Ref.TH.3640-CERN

1/N² CORRECTIONS IN TWO-DIMENSIONAL LATTICE GAUGE SYSTEM
WITH MIXED ACTION

J. Jurkiewicz^{*)}

CERN - Geneva

and

C.P. Korthals Altes

Centre de Physique Théorique 2, C.N.R.S.
B.P. 907, Luminy, F-13288 Marseille Cedex 9

A B S T R A C T

A form of the $1/N^2$ correction for a two-dimensional system described by the mixed fundamental-adjoint action is discussed. An exact result for the $U(N)$ gauge group is derived. The form of correction in $SU(N)$ is also discussed and the difference between the two groups is pointed out. The form of correction in two dimensions can be generalized for any number of dimensions if corrections to the plaquette free energy are to be calculated.

^{*)} On leave of absence from the Institute of Physics, Jagellonian University, Krakow, Poland.

1. INTRODUCTION

Two-dimensional systems play a special role in the lattice gauge theories. An infinite system with a total action expressible as a sum of plaquette actions can be described in two dimensions if one can solve a single plaquette problem. This in turn, by gauge invariance, can be reduced to a single integral over the gauge group. The simplicity of a two-dimensional theory can nevertheless be used to learn something about the properties of a theory in more dimensions and to check the mathematical methods used to describe the more complicated systems.

In this paper, we shall be concerned with a problem of the large N expansion¹⁾ where the properties of the gauge system with the gauge groups $U(N)$ or $SU(N)$ are obtained by expanding around the $N = \infty$ solution. The corrections are written as $1/N$ expansion and usually their derivation is more complicated than solving the problem for $N = \infty$. For a two-dimensional system described by the Wilson action, an analytic form of the single plaquette partition function at $N = \infty$ was given by Gross and Witten²⁾. An analysis of the $1/N$ corrections in the $U(N)$ Wilson theory is due to Goldschmidt³⁾ who calculated the $1/N^2$ and $1/N^4$ terms, using the orthogonal polynomial method⁴⁾.

We shall discuss the properties of the two-dimensional system described by the mixed fundamental-adjoint action. At $N = \infty$ in arbitrary number of dimensions, the averages of the Wilson loops satisfy the same set of Schwinger-Dyson equations as in the pure Wilson theory, but with an effective bare coupling B given as a function of fundamental and adjoint couplings⁵⁾. The string tension, measured in the mixed theory for given values of bare couplings, must effectively be the same as in the Wilson theory with the corresponding value of the effective coupling.

This property can be verified in two dimensions by an explicit calculation⁶⁾. We shall discuss here the structure of the mixed theory when $1/N^2$ corrections are taken into account. Their analytic form for the $U(N)$ theory will be derived in Section 2. The effect of these corrections on the string tension will be discussed in Section 3. In Section 4, we shall consider the $1/N^2$ corrections to the $SU(N)$ theory and in Section 5, implications for systems with larger number of dimensions.

2. $1/N^2$ CORRECTIONS FOR THE $U(N)$ SYSTEM

In this section, we compare the single plaquette partition functions defined for the Wilson and mixed actions

$$\begin{aligned} Z_W(\beta) &= \exp[N^2 F_W(\beta)] = \int DU \exp[N\beta R], \\ Z_M(\beta_f, \beta_a) &= \exp[N^2 F_M(\beta_f, \beta_a)] \\ &= \int DU \exp[N\beta_f R + \beta_a(R^2 + I^2)]. \end{aligned} \quad (2.1)$$

In Eq.(2.1), DU is an invariant $U(N)$ group measure and R and I denote respectively real and imaginary parts of $\text{tr}U$. $F_W(\beta)$ and $F_M(\beta_f, \beta_a)$ are the plaquette free energies for the Wilson and mixed actions.

