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PADE APPROXIMANTS AND CLOSED FORM SOLUTIONS
OF THE KdV AND MKdV EQUATIONS

F. Lambert
CERN - Geneva

A B S T R A C T

The closed form solutions of the Korteweg-de Vries (KdV) and modified MKdV equations which are obtainable by inverse scattering or Hirota's method are found to be Padé approximants to the formal series which results from iterating particular solutions of the linearized equation.

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The methods of constructing explicit solutions of IST solvable ¹⁾ non-linear equations fall into two categories :

- i) inverse methods starting from the associated eigenvalue problem, such as the Volterra factorization methods of Shabat ²⁾ and Cornille ³⁾ ;
- ii) perturbation methods which start from the original equation and which transform it into an appropriate iteration scheme, such as Hirota's method ⁴⁾ of bilinear differential forms.

Though the two approaches look different in many respects, it seems that there should also exist some important links between the above methods, at least in the case of a KdV-like non-linearity.

We first note that both the inverse and the perturbation schemes yield solutions of the KdV equation which are deduced from solutions of the linearized equation ^{4),5)}. By iterating these (first order) solutions, written as a Fourier integral over some bounded measure, Rosales ⁶⁾ has even succeeded to recast the perturbation series into a form from which the Marchenko integral equation followed naturally, or almost...

In this note we discuss some of Rosales' results in the language of Padé approximants ⁷⁾, which might be helpful in understanding the hidden links between the two schemes.

Specializing first to the KdV equation ($q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$) :

$$q_t + q_{xxx} + 6 q q_x = 0 \quad (1)$$

or rather to the related PKdV equation for the "potential" Q , $q = -\partial_x Q$:

$$Q_t + Q_{xxx} - 3 Q_x^2 = 0, \quad (2)$$

we look for solutions in the form

$$Q = \sum_{n=1}^{\infty} \varepsilon^n Q_n(x, t) \quad (3)$$

The consecutive terms can be obtained from the iteration system :

$$Q_{n,t} + Q_{n,xxx} = 3 \sum_{j=1}^{n-1} Q_{j,x} Q_{n-j,x} \quad (4)$$

The first order term Q_1 is but a solution of the linearized equation :

$$Q_t + Q_{xxx} = 0, \quad (5)$$

and can be expressed as a Fourier integral over a complex measure ⁸⁾ :

$$Q_1(x, t) = \int d\mu(k) e^{ikx + ik^3 t}. \quad (6)$$

For the n^{th} order term Q_n , it is enough to take the particular solution of Eq. (4) which can be written in terms of $d\mu(k)$ in the general form [cf. Ref. 7] :

$$Q_n(x, t) = \int [d\mu(k)]^n (-i)^{n-1} \frac{e^{i\Omega_n}}{\prod_{1 \leq j < n} (k_j + k_{j+1})} \quad (7)$$

with

$$[d\mu(k)]^n = d\mu(k_1) \dots d\mu(k_n) \\ \Omega_n = (k_1 + \dots + k_n)x + (k_1^3 + \dots + k_n^3)t.$$

Subject to appropriate convergence conditions (e prescription for real k 's), it is possible to remove the denominator in the above integrand, at the cost of $n-1$ extra one-dimensional integrations over auxiliary space variables z_j :

$$Q_n(x, t) = \int [d\mu(k)]^n e^{i(k_1^3 + \dots + k_n^3)t} \frac{(-i)^{n-1}}{2^{n-1}} \int_x^\infty dz_1 \dots \int_x^\infty dz_{n-1} e^{\frac{i}{2}[k_1(x+z_1) + \dots + k_n(z_{n-1}+x)]} \quad (8)$$

Changing the order of integrations we get :

$$Q_n(x, t) = 2(-i)^{n-1} \int_x^\infty dz_1 \dots \int_x^\infty dz_{n-1} \left[\frac{1}{2} \int d\mu(k_1) e^{ik_1^3 t + \frac{i}{2}k_1(x+z_1)} \right] \dots \left[\frac{1}{2} \int d\mu(k_n) e^{ik_n^3 t + \frac{i}{2}k_n(z_{n-1}+x)} \right] \quad (9)$$

It is then natural to introduce :

$$F(x+z, t) = \frac{1}{2} \int d\mu(k) e^{ik^3 t + \frac{i}{2}k(x+z)}, \quad (10)$$

in order to end up with the perturbation solution :

$$Q = -2 \sum_{n=1}^{\infty} (-\varepsilon)^n \int_x^{\infty} dz_1 \dots \int_x^{\infty} dz_{n-1} F(x+z_1, t) F(z_1+z_2, t) \dots F(z_{n-1}+x, t), \quad (11)$$

which, up to the factor -2 , can be recognized as the Neumann series solution of the Marchenko equation :

$$K(x, y; t) + \varepsilon F(x+y, t) + \varepsilon \int_x^{\infty} dz K(x, z; t) F(z+y, t) = 0 \quad (12)$$

taken at $x=y$:

$$K(x, x; t) = \sum_{n=1}^{\infty} (-\varepsilon)^n \int_x^{\infty} dz_1 \dots \int_x^{\infty} dz_{n-1} F(x+z_1, t) \dots F(z_{n-1}+x, t). \quad (13)$$

