MONTE CARLO SIMULATION OF NON-COMPACT QCD
WITH STOCHASTIC GAUGE FIXING

E. Seiler
Max-Planck-Institut für Physik und Astrophysik
Werner-Heisenberg Institut für Physik
D-8000 Munich 40, Fed. Rep. of Germany

I. O. Stamatescu *)
CERN -- Geneva

and

D. Zwanziger **)
Physics Dept., New York University
New York, NY 10003, U.S.A.

ABSTRACT

A non-compact lattice model of quantum chromodynamics is studied numerically. Whereas in Wilson's lattice theory the basic variables are the elements of a compact Lie group, the present lattice model resembles the continuum theory in that the basic variables A are elements of the corresponding Lie algebra, a non-compact space. The lattice gauge invariance of Wilson's theory is lost. As in the continuum, the action is a quartic polynomial in A , and a stochastic gauge fixing mechanism - which is covariant in the continuum and avoids Faddeev-Popov ghosts and the Gribov ambiguity - is also transcribed to the lattice. It is shown that the model is self-compactifying, in the sense that the probability distribution is concentrated around a compact region of the hyperplane $\text{div } A = 0$ which is bounded by the Gribov horizon. The model is simulated numerically by a Monte Carlo method based on the random walk process. Measurements of Wilson loops, Polyakov loops and correlations of Polyakov loops are reported and analyzed. No evidence of confinement is found for the values of the parameters studied, even in the strong coupling regime.

*) Address after 1 July 1983: Max-Planck-Inst. f. Physik und Astrophysik, Postfach 401212, D-8000 Munich 40, Fed. Rep. of Germany.

**) This research was supported in part by National Science Foundation Grant No. PHY-8116102.

1. INTRODUCTION

Wilson's theory is remarkable because it preserves local gauge invariance on a discrete space-time lattice⁽¹⁾. It has moreover achieved considerable computational success as a physical model of confining QCD⁽²⁾. This raises the question whether the local gauge invariance is essential to its success or whether it is possible to construct a model that is a lattice analog of the gauge-fixed continuum theory and which would lie in the same universality class as Wilson's. We report here a numerical investigation of a model which is the lattice analog of continuum QCD with a stochastic gauge fixing mechanism. As in the continuum theory, its basic variables, A , are elements of a Lie algebra, a non-compact space, whereas in Wilson's theory they are the elements U of the corresponding compact Lie group. As the continuum limit is approached they should be related by $U = \exp A$.

The study of a lattice analog of non-compact QCD was initiated by Patrascioiu, Seiler and Stamatescu⁽³⁾ who used the temporal gauge^{*)}. They found numerical evidence for asymptotic freedom but no evidence of confinement. Thus the results are quite different from Wilson's theory. It should be noted, however, that if the temporal gauge is imposed on Wilson's theory then not all the link variables $U = \exp A$ have to be close to unity, even in the weak coupling régime, as pointed out by Baaquie⁽⁴⁾. Therefore, it might not be too surprising to find different behavior in the two models since the truncation of the expansion of U in powers of A which would lead from Wilson's model to the non-compact temporal gauge model is not justified. Baaquie also found, however, that if the Landau gauge, $\text{div } A = 0$ is imposed in Wilson's theory, then all the U 's are close to unity in the weak coupling limit.

An obstacle to Monte Carlo simulations in the Landau gauge or other covariant gauges has been the presence of fermionic ghosts in the standard Faddeev-Popov method which, when viewed non-perturbatively, also suffers from the Gribov ambiguity. However, stochastic gauge fixing, which is a modification of the stochastic quantization method of Parisi and Wu⁽⁶⁾, avoids these difficulties. In Section 2.A we shall review stochastic gauge fixing

*) More precisely, the link variables were fixed on a maximal tree that most closely resembles the temporal gauge.

in the continuum theory and define a lattice model based on it which allows Monte Carlo simulation in a class of covariant stochastic gauges characterized by a gauge parameter α . In the limit $\alpha \rightarrow \infty$ the Landau gauge is approached and one would expect the present model to best approximate Wilson's theory on a finite lattice at large α ^{*)}. However, in the present study, which is the first numerical investigation of the model, α has been set at a moderate value, as discussed in Sections 3 and 4. In precise analogy to the continuum theory, the present model is characterized by an action that is a quartic polynomial in the link variables A of the form $S = \frac{1}{4} \sum (F_{\mu\nu}^a)^2$. Moreover, in its quadratic and cubic terms, it agrees with the expansion of Wilson's action that results when $U_\mu = \exp A_\mu$ is expanded in powers of A .