Knowledge of $F_W(\beta)$ for the Wilson $U(N)$ action is sufficient to calculate expectation values of real and imaginary parts of $\text{tr}U$ and their correlations. Using the invariance of the measure under the change of variable $U \rightarrow Ue^{i\mu}$, DU remains unchanged, so

$$Z_W(\beta) = \int DU \exp[N\beta(R \cos \mu - I \sin \mu)] \quad (2.2)$$

for any μ . Writing

$$\begin{aligned} N\beta \cos \mu &= NB + \varepsilon, \\ N\beta \sin \mu &= \lambda, \end{aligned} \quad (2.3)$$

or

$$\beta = \sqrt{(B + \frac{\varepsilon}{N})^2 + (\frac{\lambda}{N})^2},$$

we can rewrite $Z_W(\beta)$ as a generating function of averages of powers of R and I :

$$Z_W\left(\sqrt{(B + \frac{\varepsilon}{N})^2 + (\frac{\lambda}{N})^2}\right) = \int DU \exp[NBR + \varepsilon R - \lambda I]. \quad (2.4)$$

Thus differentiating $F_W[\sqrt{(B+\varepsilon/N)^2 + (\lambda/N)^2}]$ at $\varepsilon = \lambda = 0$, gives

$$\begin{aligned} \langle \frac{R}{N} \rangle &= F'_W(\beta), \\ \langle \frac{I}{N} \rangle &= 0, \\ \langle R^2 \rangle_c &= F''_W(\beta) \\ \langle I^2 \rangle_c &= \frac{F'_W(\beta)}{\beta} \\ \langle RI \rangle_c &= 0 \end{aligned} \quad (2.5)$$

etc.

We shall use Eq.(2.4) to re-express $Z_M(\beta_f, \beta_a)$ in terms of Z_W . We first rewrite the mixed action in Eq.(2.1), as

$$\begin{aligned} N\beta_f R + \beta_a(R^2 + I^2) \\ = N(\beta_f + 2\beta_a w)R + \beta_a[(R - Nw)^2 + I^2] - N^2\beta_a w^2 \end{aligned} \quad (2.6)$$

with quantity w to be defined later. Using

$$B = \beta_f + 2\beta_a w \quad (2.7)$$

as a new coupling in the Wilson theory, we get from Eq.(2.4)

$$\begin{aligned} Z_M(\beta_f, \beta_a) &= \exp[-N^2\beta_a w] \times \\ &\times \exp[Nw\varepsilon] \exp\left[\beta_a\left(\frac{\partial^2}{\partial \varepsilon^2} + \frac{\partial^2}{\partial \lambda^2}\right)\right] \exp[-Nw\varepsilon] \exp\left[N^2 F_W\left(\sqrt{(B+\frac{\varepsilon}{N})^2 + (\frac{\lambda}{N})^2}\right)\right] \bigg|_{\varepsilon=\lambda=0} \end{aligned} \quad (2.8)$$

Expanding F_W around $\varepsilon = \lambda = 0$ gives

$$\begin{aligned} N^2 F_W\left(\sqrt{(B+\frac{\varepsilon}{N})^2 + (\frac{\lambda}{N})^2}\right) \\ = N^2 F_W(B) + N\varepsilon F'_W(B) + \frac{1}{2}\varepsilon^2 F''_W(B) + \frac{1}{2}\mu^2 \frac{F'_W(B)}{B} \\ + O\left(\frac{\lambda}{N}\right). \end{aligned} \quad (2.9)$$

Calculating $F_M(\beta_f, \beta_a)$ to order $1/N^2$ will require keeping in Eq.(2.9) only terms up to $O(1)$. Putting

$$w = w(B) = F'_w(B) \quad (2.10)$$

in Eq.(2.8), reduces the $O(1/N)$ term in F_M . Calculation of Eq.(2.8) to order $1/N^2$ in F_M becomes now straightforward and we get

$$F_M(\beta_f, \beta_a) = F_w(B) - \beta_a w^2(B) + \frac{1}{N^2} \Delta_M(B, \beta_a) \quad (2.11)$$

with

$$\Delta_M(B, \beta_a) = -\frac{1}{2} \ln \left[(1 - 2\beta_a w'(B)) (1 - 2\beta_a \frac{w(B)}{B}) \right].$$

Term Δ_M is clearly of order $1/N^2$ as compared to the first two terms in Eq.(2.11). Corrections to Eq.(2.11) will necessarily be of higher order in $(1/N)$. Definition of B [Eq.(2.7)] is the same as follows from the general argument⁵⁾ for $N = \infty$, since from Eq.(2.5) follows

$$w = \left\langle \frac{R}{N} \right\rangle_w = \left\langle \frac{tr U}{N} \right\rangle_w. \quad (2.12)$$

We conclude this section by writing Δ_M in terms of correlations in the Wilson theory with an effective coupling B . From Eq.(2.5)

$$\Delta_M(B, \beta_a) = -\frac{1}{2} \ln \left[(1 - 2\beta_a \langle R^2 \rangle_c) (1 - 2\beta_a \langle I^2 \rangle_c) \right]. \quad (2.13)$$

This form will be useful when $U(N)$ and $SU(N)$ theories are compared in Section 4. For $SU(N)$, some of the relations in Eq.(2.5) are not valid, in particular the expression for $\langle I^2 \rangle_c$ in terms of w , but Eq.(2.13) remains true (see Section 4).

