard metric).

Spaces of constant scalar curvature are probably unclassifiable indeed, any compact manifold of dimension bigger than two admits an infinite dimensional set of Riemannian metrics with constant scalar curvature). Homogeneous spaces $G/H$ with a $G$ invariant metric (and Lie groups with a left invariant metric) are manifolds of constant scalar curvature but the previous comment shows that they are far from being the only ones! It should be noticed that such metrics on $G/H$ which have constant scalar curvature but are not $G$ invariant could be used to perform "dimensional reduction", but have never been used or even referred to in the physical literature. Spaces of constant Ricci curvature are more often called Einstein spaces; they are "intermediate" between spaces of constant curvature and those of constant scalar curvature, and provide the subject of the next six comments.
Comment 24

A Riemannian manifold of constant Ricci curvature $\rho$ is called an Einstein space; this is equivalent to the usual definition ($\text{Ric} = \kappa g$). It is very often convenient to introduce the Einstein tensor $\mathcal{E}^i_{\ j} = (\text{Ric}^i_{\ j} - \frac{1}{2} g^i_{\ j})$; notice that $\text{Ric}^i_{\ j} = \mathcal{E}^i_{\ j} - \frac{1}{n-2} \text{Tr}(\mathcal{E}^i_{\ j}) \delta^i_{\ j}$, where $n = \text{dim } M$. Therefore, $\mathcal{E}$ and $\text{Ric}$ determine each other. If $(M, \mathcal{G})$ is an Einstein space with $\text{Ric} = \rho g$, then $\mathcal{E} + \Lambda g = 0$ with $-\Lambda = k - \frac{T}{2}$; besides, $\text{Tr}(\mathcal{E}^i_{\ j}) = -\Lambda n$. $k$ is called the Einstein constant and $\Lambda$ the cosmological constant (it is quite misleading to call $k$ itself the cosmological constant). Notice that the number $k$ is sometimes also called the mean curvature $\bar{\kappa}$, indeed, if the Einstein space is of constant curvature $K_0$ (which is by no means a necessary condition!), then $k = (n-1)K_0$.

In general, for an Einstein space of dimension $n$, $k = nk = 2\Lambda n/(n-2)$; for example, the standard $S^n$ sphere (with its standard Riemannian metric) is an Einstein space with $k = n - 1$, $\tau = n(n-1)$ and $\Lambda = -(n-1)(n-2)/2$ (and, of course, $K_0 = 1$). Here we suppose the volume normalized to its standard value $[2(\pi)^{n+1}/n!]$ for $S^{2n+1}$ and $2(2n)^n/(2n-1)(2n-3), \ldots$, $3.1$ for $S^{2n}$. The sign of an Einstein metric is by definition the sign of $k$.

Comment 25

The problem of classifying compact Einstein spaces splits naturally into two main questions: the first one is to find compact manifolds which admit at least one Einstein metric and the second one is to describe the set of Einstein metrics on such a manifold. These problems are difficult and far from being answered; for a review of what is known, see Ref. 38). In dimensions two and three, almost everything is known (see comment 29), the real problem beginning in dimension four. Rather than trying to answer the previous questions, one can try to see when a manifold cannot admit an Einstein structure: there are topological obstructions which usually involve the characteristic numbers of the manifold. One can also find out the conditions for an Einstein structure $g$ on a manifold to be deformable or not, i.e., if there is (or not) a one-parameter family $g(t)$ of Einstein metrics such that $g(0) = g$ and the volume of $g(t)$ is constant for $t > 0$. If we denote by $\mathcal{E}$ the space of all Einstein metrics on $M$ whose volume is constant, a deformation of metrics contained in $\mathcal{E}$ is called an Einstein deformation; notice that the usual Einstein metric on the standard $n$ sphere $S^n$ is not deformable. One can also notice that Einstein metrics are the critical points of the functional $\mathcal{E} = \int_M \text{Tr} \text{vol}(g)$, where $\mathcal{E}$ is the space of all Riemannian metrics of fixed volume. In general, this critical point is neither maximal nor minimal when $g$ varies in $\mathcal{E}$; however, when $g$ varies in the subset $C$ of $\mathcal{E}$ constituted by those Riemannian metrics of constant scalar curvature $\kappa$, then some Einstein metrics are maximal in $C$, and are called stable [see the study in Ref. 43)].
As we mentioned in comment 20, the physically relevant object is the space \( \mathcal{R} \) of Riemannian structures, and what is interesting here is not really the space of Einstein metrics but the space of Einstein structures (called moduli of Einstein manifolds). Little is known [Ref. 44] and references therein]; let us quote that, for tori (and their quotients) the space of moduli of Einstein manifolds coincide with their space of moduli (as flat manifolds). The use of K3 surfaces has been advocated in Ref. 45 in the context of Kaluza-Klein theory, and used in Ref. 46 to provide a new solution to supergravity; it is maybe useful to notice that this is the only non-trivial case where the space of moduli is known: it can be identified with a quotient of \( \text{SO}(19,3)/\text{SO}(19) \times \text{SO}(3) \) by a discrete group. All Einstein metrics on K3 surfaces are Ricci-flat\(^{44}\).