In Section 2A we describe a lattice calculus which provides a natural vehicle for transcribing continuum quantities to the discrete lattice. This is an extension of a standard formalism ⁽⁷⁾. Our model is then defined. In Section 2B we make a detailed study of the gauge fixing force on the lattice. The principal conclusion is that as the Landau gauge is approached, $\alpha \rightarrow \infty$, the stochastic gauge fixing force confines the probability to a compact region of the hyperplane $\text{div } A = 0$ that, for $SU(2)$, is contained in a poly-sphere of radius 16. Thus, in the Landau gauge, the non-compact lattice theory may be said to be self-compactifying. This result is a manifestation in our model of the Gribov horizon ⁽⁸⁾. For it has been demonstrated in the continuum theory, that as $\alpha \rightarrow \infty$, the stochastic gauge fixing force confines the probability to a convex region of the hyperplane $\text{div } A = 0$ that intersects every gauge orbit and that is bounded in every direction by the Gribov horizon ^(9,10). In the continuum theory such a region need not be compact, but it becomes compact on the finite lattice.

In Section 3 our numerical investigation is described. It is based on a random walk approximation of the diffusion equation for a stochastic process that was described previously ⁽¹¹⁾. The computations were done on the CRAY-1 of the Max Planck-Institute for Plasmaphysics in Garching.

Section 4 contains interpretation and analysis of our results.

*) The β -function of our present model may be strongly α -dependent on a finite lattice, and a calculation of this quantity is called for to properly define the continuum limit of our model. To be relevant for physics the model should possess a gauge invariant scaling limit.

2A. Definition of the Model

We consider a 4-dimensional simple cubic lattice with N_μ sites along the edge in the μ -direction (in most cases $N_\mu = N$ for all μ). The sites are labelled m, n etc. where $m = (m_\mu)$ ($\mu = 0, 1, 2, 3$) and m_μ runs through $(1, \dots, M_\mu)$. To minimize boundary effects we take the lattice to be periodic, so the sites m and $m + N_\mu e_\mu$ are identified where e_μ is the unit vector in the positive μ -direction (we ignore twisted boundary conditions). The basic variables are the link variables $A(m, n)$, defined for each ordered pair of nearest neighbors (m, n) and taking values in a Lie algebra $\mathfrak{g}$. So if $\{X_a\}$ is a basis of $\mathfrak{g}$ we have

$$A(m, n) = A^a(m, n) X_a$$

$$[X_a, X_b] = f^{abc} X_c$$

For $\mathfrak{g} = \text{su}(2)$, we choose $X_a = -i\sigma_a/2$ where σ_a are the Pauli matrices and $a = 1, 2, 3$, so f^{abc} is the antisymmetric symbol with $f^{123} = 1$. Changing the orientation of a link reverses the sign of a link variable, $A(m, n) = -A(n, m)$, so a non-redundant but asymmetric labelling is provided by $A_\mu(m) \equiv A(m, m + e_\mu)$. The link variables may be thought of as the line integral of the continuum vector potential or connection $A_\mu^{(c)}(x)$ along the link:

$$A_\mu(m) = \int_{am}^{a(m+e_\mu)} A^{(c)} \cdot dx = a \int_0^1 A_\mu^{(c)}(am + ae_\mu t) dt \quad (2.1a)$$

Alternatively, one may think of the link variables as related to $A_\mu^{(c)}$ by the formula

$$\exp A_\mu(m) = P \exp \int_{am}^{a(m+e_\mu)} A^{(c)} \cdot dx \quad (2.1b)$$

where a is the lattice spacing and P stands for path ordering. (2.1a) and (2.1b) are only equivalent up to terms of order $(aA^{(c)})^2$. Our model has a chance of being close to the continuum theory only if this difference is actually small, that is if $P \exp \int A \cdot dx$ is well approximated by $\exp a \int_0^1 A^{(c)}(am + ae_\mu t) dt$. This requires the components of $A_\mu(m)$ to be numerically small (in which case our model may actually be close to Wilson's). This will force us to introduce gauge fixing.

