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**SOLUTIONS OF EINSTEIN'S EQUATIONS INVOLVING
ARBITRARY FUNCTIONS**

by

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SOLUTIONS OF EINSTEIN'S EQUATIONS INVOLVING
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by

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ABSTRACT

Necessary and sufficient conditions are derived for a solution of Einstein's vacuum equations to depend on an arbitrary function of some scalar function ϕ . Unlike the case of the scalar wave equation the constant surfaces of the function ϕ need not be null. This apparent anomaly is discussed.

1. INTRODUCTION

While many exact solutions of Einstein's equations have been discovered by now, these rarely depend on more than one or two parameters. The only exception known to the author are the plane-fronted waves¹, which can depend on an arbitrary function of a null coordinate u . Yet it is a feature of hyperbolic differential equations that they often have classes of solutions with arbitrary functions in them. It is the purpose of this paper to investigate this situation in general relativity. We shall find necessary and sufficient conditions for a one-parameter family of metrics to be generalizable, by which we mean that the metrics are still solutions of Einstein's equations after the parameter is replaced by an arbitrary function of some given function ϕ .

Section 2 sets the scene. The scalar wave equation in Minkowski space is discussed in detail, conditions being derived for a solution of the form $F(x, f(\phi))$, where f is an arbitrary function of a given function ϕ : a similar treatment of Maxwell's equations is outlined. Two main features are worth noticing in these cases. Firstly ϕ must satisfy the null condition

$$\phi_{,\mu} \phi_{,\nu} g^{\mu\nu} = 0$$

i.e. the surfaces $\phi = \text{const}$ are null hypersurfaces. Secondly, no solutions of the form

$$F = F(x, f(\phi, \psi))$$

depending in an arbitrary way on two functions ϕ, ψ can exist.

In Section 3 we consider a one-parameter family of metrics $g_{\mu\nu}(x, \lambda)$. The Riemann tensor is computed for its generalization, which consists of replacing λ by an arbitrary function $f(\phi)$. Conditions are then derived for the one-parameter family to be Riemann-generalizable, i.e. in order that the resulting Riemann tensor should be obtained from the original one-parameter family of Riemann tensors by simply the same replacement of λ by $f(\phi)$.

Section 4 is the key section. Here the same procedure is applied to the Ricci tensor, and somewhat surprisingly it is found that the conditions for Ricci-generalizability are identical with those for Riemann-generalizability, at least in the case of Riemannian or Lorentzian metrics. We now have all the conditions necessary for a one-parameter family of vacuum metrics to be generalizable.

In Section 5 we discuss one-parameter families of vacuum metrics which arise by simply applying a one-parameter family of coordinate transformations to a given metric. The example of plane waves shows that this seemingly trivial procedure may lead to non-equivalent generalizations.

Section 6 discusses a particular specialization of the conditions derived in Section 4, in complete detail. All exact solutions are found which fall into this subcase, but unfortunately all turn out to depend on the arbitrary function in a trivial way whenever $\phi_{,\mu}$ is non-null.

Finally in Section 7 we discuss in greater detail the unexpected conclusion that the generalizing function ϕ is not necessarily a null coordinate in the case of general relativity. At first sight this seems to be at variance with what is known of the characteristic surfaces of Einstein's equations. It is shown that while in the case of the scalar wave equation or Maxwell's equations the constant surfaces of the generalizing function ϕ must be characteristic surfaces, this conclusion does not follow in the case of Einstein's equations. The way is therefore open for ϕ to be space-like or time-like. However non-trivial examples will have to be more complicated than any analysed in this paper.

2. GENERALIZABLE SOLUTIONS OF THE WAVE EQUATION AND MAXWELL'S EQUATIONS

Suppose we search for solutions of the wave equation

$$\square \psi \equiv \psi_{,\mu}{}^{\mu} = 0 \quad (1)$$

of the form

$$\psi = F(x, f(\phi)) \quad (2)$$

where $f(\phi)$ is an arbitrary function (subject to suitable differentiability conditions) of some scalar function ϕ on Minkowski space (the argument x is shorthand for the four arguments x^0, x^1, x^2, x^3).

On setting $f = \lambda = \text{const}$ we see that

$$\psi = F(x, \lambda) \quad (3)$$

is a one-parameter family of solutions of the wave equation

$$\square F = 0.$$

We shall say that such a one-parameter family of solutions (3) is generalizable by a (non-constant) function ϕ if $\bar{\psi} = F(x, f(\phi))$ is also a solution of the wave equation for arbitrary functions f .

What conditions must $F(x, \lambda)$ satisfy in order for it to be generalizable by ϕ ? From (2) we clearly have

$$\psi_{,\mu} = F_{,\mu}(x, f(\phi)) + \frac{\partial F}{\partial \lambda}(x, f(\phi)) f'(\phi) \phi_{,\mu}$$

where

$$f'(\phi) = df/d\phi, \quad \phi_{,\mu} = \phi_{,\mu}.$$

A more compact way of writing this equation is

$$\bar{\psi}_{,\mu} = \bar{F}_{,\mu} + \bar{K} f'(\phi) \phi_{,\mu} \quad (4)$$

where

$$\begin{aligned} F_{,\mu} &= \partial F(x, \lambda) / \partial x^{\mu}, \\ K(x, \lambda) &= \frac{\partial}{\partial \lambda} F(x, \lambda) \end{aligned} \quad (5)$$

and placing a bar over a function involving the parameter λ means that all occurrences of λ are replaced by an arbitrary function $f(\phi)$.