Comment 27 - A few unrelated short comments

1) Left invariant metrics on an Abelian torus (like \( T^7 \)) are all flat and all Einstein (\( \text{Ric} = 0 \)). In the case of \( T^7 \), the manifold of these metrics (with volume fixed) can, of course, be identified with \( \text{SL}(7,\mathbb{R})/\text{SO}(7,\mathbb{R}) \).

ii) A homogeneous Einstein metric on a manifold \( M \) is, by definition, an Einstein, \( G \) invariant metric on \( M \) where \( G \) is transitive on \( M \) (i.e., \( M \) can be written as \( G/H \)). It should not be believed that any Einstein metric is homogeneous! For example, one knows two distinct homogeneous Einstein metrics on \( S^{4n+3} \) spheres\(^{47}\) (the volume is, of course, fixed), the "second Einstein metric on \( S^7 \)" has been used in Ref. 14) for supergravity. It can be proved\(^{48}\) that these two metrics are the only homogeneous Einstein metrics on \( S^{4n+3} \), \( n \neq 3 \), but there exist maybe (and probably) many others which are not homogeneous but are nevertheless Einstein metrics. \( S^{15} \) is a special case\(^{48}\).

iii) The (unique, up to scale) \( G \) invariant metric on an irreducible symmetric space \( G/H \) is Einstein.

iv) The (unique, up to scale) \( G \) invariant metric on an isotropy irreducible space \( G/H \) is Einstein\(^{11}\). A \( G \) invariant metric on a non-isotropy irreducible space \( G/H \) is usually not Einstein, even if it is a normal metric.

v) If \( (M,g) \) and \( (N,h) \) are two Einstein manifolds, their product \( (M \times N, g \times h) \) will also be an Einstein manifold if their Einstein constant \( k \) is the same. Such an adjustment of the Einstein constants is obviously always possible (by rescaling) if the sign of \( k \) is the same for \( M \) and \( N \).

vi) It should not be believed that the Einstein metrics on compact manifolds necessarily have a positive sign; counter-examples are known\(^{37}\). Notice that the isometry group of a compact manifold with negative Ricci curvature is finite.
Comment 28

Rather than trying to solve the difficult problems mentioned in comment 24, one can limit oneself to the study of finding particular Einstein metrics on special manifolds like groups or homogeneous spaces. The basic strategy is more or less always the same: one first chooses some fiber of the manifold $M$ under consideration (for example, the group $G$ as a $H$ bundle over $G/H$, $S^7$ as an $SU(2)$ bundle over $S^5$, etc.), then chooses a particular metric on $M$ and begins to "distort" it in a way appropriate to the fiberings; in the obtained family of new metrics, one looks for those where the Einstein condition is satisfied, either by computing the Ricci tensor or by looking at saddle points of some functional (see comment 24). A complete classification of homogeneous Einstein metrics on spheres $S^n$ and projective spaces $CP^n$ is available$^{68}$; of course, there exist certainly other (non-homogeneous) Einstein metrics on those spaces. The obtained homogeneous metrics in Ref. 48) are all naturally reductive (except for a new example in $S^5$ and in $CP^{2n+1}$ - see also Ref. 49). A complete classification of naturally reductive, left invariant, Einstein metrics on Lie groups is available$^{13}$ in the subcase where the subalgebra defined by the reduction acts irreducibly on its orthogonal complement. Although these restrictions may look quite severe, there is enough room to exhibit, for any simple compact Lie group - but $SU(2)$ - at least one (and usually more) such Einstein metrics, besides the bi-invariant one (which is always Einstein). Many non-trivial examples of Einstein metrics on homogeneous spaces which are not isotropy irreducible can be found by using (or generalizing) the methods of Ref. 47). For example, when the normalizer of $H$ in $G$ is such that $N/H$ is not a discrete group and when $G/N$ is symmetric, the formula in Ref. 10 (par. 4.3) allows one to find immediately two non-isometric Einstein structures on $G/H$ [only one if $N/H$ is Abelian].