3. $1/N^2$ CORRECTIONS TO THE STRING TENSION

In this section, we shall derive an explicit form of the $1/N^2$ corrections in the $U(N)$ theory. In two dimensions, the string tension σ is simply related to $\langle \text{tr} U \rangle$:

$$\sigma = \ln \left\langle \frac{\text{tr} U}{N} \right\rangle. \quad (3.1)$$

We shall therefore consider a form of $\langle \text{tr} U/N \rangle_M = \langle R/N \rangle_M$ in the mixed theory. From Eqs.(2.1) and (2.12) follows

$$\begin{aligned} \left\langle \frac{R}{N} \right\rangle_M &= \frac{\partial}{\partial \beta_f} F_M(\beta_f, \beta_a) \\ &= w(B) + \frac{1}{N^2} \frac{\partial \Delta_M(B, \beta_a)}{\partial \beta_f} + \dots \end{aligned} \quad (3.2)$$

To get a systematic $1/N$ expansion for the $\langle R/N \rangle_M$, we shall use the known expansion³⁾ of $w(B)$. The form of this expansion depends on whether $B < 1$ or $B > 1$. For $B < 1$ (strong coupling)

$$w(B) = \frac{B}{2} \quad (3.3)$$

with exponentially small corrections in N . For $B > 1$

$$w(B) = w_0(B) + \frac{1}{N^2} w_1(B) + O\left(\frac{1}{N^4}\right) \quad (3.4)$$

with

$$\begin{aligned} w_0 &= 1 - \frac{1}{2B} \\ w_1 &= -\frac{1}{8B^2} \left(1 - \frac{1}{B}\right)^{-1}. \end{aligned} \quad (3.5)$$

To calculate the $1/N^2$ correction to $\langle R/N \rangle_M$, we need in both cases only the $N = \infty$ result. As can easily be seen for $B < 1$, correction vanishes. This in fact remains true to all orders in $1/N^2$: corrections are exponentially small in N . For $B > 1$, Eq.(3.2) yields a non-trivial result. We observe here that Δ_M is continuous when crossing the $B = 1$ phase transition line in (β_f, β_a) plane, but its derivatives have finite discontinuities, in particular

$$\frac{\partial \Delta_M}{\partial \beta_f} = \begin{cases} 0 & \text{for } B = 1^- \\ -\frac{\beta_a}{(1 - \beta_a)^2} & \text{for } B = 1^+. \end{cases} \quad (3.6)$$

In the weak coupling, Eq.(3.2) can be interpreted as a redefinition of the coupling $B \rightarrow B + (1/N^2)\Delta$ in the pure Wilson theory, where

$$\Delta = \frac{1}{w'_0(B)} \frac{\partial \Delta_M(B, \beta_a)}{\partial \beta_f} \quad (3.7)$$

For $\beta_a \ll \beta$, Δ becomes

$$\Delta = -2\beta_a - \frac{4\beta_a^2}{B} \left(1 + O\left(\frac{1}{B}\right)\right) + O(\beta_a^3). \quad (3.8)$$

As can easily be seen, Eq.(3.8) is dominated in this limit by the terms resulting from the large B behaviour of $\langle I^2 \rangle_c$. As we shall argue in the next section, this is a special feature of the $U(N)$ theory.

The redefinition (3.7) makes sense if only the string tension [or $\langle R/N \rangle_{M-}$] is considered. Other gauge invariant quantities, i.e., $\langle (R^2 + I^2)/N^2 \rangle_M$, will in general define different mappings of the Wilson theory.