It also follows from (10) and (6) that :

$$F(2x, t) = \frac{1}{2} Q, (x, t) \quad (14)$$

It is now clear that solving the Marchenko integral equation, at $x=y$, with an inhomogeneous term given by a solution of the linearized equation, as is required by Shabat's inverse procedure, will just amount to summing the above perturbation series. The summation of the series can be carried out in the very special cases in which the measure $d\mu(k)$, which characterizes the first order term, is discrete with an appropriate support. An important example treated by Rosales is that of measures of the type :

$$\int d\mu(k) f(k) = \sum_{j=1}^N a_j^2 f(ik_j), \quad a_j \text{ and } k_j \in \mathbb{R}, k_j > 0. \quad (15)$$

Writing then :

$$F(x+y, t) = \frac{1}{2} \int d\mu(k) e^{ik^3 t + \frac{i}{2} k(x+y)} = \frac{1}{2} \sum_{j=1}^N a_j^2 e^{k_j^3 t - \frac{k_j}{2}(x+y)} \quad (16)$$

in the form

$$F(x+y, t) = \frac{1}{2} \mathbb{P}^T(x, t) \mathbb{P}(y, t), \quad (17)$$

where $\mathbb{P}$ denotes the column matrix :

$$\mathbb{P}(x, t) = \begin{pmatrix} p_1 = a_1 e^{-\frac{1}{2} k_1 x + \frac{1}{2} k_1^3 t} \\ \vdots \\ p_N = a_N e^{-\frac{1}{2} k_N x + \frac{1}{2} k_N^3 t} \end{pmatrix} \quad (18)$$

we get :

$$Q = - \sum_{n=1}^{\infty} (-\varepsilon)^n Q_n(x, t)$$

with

$$Q_n(x, t) = \frac{1}{2^{n-1}} \int_x^{\infty} dz_1 \dots \int_x^{\infty} dz_{n-1} \left[\bar{\rho}(x, t) [\rho(z_1, t) \bar{\rho}(z_1, t) \dots \rho(z_{n-1}, t) \bar{\rho}(z_{n-1}, t)] \rho(x, t) \right] \quad (19)$$

and thus

$$Q_n = \bar{\rho}^T(x, t) A_N^{n-1} \rho(x, t) \quad (20)$$

with

$$A_N = \frac{1}{2} \int_x^{\infty} dz \rho(z, t) \bar{\rho}^T(z, t) = \begin{pmatrix} \frac{p_1^2}{2k_1} & \dots & \frac{p_1 p_N}{k_1 + k_N} \\ \vdots & & \vdots \\ \frac{p_1 p_N}{k_1 + k_N} & & \frac{p_N^2}{2k_N} \end{pmatrix}$$

As A_N is a real, symmetric and positive definite $N \times N$ matrix, we may sum the perturbation series (19) and obtain the closed form :

$$Q = \varepsilon \bar{\rho}^T(x, t) (I + \varepsilon A_N)^{-1} \rho(x, t), \quad (21)$$

or

$$Q = \varepsilon \text{Tr} [(I + \varepsilon A_N)^{-1} \rho \bar{\rho}^T] = -2\varepsilon \text{Tr} [(I + \varepsilon A_N)^{-1} A_{N,x}]$$

and thus

$$Q = -2 \partial_x \ln \det(I + \varepsilon A_N), \quad (22)$$

which corresponds to Hirota's multisoliton formula ⁹⁾.

Yet, we remark that in despite of its elegance, the formula (22) does not exhibit the explicit dependence of the closed form solution on the successive perturbation terms (20).

In order to establish this dependence, we set $|\varphi\rangle = \rho(x, t)$, and rewrite (19) or

$$Q = - \sum_{n=1}^{\infty} (-\varepsilon)^n \langle \varphi, A_N^{n-1} \varphi \rangle = \varepsilon \langle \varphi, (I + \varepsilon A_N)^{-1} \varphi \rangle \quad (23)$$

as the scalar product :

$$Q = - \langle \varphi, \psi_N \rangle, \quad (24)$$

where $|\psi_N\rangle$ denotes the solution of the (truncated) functional equation :

$$|\psi_N\rangle = -\varepsilon |\varphi\rangle - \varepsilon A_N |\psi_N\rangle \quad (25)$$

i.e.,

$$|\psi_N\rangle = -\varepsilon (I + \varepsilon A_N)^{-1} |\varphi\rangle. \quad (26)$$