Differentiating again and substituting in (1) gives

$$\begin{aligned} 0 = \square \bar{F} + f'(\phi) (2\bar{K}_{,\mu} \phi^{\mu} + \bar{K} \square \phi) + f'^2(\phi) \frac{\partial \bar{K}}{\partial \lambda} \phi_{,\mu} \phi^{\mu} \\ + f''(\phi) \bar{K} \phi_{,\mu} \phi^{\mu}. \end{aligned} \quad (6)$$

Since $f(\phi)$ is to be arbitrary it is clear that in order for $F(x, \lambda)$ to be generalizable by ϕ the coefficients of $f'(\phi)$, $f'^2(\phi)$, $f''(\phi)$ must vanish

separately. This results in the following conditions

$$F_{\mu}^{\mu} = 0 \quad (7)$$

$$\phi_{\mu} \phi^{\mu} = 0 \quad (8)$$

$$2 K_{\mu} \phi^{\mu} + K \phi_{\mu}^{\mu} = 0 . \quad (9)$$

Eq. (7) is of course merely a restatement of the fact that F is a one-parameter family of solutions of the wave equation. Eq. (8) says that the $\phi = \text{const}$ surfaces are null hypersurfaces, i.e. characteristic surfaces. This feature is not surprising in view of the fact that discontinuities of solutions of the wave equation (e.g. choosing f'' discontinuous) can only occur across such characteristic surfaces. Eq. (9) is a first order linear partial differential equation for K , which can be solved for any given ϕ . Note that it has been possible to remove bars from the equations arising out of Eq. (6), since for any one-parameter set of equations it is evidently true that

$$H(x, \lambda) = 0 \iff \bar{H} = H(x, f(\phi)) = 0 .$$

Some typical examples of generalizable solutions are

- (i) $F = F(y, z, \lambda)$ any one-parameter solution of the 2-dimensional Laplace equation

$$F_{yy} + F_{zz} = 0$$

is generalizable by the functions $\phi = t \pm x$.

- (ii) $F = \lambda r^{-1}$ where $r = (x^2 + y^2 + z^2)^{1/2}$ is generalizable by functions

$$\phi = t \pm r .$$

- (iii) $F = u(x, y, z) \lambda$, $\phi = t - v(x, y, z)$. This general case has been solved by Friedlander².

If we had posed the above problem with f an arbitrary function of two independent variables ϕ, ψ we would have obtained from the coefficients of $f_{\phi\phi}$, $f_{\psi\psi}$ and $f_{\phi\psi}$ respectively

$$\phi_{\mu} \phi^{\mu} = \psi_{\mu} \psi^{\mu} = \phi_{\mu} \psi^{\mu} = 0 .$$

Since no two linearly independent null vectors may be orthogonal to each other in a Lorentzian metric, this condition is impossible to fulfil. Thus no solution of the wave equation may contain a general function of two variables. In non-Lorentzian metrics this is not true, since

$$\psi_{tt} + \psi_{ww} - \psi_{xx} - \psi_{yy} = 0$$

has solutions

$$\psi = f(t \pm x, w \pm y) \quad , \quad f \text{ arbitrary} .$$

A similar treatment for Maxwell's equations

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$

$$F_{\mu\nu}{}^{,\nu} = 0$$

yields that a one-parameter family of solutions $F_{\mu\nu}(x, \lambda)$ is generalizable by a function ϕ if

$$K_{\mu\nu} \equiv \frac{\partial F_{\mu\nu}}{\partial \lambda} = -K_{\nu\mu}$$

satisfies

$$K_{\mu\nu} \phi_{,\rho} + K_{\nu\rho} \phi_{,\mu} + K_{\rho\mu} \phi_{,\nu} = 0$$

and

$$K_{\mu\nu} \phi^{,\nu} = 0 .$$

It follows from these equations again that

$$\phi_{,\rho} \phi^{,\rho} = 0$$

and also that

$$K_{\mu\nu} = \phi_{[\mu} q_{\nu]} \equiv \frac{1}{2}(\phi_{\mu} q_{\nu} - \phi_{\nu} q_{\mu})$$

where q_{ν} is a space-like vector satisfying

$$\phi_{\mu} q^{\mu} = 0 .$$

3. RIEMANN-GENERALIZABLE METRICS

Let $g_{\mu\nu} = g_{\mu\nu}(x, \lambda)$ be a one-parameter family of general-relativistic metrics (i.e. 4-dimensional with signature $-+++$). Again we adopt the

convention that if ϕ is a given function on the space-time we set

$$\bar{g}_{\mu\nu}(x) = g_{\mu\nu}(x, f(\phi))$$

where f is an arbitrary function (i.e. $\bar{g}_{\mu\nu}$ actually represents a class of metrics arising from $g_{\mu\nu}(x, \lambda)$ and ϕ).