Before defining our model it is helpful to describe a convenient lattice formalism. To a large extent this formalism is well-known⁽⁷⁾ but we

will need some special features and also introduce a notation suited for our purposes. For this reason we give a complete description. Our lattice contains sites (0-cells), links (1-cells), plaquettes (2-cells), etc.; all of them can occur in two orientations (the "orientation" of a site is an arbitrary assignment of a sign).

As mentioned already, we sometimes find it useful to label the link $(m, m + e_\mu)$ in an asymmetric way by a site m and a positive direction μ . Similarly, a plaquette may also be labelled by a site m and an ordered pair (μ, ν) of positive directions; the plaquette can then be described as the oriented square enclosed by the closed loop starting at m and first moving one lattice unit in $+\mu$ direction, then 1 lattice unit in $+\nu$ direction, then back in the $-\mu$ and $-\nu$ direction.

We will consider certain (real valued) functions of the p-cells, called p-chains, which reverse their sign if we flip the orientation of the argument. On the space of these p-chains there is a natural ℓ^2 inner product: If v, w are two p-chains, define

$$(v, w) \equiv \sum_{c_p}' v(c_p) w(c_p) \quad (2.2)$$

where the sum $\sum_{c_p}'$ is over all positively oriented p-cells c_p . Thus, p-chains can be naturally identified with linear forms on p-chains (the so-called p-cochains). We may think of the p-chains as lattice analogs of differential p-forms and we often call them lattice p-forms.

Many operations known from differential geometry have their counterparts on the lattice as we will explain.

We begin by defining the lattice exterior derivative d as a linear operator from p-forms to $(p + 1)$ -forms. If v is a p-form

$$(dv)(c_{p+1}) = \sum_{c_p \in \partial c_{p+1}}' v(c_p) \quad (2.3)$$

where the sum is over the p-cells contained in the oriented boundary with the proper orientation. For a site variable v this means

$$\begin{aligned}
 (dv)(m, m+e_\mu) &\equiv v(m+e_\mu) - v(m) \\
 &\equiv (dv)_\mu(m) \equiv \Delta_\mu v(m)
 \end{aligned}
 \tag{2.4}$$

where Δ_μ is the forward lattice derivative. Sometimes we will also use the backward lattice derivative $\Delta_{-\mu}$ defined by

$$\Delta_{-\mu} v(m) \equiv v(m) - v(m-e_\mu) \tag{2.5}$$

For a link variable L we have in explicit notation

$$\begin{aligned}
 (dL)_{\mu\nu}(m) &= - (dL)_{\nu\mu}(m) \\
 &= \Delta_\mu L_\nu(m) - \Delta_\nu L_\mu(m) \\
 &= L_\mu(m) + L_\nu(m+e_\mu) - L_\mu(m+e_\nu) - L_\nu(m)
 \end{aligned}
 \tag{2.6}$$

as illustrated in Fig. 1.

The lattice divergence operator δ is defined as minus the adjoint of d : For any p -chain v and $(p+1)$ -chain w we require

$$(dv, w) = - (v, \delta w) \tag{2.7}$$

which determines δ as an operator from $(p+1)$ -chains to p -chains. Explicitly

$$(\delta w)(c_p) = \sum'_{\partial c_{p+1} \ni c_p} w(c_{p+1}) \tag{2.8}$$

where the sum is over all $(p+1)$ -chains containing c_p properly oriented in their boundary.

In explicit notation we have for a link variable L :

$$\delta L(m) = \sum_{\mu} \Delta_{\mu} L_{\mu}(m) \quad (2.9)$$

We note the well-known and easily proven fact

$$d^2 = 0 = \delta^2 \quad (2.10)$$

which leads to Bianchi identities. Next we want to define the analog of the wedge (exterior) product of p-forms. To keep the formulae transparent we do this explicitly only for "low" p; this is all that is needed in the following.

The exterior product of two site variables v and w is nothing but the ordinary pointwise product:

$$(vw)(m) \equiv v(m) w(m) \quad (2.11)$$

The exterior product of a site variable v with a p-form G we define as

$$(vG)(c_p) \equiv s_{c_p} v \cdot G(c_p) \quad (2.12)$$

where $s_{c_p} v$ is the average of all $v(m)$ over the (positively oriented) sites m occurring in the boundary of c_p .