4. $1/N^2$ CORRECTIONS IN THE $SU(N)$ THEORY

In this section, we shall discuss the form of the $1/N^2$ correction for the $SU(N)$ theory. We shall be concerned here only with the weak coupling ($B > 1$) regime. We first observe that the form (2.13) of the correction Δ_M can be given a simple graphical interpretation. Consider a sum of Feynman type ring diagrams (Figure). Each vertex on these diagrams is represented by two semicircles corresponding to $\beta_a \text{tr} U \cdot \text{tr} U^+$ and a summation over both orderings of $\text{tr} U$ and $\text{tr} U^+$ in each vertex of the chain is understood. Double lines represent correlations among semicircles [$\langle (\text{tr} U)^2 \rangle_c$, $\langle \text{tr} U \cdot \text{tr} U^+ \rangle_c$ or $\langle (\text{tr} U^+)^2 \rangle_c$ depending on the ordering]. Using

$$\langle R I \rangle_c = 0 \quad (4.1)$$

it is easy to show that a sum shown in the Figure will separate into two disjoint sums involving only real or only imaginary parts of $\text{tr} U$ and $\text{tr} U^+$, yielding thus expression (2.13).

As can be checked the form of the $1/N^2$ correction to the free energy remains true also for $SU(N)$ if the mixed action is written as in (2.6) and the second term is treated as a perturbation. All other diagrams, in particular those where more complicated correlations occur, will be suppressed by the powers of $1/N$.

For the $SU(N)$ theory $\langle R^2 \rangle_c$ is again related to w' as in the $U(N)$ case. The term $\langle I^2 \rangle_c$ must, however, be different. By a simple perturbative argument using the tracelessness of the $SU(N)$ generators, one expects $\langle I^2 \rangle_c = 0(1/B^3)$ for large B in contrast to the $U(N)$ situation where $\langle I^2 \rangle_c = 0(1/B)$. The large B behaviour of Eq.(2.13) will effectively be dominated for $SU(N)$ by the behaviour of $\langle R^2 \rangle_c \sim 0(1/B^2)$. For this case again, we can interpret a $1/N^2$ effect on the string tension as a redefinition of the effective coupling $B \rightarrow B + (1/N^2)\Delta$. Now, however, for $\beta_a \ll B$

$$\Delta = - \frac{2\beta_a}{B} \left(1 + O\left(\frac{1}{B^2}\right)\right) - \frac{4\beta_a^2}{B^3} \left(1 + O\left(\frac{1}{B^2}\right)\right) + O(\beta_a^3) \quad (4.2)$$

which shows similar behaviour to that observed in four dimensions⁷⁾.

5. CONCLUSIONS

The form of the $1/N^2$ correction to the free energy calculated for the $U(N)$ group can be used to describe similar effects in $SU(N)$. For the weak coupling, its effect in $SU(N)$ is much smaller than in $U(N)$ and at least for large B and $\beta_a \ll B$, the $N = \infty$ result seems to be a better approximation in $SU(N)$ than in $U(N)$ theory.

The topological structure of the $1/N^2$ correction for a two-dimensional theory can easily be extended to more dimensions. By arguments similar to those used in the preceding section, we expect it to be true in any number of dimensions. Then, however, one must take into account that correlations are not purely local as in two dimensions and that different orientations of plaquette must be taken into account. The $1/N^2$ correction to the free energy can again be represented by diagrams of the Figure but with additional summations over positions and orientations of plaquette appearing in vertices. The resulting expression is perfectly accessible in the perturbative expansion.

REFERENCES

- 1) G. 't Hooft, Nucl. Phys. B72 (1974) 461.
- 2) D.J. Gross and E. Witten, Phys. Rev. D21 (1980) 446.
- 3) Y.Y. Goldschmidt, J. Math. Phys. 21 (1980) 1842.
- 4) J.-M. Drouffe and C. Itzykson, Physics Reports, 38C (1978) 133.
- 5) Yu. Makeenko and M.I. Polikarpov, Nucl. Phys. B205 [FS 5] (1982) 386;
S. Samuel, Phys. Lett. 112B (1982) 237.
- 6) T.L. Chen, Ch. I-Tan and X.T. Zheng, Phys. Lett. 109B (1982) 383.
- 7) J. Jurkiewicz, C.P. Korthals Altes and J.W. Dash, CERN preprint TH-3621 (1983).

$$\Delta_M = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \dots$$

FIGURE

Topological structure of the $1/N^2$ corrections to the mixed action free energy.