We set

$$K_{\mu\nu}(x, \lambda) = \frac{1}{2} \frac{\partial}{\partial \lambda} g_{\mu\nu}(x, \lambda) \quad (10)$$

and raise and lower indices by $g_{\mu\alpha}$, $g^{\mu\alpha}$:

$$K^\mu_{\nu} = g^{\mu\alpha} K_{\alpha\nu},$$

$$K^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} K_{\alpha\beta} = -\frac{1}{2} \frac{\partial}{\partial \lambda} g^{\mu\nu}.$$

Then

$$\frac{\partial}{\partial \lambda} \Gamma^\rho_{\mu\nu} = P^{\rho\alpha}_{\mu\nu;\alpha} \quad (11)$$

where

$$P^{\rho\alpha}_{\mu\nu} = K^\rho_{\mu} \delta^\alpha_{\nu} + K^\rho_{\nu} \delta^\alpha_{\mu} - K_{\mu\nu} g^{\rho\alpha}. \quad (12)$$

Now if $F = F(x, \lambda)$ is any function then clearly

$$\bar{F}_{,\rho} = \overline{F_{,\rho}} + \frac{\partial F}{\partial \lambda} f'(\phi) \phi_\rho \quad (13)$$

whence

$$\begin{aligned} \Gamma^\rho_{\mu\nu}(\bar{g}) &\equiv \frac{1}{2} \bar{g}^{\rho\sigma} (\bar{g}_{\sigma\mu, \nu} + \bar{g}_{\sigma\nu, \mu} - \bar{g}_{\mu\nu, \rho}) \\ &= \bar{\Gamma}^\rho_{\mu\nu} + \bar{P}^{\rho\alpha}_{\mu\nu} \phi_\alpha f'(\phi). \end{aligned}$$

Applying (13) again to $\Gamma^\rho_{\mu\nu, \rho}(\bar{g})$ and using (11) we obtain the following expression for the generalized Riemann tensor

$$\begin{aligned} R^\rho_{\mu\sigma\nu}(\bar{g}) &= \bar{R}^\rho_{\mu\sigma\nu}(\bar{g}) + f'(\phi) [P^{\rho\alpha}_{\mu\nu;\alpha} \phi_\sigma - P^{\rho\alpha}_{\mu\sigma;\alpha} \phi_\nu \\ &\quad + (P^{\rho\alpha}_{\mu\nu} \phi_\alpha)_{;\sigma} - (P^{\rho\alpha}_{\mu\sigma} \phi_\alpha)_{;\nu}] + \\ &\quad + f'^2(\phi) \phi_\alpha \phi_\beta \left[P^{\tau\alpha}_{\mu\nu} P^{\rho\beta}_{\tau\sigma} - P^{\tau\alpha}_{\mu\sigma} P^{\rho\beta}_{\tau\nu} + \frac{\partial P^{\rho\alpha}_{\mu\nu}}{\partial \lambda} \delta^\beta_{\sigma} - \frac{\partial P^{\rho\alpha}_{\mu\sigma}}{\partial \lambda} \delta^\beta_{\nu} \right] + \\ &\quad + f''(\phi) \phi_\alpha \phi_\beta [P^{\rho\alpha}_{\mu\nu} \delta^\beta_{\sigma} - P^{\rho\alpha}_{\mu\sigma} \delta^\beta_{\nu}]. \end{aligned} \quad (14)$$

We call the family of metrics Riemann-generalizable by the function ϕ

if

$$R_{\mu\sigma\nu}^0(\bar{g}) = \bar{R}_{\mu\sigma\nu}^0(g) , \quad (15)$$

i.e. if replacing λ by an arbitrary function $f(\phi)$ implies the Riemann tensor components are obtained simply by replacing all occurrences of λ by $f(\phi)$.

Since f is arbitrary we may set the coefficients of f'' , f'^2 , and f' separately to zero, and as in Section 2, we may remove all bars in these equations. Using Eq. (12) the coefficient of $f''(\phi)$ results in

$$\phi_{[\mu} K_{\rho]} [\nu \phi_{\sigma]} = 0 \quad (16)$$

which holds if and only if there exists a vector field $\psi_\nu(x, \lambda)$ such that

$$K_{\mu\nu} = \phi_\mu \psi_\nu + \psi_\mu \phi_\nu . \quad (17)$$

The last two terms in the coefficient of f'^2 are just $\partial/\partial\lambda$ applied to the coefficient of f'' , hence they vanish as a consequence of Eq. (16). On the other hand from Eq. (17) we obtain

$$\phi_\alpha P_{\mu\nu}^{\tau\alpha} = 2\psi^\tau \phi_\mu \phi_\nu \quad (18)$$

whence the first two terms of the coefficient of f'^2 also vanish. Hence no further information arises by setting the coefficient of f'^2 to zero.

Turning now to the coefficient of $f'(\phi)$, when Eqs (17) and (18) are substituted in this equation we obtain

$$\phi_\mu L^\rho_{\nu\sigma} [\nu \phi_\sigma] = L_{\mu[\nu} \phi_{\sigma]} \phi^\rho , \quad (19)$$

where

$$L_{\mu\nu} = \psi_{\mu;\nu} + \psi_{\nu;\mu} \equiv \mathcal{L}_\psi g_{\mu\nu} . \quad (20)$$

That is, $L_{\mu\nu}$ satisfied the same equation as $K_{\mu\nu}$, whence

$$L_{\mu\nu} = \phi_\mu \alpha_\nu + \alpha_\mu \phi_\nu \quad (21)$$

for some vector field

$$\alpha_\mu = \alpha_\mu(x, \lambda) .$$

4. RICCI-GENERALIZABLE METRICS

Contracting Eq. (14) over ρ and σ we obtain conditions for a one-parameter family of metrics $g_{\mu\nu}(x, \lambda)$ to be Ricci-generalizable by a function ϕ , i.e. for

$$R_{\mu\nu}(\bar{g}) = \bar{R}_{\mu\nu}(g) . \quad (22)$$

In this case the coefficient of $f''(\phi)$ gives rise to the equation

$$\phi_\beta K^\beta_\mu \phi_\nu + \phi_\beta K^\beta_\nu \phi_\mu - K_{\mu\nu} \phi^\alpha_\alpha - K \phi_\mu \phi_\nu = 0 \quad (23)$$

where

$$K = K^\alpha_\alpha .$$

If $\phi_\alpha \phi^\alpha \neq 0$ then it follows at once that

$$K_{\mu\nu} = \phi_\mu \psi_\nu + \psi_\mu \phi_\nu \quad (24)$$

for some $\psi_\mu = \psi_\mu(x, \lambda)$.