Comment 29

We said in comment 25 that (almost) everything is known about Einstein spaces in dimensions two and three; indeed, it can be easily proved that in those cases, being Einstein is equivalent to being a manifold of constant (sectional) curvature [see for example Ref. 33)]. But in dimension two any compact manifold admits Riemannian metrics of constant curvature, hence an Einstein metric via the above theorem. (Such is the case in particular for the sphere $S^2$, the real projective space $RP^2$, the torus $T^2$, the Klein bottle and all surfaces of genus $\gamma$, orientable or not.) In dimension three one has just to classify (in the compact case) the compact quotients of the three-sphere [see for example Ref. 7)]. "Almost" everything is known in those dimensions because people also know the structure of moduli of Einstein spaces [everything is known in dimension two and a complete answer is at hand in dimension three - see Ref. 38] and references therein].
Comment 30

Examples of Einstein spaces of low dimensions \( (n \leq 7) \). Our aim in this comment is just to provide the interested reader with few examples which, on the one hand, are interesting per se but which can also be used to obtain new solutions of 11-dimensional supergravity (see comment 33), indeed, all such spaces provide solutions to these equations. However, the feeling of the author is not that these solutions should be used but rather that the important question is: why should these solutions or others be used? Physical motivations seem quite unclear at this time. We first list simply connected irreducible, non-flat, homogeneous manifolds. Spheres \( S^n \) \( (1 < n < 7) \): only one homogeneous Einstein metric; its full isometry group is \( O(n+1) \). Sphere \( S^7 \): two homogeneous Einstein metrics; beside the usual one which is \( O(8) \) invariant, there is another one which is \( USp(4) \times SU(2) \) invariant. Complex projective space \( CP^2 \): only one such metric known, the one associated with its expression as a symmetric space \( SU(3)/SU(2) \times U(1) \); it is \( SU(3) \) invariant. Complex projective space \( CP^3 \): two homogeneous Einstein metrics, the first is \( SU(4) \) invariant and is associated with its \( SU(4)/SU(3) \times U(1) \) structure, the other one (which is not even naturally reductive\(^{48}\)) has an isometry group \( USp(4) \). The normal metric on the irreducible symmetric spaces \( SU(3)/SO(3) \), \( SU(4)/USp(4) \), \( SO(5)/(SO(3) \times SO(2)) \), \( SO(6)/U(3) \) of dimensions five, five, six, six. There is also one (at least) homogeneous Einstein metric on the real Stiefel manifold \( V_{3,5} = SO(5)/SO(3) \) which is not a symmetric space \( [\text{this last example, of dimension seven is not homeomorphic to } S^7 \text{ nor to the product } S^3 \times S^4; \text{ it has nothing to do with } USp(4)/USp(2)] \). One can now use the remark \( v \) of comment 27 and construct non-irreducible seven-dimensional Einstein spaces by making products like \( S^2 \times S^5 \), \( S^3 \times S^4 \), \( CP^2 \times S^3 \), \( S^2 \times S^2 \times S^3 \), \( SU(4)/USp(4) \times S^4 \), etc.. Care has to be taken if one wants to use \( S^1 \) factors; indeed, it is difficult to adjust the magnitude of the Einstein constant since \( S^1 \) is always flat and moreover there exist topological obstructions \( [\text{for example, } S^1 \times S^3 \text{ does not admit any Einstein metric}\(^{39}\)] \). Of course, one can also consider non-simply-connected Einstein manifolds (like \( RP^2 \)) and make products with them. One can also consider manifolds like \( CP^2 \# CP^2 \) or \( CP^3 \# CP^3 \) which admit Einstein structures\(^{38}\) (here \( \# \) denotes the connected sum). One can also use spaces like \( K3 \) surfaces which admit a lot of Einstein structures, ... Besides, one can also construct flat Einstein manifolds by taking products of flat Einstein manifolds (for example, the product of a flat three-dimensional Klein hyperbottle by a flat Einstein \( K3 \) surface, ..., is a seven-dimensional Einstein manifold \( ! \)). We hope to have convinced the reader that the important question is not to find new seven-dimensional Einstein spaces, but to find physical justifications for this search.
There are essentially two ways of obtaining four-dimensional physical (?) theories from gravity in higher dimensions; a clear distinction should be established between them.

