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PSEUDO-ABELIAN OPERATORS IN THE LARGE N LIMIT

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A B S T R A C T

The large N limit of the determinants of gauge invariant products of $U(N)$ variables for lattice gauge and spin models is reconsidered. The corresponding external field problem is completely solved and the resulting expectation values are shown to violate factorization and to behave as if being generated from an effective continuum $U(1)$ theory.

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A fruitful approach to systems characterized by a $U(N)$ symmetry, like chiral models and gauge theories, is the $1/N$ expansion. It is therefore useful to improve our knowledge of the properties of the large N limit of these systems.

Most of our prejudices about these properties are based on the factorization of gauge invariant amplitudes, stated as follows : whenever we consider a product of operators, each admitting a finite expectation value in the limit $N \rightarrow \infty$, then the expectation value of this product is the product of the single expectation values :

$$\langle AB \dots C \rangle = \langle A \rangle \langle B \rangle \langle \dots \rangle \langle C \rangle \quad (1)$$

However, there are limitations to the validity of Eq. (1) : in particular, it is known ¹⁾ that factorization does not hold when one considers the determinant of a (gauge covariant) product of $U(N)$ variables :

$$\Delta(C) \equiv \det \prod_{l \in C} U_l \quad (2)$$

in lattice gauge theories, where C is a Wilson loop, and

$$\Delta(x) \equiv \det U(o) U^+(x) \quad (3)$$

in chiral models.

Green and Samuel ¹⁾ have stressed the importance of understanding the large N properties of this operator : they claim that it behaves as an order parameter for the large N phase transition characterizing the class of models we are discussing in arbitrary dimensions.

Its non-trivial behaviour is related to the fact that the determinant picks up the phase characterizing the $U(1)$ subgroup that constitutes the centre of $U(N)$ and since

$$U(N) \cong U(1) \times \frac{SU(N)}{Z_N} \quad (4)$$

$SU(N) \rightarrow U(N)$ as $N \rightarrow \infty$ because $Z_N \rightarrow U(1)$; therefore the determinant in the large N limit reflects properties of the centre of $SU(N)$ in the same limit. This argument shows that the determinant has some properties of Abelian operators. However, it is crucial that this Abelian $U(1)$ subgroup whose properties we are studying is not decoupled, as happens in the continuum theory, and therefore

$\langle \Delta \rangle$ does not, in general, have on the lattice the free theory behaviour it has in the continuum.

We are going to show that the Abelian nature of Δ does, however, show up in a very subtle way in the large N limit of this operator. This is the reason why we define it as a "pseudo-Abelian" quantity.

In order to develop our argument, we consider a special class of determinant operators, those defined by

$$\Delta_\ell = \langle \det U^\ell \rangle = \frac{\int dU \det U^\ell \exp N \text{Tr} (U^\dagger A + A^\dagger U)}{\int dU \exp \text{Tr} N (U^\dagger A + A^\dagger U)} \quad (5)$$

where A is an arbitrary $N \times N$ matrix source. The quantities Δ_ℓ appear in different contexts : when one chooses $A = \beta I$ then Δ_ℓ is the one-plaquette (one-link) expectation value for two-dimensional gauge theories (one-dimensional chiral models). When A is determined in a self-consistent way Δ_ℓ is the mean field approximation to $\Delta(C)$ ($\Delta(x)$). Therefore the properties of Δ_ℓ are relevant to the more general problem we have discussed.

Let us sketch the main known results concerning Δ_ℓ ²⁾. We start by defining the variables z_k eigenvalues of the Hermitian matrix

$$2N (A^\dagger A)^{1/2} = R \text{diag} (z_1, \dots, z_N) R^\dagger \quad (6)$$

(R is a rotation matrix) and

$$\theta = \frac{i}{2N} [\ln \det A^\dagger - \ln \det A] \quad (7)$$

Δ_ℓ can depend only on z_k and θ , and it was found that, for arbitrary $U(N)$ matrices U

$$\Delta_\ell = e^{iN\ell\theta} \frac{W_\ell}{W_0} \quad (8)$$

where

$$W_\ell = \left(\prod_k z_k \right)^{\ell\ell} \left(\prod_k \frac{1}{z_k} \frac{\partial}{\partial z_k} \right)^{\ell\ell} W_0 = \det \left(z_i^{j-1} I_{j-1-\ell}(z_i) \right) \quad (9)$$

and W_ℓ satisfies the following differential Schwinger-Dyson equation

$$\frac{1}{N} \left[\sum_k z_k^2 \frac{\partial^2}{\partial z_k^2} + (3-2N) \sum_k z_k \frac{\partial}{\partial z_k} - \sum_k z_k^2 + \frac{2}{3} N(N-1)(N-2) \right] W_\ell = \ell^2 W_\ell \quad (10)$$

The large N limit of W_ℓ can easily be extracted from Eq. (9) 3)-5) :

$$W_\ell \xrightarrow{N \rightarrow \infty} W^{(\infty)} = \exp \left[\sum_k z_k - \frac{1}{2} \sum_k \ell_n 2n z_k + \sum_{i < k} \ell_n (z_i - z_k) \right] \quad (11)$$

when the weak coupling condition

$$\sum_k \frac{1}{z_k} \leq 1 \quad (12)$$

is satisfied.