If $\phi_\alpha \phi^\alpha = 0$ however we can only conclude from Eq. (23) that ϕ_μ is a (null) eigenvector of $K_{\mu\nu}$,

$$\phi_\rho K^\rho_\mu = \frac{1}{2} K \phi_\mu . \quad (25)$$

Again the last two terms of the coefficient of f'^2 vanish on taking $\partial/\partial\lambda$ of the f'' equation, while the first two terms give rise to the equation

$$\phi_\alpha \phi_\beta (P^{\tau\alpha}_{\mu\nu} P^{\rho\beta}_{\tau\rho} - P^{\tau\alpha}_{\mu\rho} P^{\rho\beta}_{\tau\nu}) = 0 .$$

In the case $\phi_\alpha \phi^\alpha \neq 0$, substitution of (24) immediately guarantees that this equation is fulfilled, but in the case $\phi_\alpha \phi^\alpha = 0$ one obtains from (25) the condition

$$(\frac{1}{2}K^2 - K_{\alpha\beta} K^{\alpha\beta}) \phi_\mu \phi_\nu = 0 . \quad (26)$$

If we assume the metric is Lorentzian (signature $n-2$, where n = dimension of space) then (24) again follows from (25) and (26). To see this, let σ_ν be a second null vector normalized such that

$$\sigma_\nu \sigma^\nu = \phi_\nu \phi^\nu = 0 , \quad \sigma_\nu \phi^\nu = 1 ,$$

and let e_i^μ ($i = 2, \dots, n-1$) be an orthonormal basis of the tangent subspace orthogonal to ϕ_ν and σ_ν , i.e.

$$\sigma_\mu e_i^\mu = \phi_\mu e_i^\mu = 0, \quad e_{i\mu} e_j^\mu = \delta_{ij}.$$

It is clear that $\phi_\mu, \rho_\mu, e_i^\mu$ form a basis of the tangent space and $K_{\mu\nu}$ may be expanded in this basis

$$\begin{aligned} K_{\mu\nu} = & K_{00} \phi_\mu \phi_\nu + K_{01} (\phi_\mu \sigma_\nu + \sigma_\mu \phi_\nu) + K_{11} \sigma_\mu \sigma_\nu \\ & + \sum_i K_{0i} (\phi_\mu e_{i\nu} + e_{i\mu} \phi_\nu) + K_{1i} (\sigma_\mu e_{i\nu} + e_{i\mu} \sigma_\nu) \\ & + \sum_{i,j} K_{ij} e_{i\mu} e_{j\nu}. \end{aligned}$$

Eq. (25) then gives

$$K_{11} = K_{1i} = 0, \quad \sum_i K_{ii} = 0$$

while (26) implies

$$\sum_{i,j} K_{ij}^2 = 0.$$

Hence $K_{ij} = 0$ for all i, j and Eq. (24) holds with

$$\psi_\mu = \frac{1}{2} K_{00} \phi_\mu + K_{01} \sigma_\mu + \sum_i K_{0i} e_{i\mu}.$$

It is easy to convince oneself that if the signature were other than Lorentzian (or Riemannian) the conclusion (24) would not in general be justified. In general relativity ($n=4$, Lorentzian) we may however adopt Eq. (24) as being equivalent to the f'' and f'^2 equations.

There remains the equation arising from setting the coefficient of $f'(\phi)$ equal to zero,

$$P_{\mu\nu;\alpha}^{\rho\alpha} \phi_\rho - P_{\mu\rho;\alpha}^{\rho\alpha} \phi_\nu + (P_{\mu\nu}^{\rho\alpha} \phi_\alpha)_{;\rho} - (P_{\mu\rho}^{\rho\alpha} \phi_\alpha)_{;\nu} = 0.$$

On substituting Eqs (12) and (24) we obtain

$$\phi_\rho \phi^\rho L_{\mu\nu} - \phi_\nu L_{\mu\rho} \phi^\rho - \phi_\mu L_{\nu\rho} \phi^\rho + L^\rho_\rho \phi_\mu \phi_\nu = 0$$

where $L_{\mu\nu}$ is given by Eq. (20). That is, $L_{\mu\nu}$ satisfies the same equation

as $K_{\mu\nu}$ (Eq. (23)) and again we may conclude that $L_{\mu\nu}$ has the form given in (21). In summary then

Theorem 1: A one-parameter family of Riemannian or Lorentzian metrics

$g_{\mu\nu}(x, \lambda)$ is Ricci-generalizable by a function ϕ if and only if there exist vector fields $\psi_\mu(x, \lambda)$ and $\rho_\mu(x, \lambda)$ such that

$$K_{\mu\nu} \equiv \frac{1}{2} \frac{\partial}{\partial \lambda} g_{\mu\nu} = \phi_\mu \psi_\nu + \psi_\mu \phi_\nu \quad (27)$$

and

$$L_{\mu\nu} \equiv \psi_{\mu;\nu} + \psi_{\nu;\mu} = \phi_\mu \rho_\nu + \rho_\mu \phi_\nu. \quad (28)$$

Since these conditions are exactly the same as those obtained in Section 3 we also have the following result

Theorem 2: A one-parameter family of Riemannian or Lorentzian metrics

$g_{\mu\nu}(x, \lambda)$ is Ricci-generalizable by a function ϕ if and only if it is Riemann-generalizable.