Comment 31

The first kind of theory consists of choosing first an \( n (>4) \) dimensional manifold \( E \) written locally as \( M \times S \) (\( M \) being space-time and \( S \) being the internal space), and choosing a metric \( g^{AB} \) on \( E \) invariant under the action of a group \( G \) transitive on \( S \). One then computes the scalar curvature of this metric and realizes that it depends only on \( x \in M \) and not on \( y \in S \). The "mixed" component \( g^{ij} \) of this metric leads naturally to the emergence of a Yang-Mills field (connection) which is valued in the Lie algebra of \( G \) when \( S \) is a group \( G \), and in the Lie algebra of \( N/H \) when \( S \) is a homogeneous space \( G/H \) (\( N \) being the normalizer of \( H \) in \( G \)). The \( g^{ij} \) components give rise to the gravitational field and the "\( g^{ij} \) components" give rise to a collection of scalar fields. The scalar curvature of \( E \) - a function of \( x \in M \) only - therefore splits naturally into a sum of several pieces: the usual Einstein-Lagrangian describing four-dimensional gravity, a generalized Yang-Mills-Lagrangian, a scalar potential for the scalar fields and a coupling term between scalars and Yang-Mills fields. If an expansion of the scalar fields is made around some value which is a critical point for the scalar potential (for this value, the "internal space" is an Einstein space), "spontaneous" symmetry breaking may occur and some gauge field acquire masses (this is not the usual Higgs mechanism).

In the case where \( S \) is a Lie group \( G \), the four-dimensional Lagrangian can be written as

\[
\mathcal{L} = [\text{Einstein lagrangian}] - \frac{1}{4} \int d^4x F_{\mu \nu}^{\text{eff}} F_{\mu \nu}^{\text{eff}} - \frac{1}{4} \sum_{i,j} \left( D_i h_{ij} D^i h_{ij} + D_i h_{ij} D^i h_{ij} \right) - V(h).
\]

where \( h_{ij}(x) \) are scalar fields describing the \( x \) dependent shape of the internal space and \( -V(h) = [\text{scalar curvature of the internal space at the point } x] \). Moreover, \( D_i h_{ij} = \partial_i h_{ij} + c^k_{ij} A^k_i h_{ij} + c^k_{ij} A^k_j h_{ij} \). The topology of \( E \) (which need not be a direct product), and the most general case where \( S \) is not necessarily a Lie group, are studied in Ref. 10). Even if one does not believe in the "existence" of extra dimensions (but this is probably only a philosophical problem), the above Lagrangian is quite rich in possibilities and predictive consequences, and deserves further study, both at the classical and quantum levels. References 50), 51) and 52) fall into the category of Kaluza-Klein theory of the first kind; indeed, depending upon their taste, different authors may have several criteria for choosing a \( G \) invariant metric \( g^{AB} \) on \( E \); for example,
in Ref. 51), the authors choose a metric on \( M \times T^7 \) which is invariant under the

\[ T^7 = U(1) \times \ldots \times U(1) \] (seven times) and which is also a solution of the
equations of 11-dimensional supergravity. This metric can be built out of several
pieces: a metric in space-time \( M \), a \( T^7 \) invariant metric on \( T^7 \) (there is a 21
parameter family of them), one can now use a connection with value in \( \text{Lie} (T^7) \)
and construct a new metric, still \( T^7 \) invariant.

**Comment 32**

The second kind of theory, initiated with Ref. 45), is quite different in

spirit: one also starts from a metric \( g \) on \( E = M \times S \) (called the background

metric), usually obtained also by looking at a trivial solution of some field

equations and usually invariant under a group \( G \) acting on \( S \). However, this

background metric usually does not incorporate any Yang-Mills field: the "mixed"

components \( g_{\mu \lambda} \) are just zero in the \( E = M \times S^7 \) example\(^{53} \), \( g \) is just the
direct product on a metric on \( M \) and the standard metric on \( S^7 \), \( \tilde{g} \) is therefore
\( \text{SO}(8) \) invariant. One then studies small fluctuations of metrics around \( \tilde{g} \) and
chooses special metrics \( g \) for which the coefficients \( g_{\mu \lambda} \) satisfy some "zero

mode" constraints. The scalar curvature of \( E \) associated with such a new metric
\( g \) is usually \( x \in M \) and \( y \in S \) dependent; the "dimensional reduction" is no

longer automatic and one has to perform a non-trivial integration over the

internal space (\( y \) co-ordinates) to obtain a four-dimensional theory. The claim

is that, after such manipulations, one recovers a theory of Yang-Mills fields valued

in the Lie algebra of \( G \). Although the idea of the method is found in Ref. 54) and

a partial analysis is made in Ref. 21) (for a very special kind of "ground state"
\( \tilde{g} \)), the author does not know any reference dealing properly with the general

case. Most of the literature on this subject, and especially in the "scalar

sector" is quite unclear since no precise geometrical description is made of what

is meant by "zero modes".