Expanding this solution on the particular base $\{|\varphi_i\rangle = A_N^i |\varphi\rangle, i=0\dots N-1\}$, we get :

$$Q = \varepsilon \sum_{i,k=0}^{N-1} \langle \varphi, \varphi_i \rangle (M^{-1})_{ik} \langle \varphi_k, \varphi \rangle, \quad (27)$$

with

$$M_{ik} = \langle \varphi_i, \varphi_k \rangle + \varepsilon \langle \varphi_i, A_N \varphi_k \rangle$$

i.e.,

$$M_{ik} = \langle \varphi, A_N^{i+k} \varphi \rangle + \varepsilon \langle \varphi, A_N^{i+k+1} \varphi \rangle$$

or

$$M_{ik} = Q_{i+k+1} + \varepsilon Q_{i+k+2} \quad (28)$$

The result thus obtained :

$$Q = \varepsilon \overbrace{(Q_1, \dots, Q_N)} \begin{pmatrix} Q_1 + \varepsilon Q_2 & \dots & Q_N + \varepsilon Q_{N+1} \\ \vdots & & \\ Q_N + \varepsilon Q_{N+1} & \dots & Q_{2N-1} + \varepsilon Q_{2N} \end{pmatrix} \begin{pmatrix} Q_1 \\ \vdots \\ Q_N \end{pmatrix} \quad (29)$$

is Nuttall's compact formula ¹⁰⁾ for the $[N/N]$ Padé approximant to the series (19) ; it involves the first $2N$ perturbation terms.

In fact, it was already clear with Eq. (21) that Q had to be a rational function of ϵ of the form :

$$Q = \frac{P_N(\epsilon)}{\tilde{P}_N(\epsilon)}, \quad (30)$$

where P_N and $\tilde{P}_N$ are polynomials of degree N .

The same result is easily obtained with the inverse scheme by noting that the kernel $F(x+y, t)$ given by Eq. (16) is separable of rank N . Application of Chisholm's theorem ¹¹⁾ shows then that the exact (Fredholm) solution of the Marchenko equation coincides with the $[N/N]$ Padé approximant to the Neumann series. Furthermore, on the diagonal $x=y$, one has the interesting property ¹²⁾ :

$$K(x, x; t) = \partial_x \log \Delta(x, t) \quad (31)$$

where $\Delta(x, t)$ denotes the Fredholm determinant of Marchenko's (diagonal) equation. Hence, we see that the multisoliton solutions can be obtained from the only knowledge of the Padé denominators.

Let us also remark that Rosales' general expression (11) for the perturbation solution is entirely analogous with Cornille's early formulas ^{12), 13)} which established a connection between the Marchenko formalism and N/D equations. In that case, the measure $d\mu(k)$ was essentially given by the discontinuity of the S matrix, while the Jost solution satisfied a functional equation which is related to our Eq. (25). As in Cornille's case, it is also possible to obtain other real closed form solutions than the multisolitons ⁵⁾, either as a diagonal Padé or as a Fredholm solution, by starting with a first order solution, or a kernel $F(x+y, t)$, given by a somewhat modified discrete measure obeying the symbolic reality condition $d\mu^*(-k^*) = d\mu(k)$. Such "ghost multisolitons" have singularities for real values of x .

Contrary to the KdV equation, the MKdV equation :

$$v_t + v_{xxx} + 6v^2 v_x = 0, \quad (32)$$

is no longer associated with a scalar eigenvalue problem. Nevertheless, it has perturbation solutions of the form ⁷⁾ :

$$v = \sum_{\substack{n=0 \\ m=2n+1}}^{\infty} \epsilon^m \alpha^n \int [d\lambda(k)]^m \frac{e^{i\Omega_m}}{\prod_{j=1}^m (k_j + k_{j+1})} \quad (33)$$

Considering again discrete measures of the form (15) one is led, in the same way as above and with the same notations, to the multisoliton solutions (if $\alpha > 0$) :

$$v = \epsilon //^T (I + \alpha \epsilon^2 A_N^2)^{-1} //^p \quad (34)$$

Writing now : $v = \epsilon \tilde{v}(x, t; \epsilon^2, \alpha)$

(35)

we see that these multisolitons will now correspond to $\boxed{N-1/N}$ Padé approximants in ϵ^2 (or α) to the perturbation series :

$$\tilde{v} = \tilde{v}_1(x, t) + \epsilon^2 \alpha \tilde{v}_2(x, t) + \epsilon^4 \alpha^2 \tilde{v}_3(x, t) + \dots \quad (36)$$

constructed with the $2N$ first terms :

$$\tilde{v}_n(x, t) = (-)^{n-1} \langle \varphi, A_N^{2(n-1)} \varphi \rangle$$

In this case there can also be real closed form solutions, without the a_j 's and the k_j 's being necessary real, but such that $d\lambda^*(-k^*) = d\lambda(k)$. If $\alpha > 0$ and if $\text{Re } k_j > 0$, they will be non-singular.

The closed form (34) can again be transformed into Hirota's formula ⁴⁾. According to his procedure one should write v immediately in the N/D form. Special bilinear differential equations must then be derived for N and D from Eq. (32). Expanding finally N and D in powers of ϵ , one determines the (Padé) coefficients by applying perturbation theory to the bilinear equations. Though his method is presented as being similar to the method of Padé approximants, we remark that it is rather a way of obtaining the Padé coefficients without calculating the $2N$ coefficients of the perturbation series, which, indeed, are rather complicated. The Padé method, on the contrary, is precisely a technique which enables us to apply the perturbation theory to the original equation, without the need of deriving first the appropriate bilinear forms, but at the cost of longer (though straightforward) calculations.

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