Perhaps the most surprising aspect of these theorems is that ϕ_μ is not necessarily a null vector. We shall return to this point later. To conclude this section we just wish to remark that as in the case of the scalar wave equation, no metrics are Ricci generalizable by arbitrary functions of two independent variables $f(\phi, \psi)$ (with ϕ_μ, ψ_μ linearly independent vector fields).

This follows by setting up the equations (22) and setting to zero all coefficients of f_ϕ, f_ψ etc. The coefficients of $f_{\phi\phi}, f_{\psi\psi}, f_{\phi\psi}$ yield respectively

$$\phi_\mu \alpha_\nu + \alpha_\mu \phi_\nu - K \phi_\mu \phi_\nu = K_{\mu\nu} \phi_\alpha \phi^\alpha \quad (29)$$

$$\psi_\mu \beta_\nu + \beta_\mu \psi_\nu - K \psi_\mu \psi_\nu = K_{\mu\nu} \psi_\alpha \psi^\alpha \quad (30)$$

$$\psi_\mu \alpha_\nu + \alpha_\mu \psi_\nu + \phi_\mu \beta_\nu + \beta_\mu \phi_\nu - K(\phi_\mu \psi_\nu + \psi_\mu \phi_\nu) = 2K_{\mu\nu} \phi_\alpha \psi^\alpha \quad (31)$$

where

$$\alpha_\mu = K_{\mu\rho} \phi^\rho, \quad \beta_\mu = K_{\mu\rho} \psi^\rho.$$

By taking suitable linear combinations of ϕ and ψ it is easy to see that for Riemannian or Lorentzian metrics there is no loss of generality in assuming that

$$\phi_\alpha \phi^\alpha \neq 0, \quad \psi_\alpha \psi^\alpha \neq 0.$$

Eqs (29) and (30) give at once that α_μ and β_μ must be linear combinations of ϕ_μ and ψ_μ

$$\alpha_\mu = a\phi_\mu + b\psi_\mu, \quad \beta_\mu = c\phi_\mu + d\psi_\mu$$

and substituting back in (29) and (30) results in

$$K_{\mu\nu} = B(\phi_\mu \psi_\nu + \psi_\mu \phi_\nu)$$

where

$$b = B\phi_\alpha \phi^\alpha, \quad a = d = B\phi_\alpha \psi^\alpha, \quad c = B\psi_\alpha \psi^\alpha.$$

Finally substituting into (30) results in $B = 0$, i.e.

$$K_{\mu\nu} = 0$$

which proves the desired result.

The most interesting case arises in general relativity when $R_{\mu\nu}(g) = 0$ leads to $\bar{R}_{\mu\nu}(\bar{g}) = 0$; we call this situation vacuum-generalizable. Theorems 1 and 2 clearly apply to this case.

5. COORDINATE TRANSFORMATIONS

A particularly simple way of generating a one-parameter family of vacuum metrics is to apply a one-parameter family of coordinate transformations

$$y^\mu = y^\mu(x, \lambda)$$

with inverse

$$x^\mu = x^\mu(y, \lambda)$$

to a given parametrized vacuum metric $\overset{\circ}{g}_{\mu\nu}(x, \lambda, \mu, \dots)$ ($\overset{\circ}{g}_{\mu\nu}$ may of course depend on no parameters at all). This new family

$$g_{\mu\nu}(y, \lambda, \mu, \dots) = \overset{\circ}{g}_{\alpha\beta}(x(y, \lambda), \lambda, \mu, \dots) \frac{\partial x^\alpha}{\partial y^\mu}(y, \lambda) \frac{\partial x^\beta}{\partial y^\nu}(y, \lambda) \quad (31)$$

is of course geometrically equivalent to the original family $\overset{\circ}{g}_{\mu\nu}$, but if it is vacuum-generalizable there is no guarantee that its generalizations by a function ϕ are so equivalent.

Unfortunately Theorem 2 implies that the simplest possible procedure, namely to apply a one-parameter coordinate transformation to flat (Minkowski) space does not lead to anything new. For in this case the Riemann tensor vanishes for the entire family, and hence all its generalizations also have vanishing Riemann tensor and are therefore flat. However for curved vacuum metrics the procedure may result in new metrics.

Differentiating (31) with respect to λ gives the transformation of

$K_{\mu\nu}$

$$K_{\mu\nu}(y, \lambda) = \overset{\circ}{K}_{\alpha\beta}(x(y, \lambda), \lambda) \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} + \xi_{(\mu; \nu)} \quad (32)$$

where

$$\xi^\beta_{(\mu; \nu)} = - \frac{\partial y^\beta}{\partial \lambda} = \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial y^\beta}{\partial x^\alpha} \quad (33)$$

and ";" refers to covariant derivative with respect to $g_{\mu\nu}$ (all other parameters $\mu, \dots$ have been suppressed). A similar analysis for any tensor $\overset{\circ}{T}^{\mu\nu\dots}_{\rho\sigma\dots}(x, \lambda)$ yields the general result

$$\frac{\partial}{\partial \lambda} T^{\mu\nu\dots}_{\rho\sigma\dots}(y, \lambda) = \left(\frac{\partial}{\partial \lambda} \overset{\circ}{T}^{\alpha\beta\dots}_{\gamma\delta\dots} \right) \frac{\partial y^\mu}{\partial x^\alpha} \dots \frac{\partial x^\gamma}{\partial y^\rho} \dots + \xi^\mu_{\gamma} T^{\alpha\beta\dots}_{\gamma\delta\dots}$$

where

$$T^{\mu\nu\dots}_{\rho\sigma\dots}(y, \lambda) = \overset{\circ}{T}^{\alpha\beta\dots}_{\gamma\delta\dots}(x(y, \lambda), \lambda) \frac{\partial y^\mu}{\partial x^\alpha} \dots \frac{\partial x^\gamma}{\partial y^\rho} \dots$$