Equation (11) represents the leading contribution and is manifestly not sufficient in order to determine the large N limit of Eq. (8) : we need to compute the $O(1)$ factor in front of the exponentially growing term represented by Eq. (11). If we parametrize W_ℓ in the form :

$$W_\ell = X_\ell W^{(\infty)} \quad (13)$$

it is easy, using the techniques presented in the Appendix of Ref. 13), to extract from Eq. (10) the following equation for X_ℓ :

$$\frac{1}{N} \left[\sum_k z_k^2 \frac{\partial^2 X_\ell}{\partial z_k^2} + 2 \sum_k z_k^2 \frac{\partial X_\ell}{\partial z_k} + \sum_{k \neq i} \frac{z_k z_i}{z_k - z_i} \left(\frac{\partial X_\ell}{\partial z_k} - \frac{\partial X_\ell}{\partial z_i} \right) \right] = \left(\ell^2 - \frac{1}{4} \right) X_\ell \quad (14)$$

The large N ansatz

$$X_\ell = X_\ell(t) \quad \text{where } t = \sum_k \frac{1}{z_k} \quad (15)$$

reduces Eq. (14) to

$$\frac{1}{N} \left(\sum_k \frac{1}{z_k^2} \right) \frac{d^2 X_\ell}{dt^2} + 2(t-1) \frac{dX_\ell}{dt} = \left(\ell^2 - \frac{1}{4} \right) X_\ell \quad (16)$$

The first term in the left-hand side is $O(N^{-2})$ and therefore can be removed, leaving us with a consistent equation whose solution (with proper boundary conditions) is

$$X_\ell = \left(1 - \sum_k \frac{1}{z_k}\right)^{\frac{1}{2}(\ell^2 - \frac{1}{4})} \quad (17)$$

It is now straightforward to compute the large N limit of Δ_ℓ :

$$\Delta_\ell \xrightarrow{N \rightarrow \infty} e^{iN\ell\theta} \left(1 - \sum_k \frac{1}{z_k}\right)^{\frac{\ell^2}{2}} \quad \text{when} \quad 1 \geq \sum_k \frac{1}{z_k} \quad (18)$$

From the standard coupling expansion, we recognize that

$$\Delta_\ell \xrightarrow{N \rightarrow \infty} 0 \quad \text{when} \quad \sum_k \frac{1}{z_k} \geq 1 \quad (19)$$

The result expressed in Eqs. (18) and (19) agrees with the exact computation presented in Ref. 6) for the case $A = \beta I$:

$$\Delta_\ell(\beta) = \left(1 - \frac{1}{2\beta}\right)^{\frac{1}{2}} \quad \beta > \frac{1}{2} \quad (20)$$

and with the perturbative weak coupling results obtained by Guha and Lee ⁷⁾ by stochastic quantization methods up to second order for Δ_1 . These authors correctly conjectured the result Eq. (18) for the special case $\ell = 1$.

We also checked our results by expanding Eq. (9) in powers of $1/z_k$ up to second order for arbitrary N . This expansion requires using the asymptotic expansion of Bessel functions :

$$I_\ell(z) = \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2z)^k} \frac{\Gamma(\ell+k+\frac{1}{2})}{\Gamma(\ell-k+\frac{1}{2}) k!} + O(e^{-z}) \quad (21)$$

decomposing the resulting determinants and representing the elementary contributions in terms of differential operators acting on the Vandemonde determinant

$$\Delta = \det(z_i^{j-1}) = \prod_{i < j} (z_i - z_j) \quad (22)$$

Extensive use of the following identities

$$\sum_k (z_k)^m \left(\frac{\partial}{\partial z_k}\right)^n \Delta = 0 \quad n > m \geq 0 \quad (23)$$

$$\sum_k \frac{1}{z_k} \frac{\partial}{\partial z_k} \Delta = - \sum_{k < \ell} \frac{1}{z_k z_\ell} \Delta \quad (24)$$

allowed us to obtain the following contributions :

$$W_\ell(\bar{z}_k) = W^{(\omega)} \left[1 - \frac{1}{2} (\ell^2 - \frac{1}{4}) \sum_k \frac{1}{z_k} + \frac{1}{8} (\ell^2 - \frac{1}{4}) (\ell^2 - \frac{9}{4}) \left(\sum_k \frac{1}{z_k} \right)^2 + O\left(\frac{1}{z_k^3}\right) \right] \quad (25)$$

We immediately recognize that Eq. (25) agrees with the expansion of Eq. (17) to second order in $\sum_k 1/z_k$ with no $O(N^{-2})$ corrections !

It should by now be clear what we mean by "pseudo-Abelian" character of Δ_ℓ . We see that Eq. (18) violates factorization : in turn it is exactly what would be predicted by an effective Abelian (Gaussian) theory governing the $U(1)$ phase of U :

$$\int_{-\infty}^{\infty} \frac{e^{-\left(\varphi - \frac{N\theta}{2\sigma}\right)^2}}{\sqrt{2\pi\sigma}} e^{i\ell\varphi} d\varphi = e^{iN\ell\theta} \left(1 - \sum_k \frac{1}{z_k}\right)^{\frac{\ell^2}{2}}$$

$$\text{if } \sigma = -\ell_n \left(1 - \sum_k \frac{1}{z_k}\right) \quad (26)$$

However, it is clear that this effective Abelian theory is not related to a trivial factorization of $U(N)$ matrices : the non-Abelian structure of the model is retained through the non-trivial dependence of σ on the parameters z_k . It is also worth observing that the effective theory described by Eq. (26) is a continuum and not a lattice theory. The lattice prediction for a $U(1)$ model follows from Eq. (9) :

$$\Delta_\ell = e^{i\ell\theta} \frac{I_\ell(\bar{z})}{I_0(\bar{z})}$$

and it agrees with Eq. (18) only up to second order in $1/z = \sum_k 1/z_k$ as shown by comparing our result Eq. (25) with Eq. (21).

We find that the "precocious scaling" of the operators $\det U^\ell$ in the large N limit in the weak coupling phase and the clear-cut nature of their phase transition make them very interesting order parameters for the lattice $N = \infty$

theory. Further study of their properties and computer simulations are probably worthwhile in order to get a better understanding of lattice gauge models in higher dimensions.

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