**Comment 33**

Solutions to 11-dimensional supergravity. If we forget spinor fields, the

basic problem is to solve the equations\(^{51} \): \( \text{d}^* F = F \wedge F, \ E = 8\pi T(F) \) where \( F =
\text{d}A \) is an antisymmetric four form, \( E \) is the Einstein tensor of a metric in

11 dimensions and \( T(F) \) is the stress energy tensor of the \( F \) field - usually

one looks for solutions which admit a natural four-dimensional interpretation.

A favourite game is to restrict oneself to solutions which have the following

structure:

\[
\begin{align*}
F_{\mu \nu} &= 3 m \, \epsilon_{\nu \rho \sigma \tau} \\
R_{\mu \nu} &= -12 m^2 \, g_{\mu \nu} \\
F_{\mu \nu} &= 0
\end{align*}
\]

\[
\begin{align*}
F_{\mu \nu} &= 0 \\
R_{\mu \nu} &= 6 m^2 \, g_{\mu \nu} \\
R_{\mu \nu} &= 0
\end{align*}
\]

\( n, m \) refer to \( S,(g_{\text{dim}}) \)
It is clear that in this case, any seven-dimensional Einstein manifold provides a solution, ..., and we saw in comment 30 that there is quite a big number of them! By choosing an arbitrary four-dimensional Einstein space-time and an arbitrary seven-dimensional Einstein internal space (with opposite signs), we can generate many solutions to these equations. It is clear that solution of the above kind do not give to E itself the structure of an Einstein space, since both Einstein constants have unequal value and sign (unless M and S are chosen flat). Here again it has to be stressed that the original theory\textsuperscript{51} falls into what we called in comment 31 dimensional reduction of the first kind, since the metric

\[
\begin{pmatrix}
\begin{bmatrix}
\gamma_{mn}(x) \\
\frac{\partial}{\partial x^m(x)}
\end{bmatrix}
&
\begin{bmatrix}
\alpha_{\mu}(x) \\
\frac{\partial}{\partial x^\mu(x)}
\end{bmatrix}

\end{pmatrix}
\]

in \( M \times T^7 \) is indeed invariant under the group \( T^7 \); however, almost all subsequent papers are not of the same kind:

\[
\begin{pmatrix}
\begin{bmatrix}
\gamma_{S^7} \\
\frac{\partial}{\partial x^S(x)}
\end{bmatrix}
&
\begin{bmatrix}
o \\
\frac{\partial}{\partial x^\nu(x)}
\end{bmatrix}

\end{pmatrix}
\]

in \( M \times S^7 \), for example, is indeed invariant under \( SO(8) \) but \( SO(8) \) invariance is lost as soon as one wants to write something non-zero in the off-diagonal blocks (even if one uses "zero modes"). The most general thing that one could do (but has not been done) on \( S^7 \) would be to insert Yang-Mills fields valued in \( U(1) \) or in \( SU(2) \); the corresponding metric would be respectively \( SU(4) \) invariant or \( USp(4) \) invariant\textsuperscript{10} but in general would not be Einstein! The geometrical setting of 11-dimensional supergravity, in the spirit of Ref. 51) and reinterpreted in a Kaluza-Klein language as a generalized Kaluza-Klein theory on a fibre bundle \( (M,T^7) \), is reasonably clear - which does not mean that all aspects of this interesting theory have been elucidated. However, the geometrical setting of this theory, in the spirit of comment 32, with its "zero mode expansion" on homogeneous spaces is far from being clear (at least as soon as one incorporates scalar fields) This does not mean, of course, that it is impossible to make a clear geometrical analysis, but this has still to be done.

CONCLUSION

Writing such an article is a difficult job, firstly because it is quite impersonal; also, it is impossible to put "everything" into such a paper. Some readers, maybe, will think that most of the enclosed comments are well known (and it is indeed the case). As already stated in the Introduction, the choice of the author in writing these comments has been guided by the use (and mostly the misuse) of existing mathematical notions among the recent physical literature.
Many interesting mathematical references have probably been forgotten and I apologize for these omissions. The idea of writing such a paper came after a discussion with A. Salam who suggested that I write a report which would act as a guide to the available mathematical literature. The above article is certainly not a report but one hope of the author is that a few readers may find, in this collection of comments, some useful pieces of information, another hope (more optimistic) is that the same readers (and others) will choose to use a standard terminology when writing articles: the community of physicists should not become a Tower of Babel!

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The present article is for reference only and (since it does not contain any new information which is not already available in books or research papers of mathematics or physics) is not intended for publication.
REFERENCES