Examples:

- (1) Suppose $g_{\mu\nu}(y, \lambda)$ is a one-parameter family of vacuum metrics, generalizable by a function ϕ , i.e. suppose Eqs (27) and (28) hold. If $\rho_\mu = 0$, then

$$\psi_{\mu; \nu} + \psi_{\nu; \mu} = 0,$$

i.e. ψ_μ is a Killing vector and from (27) we have

$$K_{\mu\nu} = \xi_{(\mu; \nu)}$$

where

$$\xi_{\mu} = 2\phi\psi_{\mu}.$$

Hence $g_{\mu\nu}(y, \lambda)$ is obtained by applying a coordinate transformation to $\overset{\circ}{g}_{\mu\nu}(x) = g_{\mu\nu}(x, 0)$, given by

$$\frac{\partial y^{\mu}}{\partial \lambda} = -\xi^{\mu} = -2\phi(y)\psi^{\mu}(y, \lambda)$$

subject to

$$y^{\mu}(x, 0) = x^{\mu}.$$

(2) Plane-fronted waves¹:

$$ds^2 = -2Hdu^2 + 2dudv + dx^2 + dy^2$$

where

$$H = H(u, x, y)$$

is subject to

$$H_{,22} + H_{,33} = 0 \quad (34)$$

($x^0 = u$, $x^1 = v$, $x^2 = x$, $x^3 = y$). All such metrics satisfy Einstein's vacuum equations. Thus it is clear that $H = H(\lambda, x, y)$ gives rise to a one-parameter family of metrics which are vacuum-generalizable by $\phi = u$. All conditions of Theorem 1 are satisfied since

$$K_{\mu\nu} = -\frac{\partial H}{\partial \lambda} u_{,\mu} u_{,\nu},$$

whence Eq. (27) holds with

$$\psi_{\mu} = -\frac{1}{2} \frac{\partial H}{\partial \lambda} u_{,\mu}$$

and (28) follows with

$$\rho_{\mu} = -\frac{\partial^2 H}{\partial x^{\mu} \partial \lambda}.$$

Theorem 2 is easily verified by computing the Riemann tensor, whose only surviving components are

$$R_{OAOB} = H_{,AB} \quad (A, B = 2, 3).$$

Clearly λ is replaced by $f(u)$ in these components, hence these solutions are all Riemann-generalizable. Flat space occurs if H is linear in x and y . Plane waves arise if H is quadratic in

x and y , and may be regarded as generalizations by $\phi = u$ of the 2-parameter metric having

$$H = \lambda_1(x^2 - y^2) - 2\lambda_2xy.$$

If we set $\lambda_1 = \cos 2\lambda$, $\lambda_2 = \sin 2\lambda$, then

$$K_{\mu\nu} = \xi_{(\mu;\nu)}$$

where $\xi_\mu = (0, 0, -y, x)$. In this case the one-parameter family is obtained from

$$d\tilde{s}^2 = -2(x^2 - y^2)du^2 + 2dudv + dx^2 + dy^2$$

by performing the one-parameter family of coordinate transformations

$$x = X \cos \lambda + Y \sin \lambda$$

$$y = -X \sin \lambda + Y \cos \lambda.$$

However the generalizations obtained by setting $\lambda = f(u)$ are not in general equivalent to $\tilde{g}_{\mu\nu}$, since the generalizations are plane waves with variable phase and amplitude while $\tilde{g}_{\mu\nu}$ has constant phase and amplitude. So here is an example where the original one-parameter family of metrics are all equivalent but their generalizations result in genuinely inequivalent metrics.

6. A SPECIAL CASE

It turns out that if we set ψ_μ proportional to ϕ_μ , the generalizable metrics can all be written down explicitly. The analysis is rather long and we will only outline the main steps here.

If ϕ_μ is a null vector field, then Eq. (28) immediately shows that it is shear-free and twist-free. All such metrics have been discovered by Kundt⁵ and need not be discussed further here.

If ϕ_μ is non-null, let us postulate it to be timelike (the spacelike case is similarly analysed) and set $x^0 = \phi$. From Eq. (27) it is easily shown that coordinates x^i ($i = 1, 2, 3$) may be found such that

$$ds^2 = - N^2(x, \lambda) (dx^0)^2 + g_{ij}(x) dx^i dx^j .$$

Since $\psi_\mu = f(x, \lambda) \phi_\mu$, Eq. (28) shows that

$$0 = \phi_{(i;j)} = - \Gamma_{ij}^0 = - \frac{1}{2} N^{-2} g_{ij,o}$$

whence $g_{ij} = g_{ij}(x^1, x^2, x^3)$. Now the spatial part of the Einstein field equations give

$$R_{ij}^{(3)} = N^{-1} N_{|ij}$$

where $|$ refers to covariant derivative with respect to the 3-metric g_{ij} .

Since $R_{ij}^{(3)}$, being constructed from the 3-metric g_{ij} , has no λ -dependence it can be verified that

$$\xi_i = N^2 (N^{-1} \partial N / \partial \lambda)_{,i}$$

satisfies

$$\xi_{(i|j)} = 0$$

and is therefore a hypersurface-orthogonal Killing 3-vector. We can therefore cast the metric in a Weyl form

$$ds^2 = - e^{2\alpha+2\mu} (dx^0)^2 + e^{2\mu} (dx^1)^2 + e^{2\nu} d\zeta d\bar{\zeta} \quad (35)$$

where

$$\zeta = x^2 + ix^3, \quad \bar{\zeta} = x^2 - ix^3,$$

$$\mu = \mu(\zeta, \bar{\zeta}), \quad \nu = \nu(\zeta, \bar{\zeta}), \quad \alpha = \alpha(x^0, x^1, \zeta, \bar{\zeta}).$$

The Einstein equations may now be written out in full:

$$\alpha_{1\zeta} + \alpha_1 \alpha_\zeta = 0, \quad (36)$$

$$\alpha_{11} + \alpha_1^2 + e^{2\mu-2\nu} (4\mu_{\zeta\bar{\zeta}} + 8\mu_\zeta \mu_{\bar{\zeta}} + 2\alpha_\zeta \mu_{\bar{\zeta}} + 2\alpha_{\bar{\zeta}} \mu_\zeta) = 0, \quad (37)$$

$$\alpha_{\zeta\bar{\zeta}} + \alpha_\zeta \alpha_{\bar{\zeta}} + \alpha_\zeta \mu_{\bar{\zeta}} + \alpha_{\bar{\zeta}} \mu_\zeta = 0, \quad (38)$$

$$\alpha_{\zeta\zeta} + \alpha_\zeta^2 + 2\alpha_\zeta (\mu_\zeta - \nu_\zeta) + 2\mu_{\zeta\zeta} + 2\mu_\zeta^2 - 4\nu_\zeta \mu_\zeta = 0, \quad (39)$$

and

$$\nu_{\zeta\bar{\zeta}} + \mu_{\zeta\bar{\zeta}} + \mu_\zeta \mu_{\bar{\zeta}} = 0. \quad (40)$$

Now Eq. (36) implies that

$$e^\alpha = G(x^0, x^1) + F(x^0, \zeta, \bar{\zeta}). \quad (41)$$

Case (i): $\alpha_1 = G_1 \equiv \partial G / \partial x^1 \neq 0$

Substitution of (41) into (39) gives

$$\mu_{\zeta\zeta} + \mu_{\zeta}^2 - 2v_{\zeta}\mu_{\zeta} = 0 .$$

Together with the results of differentiating Eqs (38) and (39) with respect to ζ and $\bar{\zeta}$ respectively, this results in

$$2v_{\zeta\bar{\zeta}} - \mu_{\zeta\bar{\zeta}} - \mu_{\zeta}\mu_{\bar{\zeta}} = 0 ,$$

which is clearly only consistent with Eq. (40) if $v_{\zeta\bar{\zeta}} = 0$. Now at this point a computation of the Riemann tensor components would show them all to vanish, so there is no need to proceed further with this case as it can only result in flat space.

Case (ii): $\alpha_1 = 0$

In this case, Eqs (37) and (38) result in

$$(e^{\alpha+2\mu})_{\zeta\bar{\zeta}} = 0 ,$$

whence

$$e^{\alpha} = e^{-2\mu} (f(x^0, \zeta) + \bar{f}(x^0, \bar{\zeta}))$$

and

$$2\mu_{\zeta\bar{\zeta}}(f + \bar{f}) + \mu_{\bar{\zeta}}f_{\zeta} + \mu_{\zeta}\bar{f}_{\bar{\zeta}} = 0 . \quad (42)$$

Let $\phi = f(0, \zeta)$ and set

$$K(x^0, \zeta, \bar{\zeta}) = (f + \bar{f}) / (\phi + \bar{\phi}) . \quad (43)$$

We are clearly only interested in the case $K \neq \text{const}$, else there is no x^0 -dependence in the metric at all. Eq. (42) is clearly equivalent to the pair of equations

$$2\mu_{\zeta\bar{\zeta}}(\phi + \bar{\phi}) + \mu_{\bar{\zeta}}\phi_{\zeta} + \mu_{\zeta}\bar{\phi}_{\bar{\zeta}} = 0 \quad (44)$$

and

$$K_{\zeta}\mu_{\bar{\zeta}} + K_{\bar{\zeta}}\mu_{\zeta} = 0 , \quad (45)$$

while Eq. (39) implies

$$2(\phi + \bar{\phi})\mu_{\zeta} - 2(v_{\zeta} + \mu_{\zeta})\phi_{\zeta} + \phi_{\zeta\zeta} = 0 \quad (46)$$

7. RELATION TO CHARACTERISTICS AND CONCLUSIONS

The structure of solutions to Einstein's equations involving arbitrary functions has been shown to possess remarkably simple properties. Perhaps the most surprising aspect of the results in Theorem 1 and 2 is that the gradient of the function ϕ need not be a null vector, as is the case for the scalar wave equation or Maxwell's equations. This is especially surprising in view of the fact that characteristic surfaces of Einstein's equations (i.e. surfaces $\phi = \text{const}$ across which the curvature tensor has a discontinuity) are known to be null surfaces^{3,4}. Since such discontinuities can apparently be generated by setting the arbitrary function f to be discontinuous, it seems at first sight paradoxical that this conclusion does not follow. A detailed look at Pirani's treatment of characteristics reveals the reason for this apparent discrepancy.

In the case of the scalar wave equation, a discontinuity in $f''(\phi)$ gives rise, via Eq. (6), to the usual characteristic condition (8). Similarly in Maxwell's equations a discontinuity in $f'(\phi)$ gives rise to the characteristic conditions

$$\Delta F_{\mu\nu} = \phi_{[\mu} \alpha_{\nu]} \quad , \quad \phi_{\mu} \phi^{\mu} = 0 \quad , \quad \phi_{\mu} \alpha^{\mu} = 0$$

obtained on replacing $K_{\mu\nu}$ by $\Delta F_{\mu\nu}$.

For Einstein's equations it is not strictly allowable to set $f(\phi)$ discontinuous since this violates the usual C^2 -differentiability conditions. If one sets $f''(\phi)$ discontinuous, this does indeed result in the discontinuity of the Riemann tensor

$$\Delta R_{\mu\nu\rho\sigma} = 2 \phi_{[\mu} K_{\rho]} [\nu \phi_{\sigma]} \Delta f''(\phi) \quad ,$$

while the condition $\Delta R_{\mu\nu} = 0$ results in

$$0 = \Delta R_{\mu\rho} = (K_{\mu\rho} \phi_{\nu} \phi^{\nu} - \phi_{\mu} K_{\rho\nu} \phi^{\nu} - \phi_{\rho} K_{\mu\nu} \phi^{\nu} + K \phi_{\mu} \phi_{\rho}) \Delta f''(\phi)$$

and

$$(e^{-2\tau} K_{\bar{\zeta}})_{\bar{\zeta}} = 0 \quad (47)$$

where

$$\tau = v + \mu - \ln(\phi + \bar{\phi}) .$$

Hence

$$K_{\bar{\zeta}} = \bar{p}(x^0, \bar{\zeta}) e^{2\tau} , \quad K_{\zeta} = p(x^0, \zeta) e^{2\tau}$$

for some function $p(x^0, \zeta)$. From Eq. (45) one sees that $p = P(x^0)b(\zeta)$ for some real function P and complex function $b(\zeta)$. Defining a complex function $Z(\zeta)$ by

$$\frac{dZ}{d\zeta} = - \frac{i}{b(\zeta)}$$

we see that $\mu = \mu(x)$ where we have set $Z = x + iy$. By changing the complex coordinates $\zeta, \bar{\zeta}$ to $Z, \bar{Z}$ it is now a relatively straightforward matter to integrate the equations (43) - (47). Apart from some removable arbitrary constants, there are three distinct cases, arising from

$$(\mu')^{-1} = x, \quad \sin x, \quad \text{or} \quad \sinh x .$$

The first case leads only to flat space whilst the second results in the metric

$$ds^2 = (1 + \cos x)^2 \{ - (A(x^0)e^Y + B(x^0)e^{-Y})^2 (dx^0)^2 + dx^2 + dy^2 \} \\ + \frac{(1 - \cos x)}{(1 + \cos x)} (dx')^2 .$$

At first sight it appears that one has here a vacuum metric of the desired kind, exhibiting arbitrary functions A, B of a timelike coordinate x^0 .

However further inspection reveals that the 2-metric

$$- (A(x^0)e^Y + B(x^0)e^{-Y})^2 (dx^0)^2 + dy^2$$

has constant curvature (curvature scalar = 1). As any such 2-metric can be brought to a canonical form, it is clear that coordinate transformations exist which eliminate the arbitrary functions. The resulting space-time is then a very special Weyl static axi-symmetric solution. The case $\mu' = (\sinh x)^{-1}$ is similar.

which implies in turn

$$\Delta R_{\rho\mu\sigma\nu} \phi_\alpha \phi^\alpha = 0.$$

At this point it is argued that if $\Delta R_{\rho\mu\sigma\nu} \neq 0$ (i.e. the surface $\phi = \text{const}$ is a characteristic) then $\phi_\alpha \phi^\alpha = 0$. However our discussion has shown that it is precisely the case $\Delta R_{\rho\mu\sigma\nu} = 0$ which occurs here since $R_{\mu\nu\rho\sigma}$ depends only on $f(\phi)$ and not its derivatives. While it is tempting on this account to permit $f(\phi)$ to be discontinuous there appears to be no guarantee that this violation of Lichnerowicz conditions can be undone by a (discontinuous) coordinate transformation. It should furthermore be recognized that these continuity conditions are an integral part of the proof of the nullness of characteristic surfaces.

Nevertheless ones feeling that this problem is intimately connected with characteristics goes deep, and this paper only goes a little way to disturbing it. The detailed example given in Section 6, at first looks most promising in its goal of obtaining a family of solutions depending in an arbitrary way on a non-null function, only to dissolve in the final analysis through a series of coordinate transformations. Whether the same phenomenon would occur in general, without the restrictive ansatz $\psi_\mu \propto \phi_\mu$, is impossible to say at this point. In conclusion it is perhaps worth pointing out the existence⁶ of a family of solutions of the Einstein dust equations, $G_{\mu\nu} = \rho u_\mu u_\nu$, which have in them arbitrary functions of a space-like coordinate. It is not inconceivable that interior solutions like this, which permit a certain amount of arbitrary variation of the matter distribution, could not lead to exterior solutions similarly dependent on an arbitrary function.

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