In explicit notation we have for the product of a site variable v and a link variable L

$$(vL)_{\mu}(m) = L_{\mu}(m) s_{\mu} v(m) \quad (2.13)$$

where s_{μ} is the forward lattice average operator in the μ -direction

$$s_{\mu} v(m) \equiv \frac{1}{2} \left(v(m+e_{\mu}) + v(m) \right) \quad (2.14)$$

If G is a plaquette variable we have similarly

$$(\nabla G)_{\mu\nu}(m) = G_{\mu\nu}(m) s_\mu s_\nu v(m) \quad (2.15)$$

The exterior product of two link variables L, M is defined as the plaquette variable

$$(L \wedge M)(c_2) \equiv \frac{1}{4} \sum' L(c_1) M(c_1') \quad (2.16)$$

where the sum is over all pairs of parallel links c_1, c_1' that are contained in the boundary of c_2 with the proper orientation. It is not hard to imagine how (2.16) should be generalized to define the exterior product of a p -form with a q -form. But we will be content with giving the explicit version of (2.16):

$$\begin{aligned} (L \wedge M)_{\mu\nu}(m) &= -(M \wedge L)_{\mu\nu}(m) \\ &= -(L \wedge M)_{\nu\mu}(m) \\ &= s_\nu L_\mu(m) s_\mu M_\nu(m) - s_\mu L_\nu(m) s_\nu M_\mu(m) \end{aligned} \quad (2.17)$$

We next consider the analog of the Leibniz rule for the exterior product thus defined. Computation produces the formulae

$$\begin{aligned} (dvw)_\mu(m) &= s_\mu v(m) (dw)_\mu(m) \\ &\quad + (dv)_\mu(m) s_\mu w(m) \\ &= s_\mu v(m) \Delta_\mu w(m) + \Delta_\mu v(m) s_\mu w(m) \end{aligned} \quad (2.18)$$

for two site variables v, w which in view of (2.13) can be written as

$$(dvw)_\mu(m) = (v dw)_\mu(m) + (w dv)_\mu(m) \quad (2.19)$$

Similarly, for a link variable L and a site variable v , we obtain

$$\begin{aligned}
 (d(vL))_{\mu\nu}(m) &= (dL)_{\mu\nu} s_\mu s_\nu v(m) \\
 &- [s_\nu L_\mu(m) s_\mu (dv)_\nu(m) - s_\mu L_\nu(m) s_\nu (dv)_\mu(m)]
 \end{aligned}
 \tag{2.20}$$

which in view of (2.17) can be written as

$$\begin{aligned}
 (dLv)_{\mu\nu}(m) \\
 = (dL \wedge v)_{\mu\nu}(m) - (L \wedge dv)_{\mu\nu}(m)
 \end{aligned}
 \tag{2.21}$$

(2.19) and (2.21) look just like their continuum counterparts, and resemble the Leibniz rule.

We will finally define an interior product by the adjoint of the exterior product. For simplicity we only consider a pair of link variables L, M . The interior product $L \cdot M$ is determined by the requirement

$$(M, Lv) = (L \cdot M, v)
 \tag{2.22}$$

for any site variable v . Explicitly this means

$$\begin{aligned}
 (M \cdot L)(m) &= \sum_\mu s_{-\mu} [M_\mu(m) L_\mu(m)] \\
 &= \frac{1}{2} \sum_n M(m, n) L(m, n)
 \end{aligned}
 \tag{2.23}$$

where $s_{-\mu}$ is the backward averaging operator (in the $-\mu$ -direction).

The definitions (2.22) and (2.23) allow us to write a compact Leibniz rule for $\delta(vL)$ (v a site, L a link variable):

$$(\delta(Lv))(m) = (\delta L \cdot v)(m) + (L \cdot dv)(m)
 \tag{2.24}$$

which in explicit notation reads

$$\begin{aligned}
 & \sum_{\mu} \Delta_{-\mu} [L_{\mu}(m) s_{\mu\nu} v(m)] \\
 &= \sum_{\mu} \Delta_{-\mu} L_{\mu}(m) v(m) \\
 & \quad + \sum_{\mu} s_{-\mu} [L_{\mu}(m) \Delta_{\mu} v(m)]
 \end{aligned} \tag{2.25}$$

The above formalism allows us to define our model by writing lattice analogs of continuum quantities. The continuum Yang-Wills field is replaced by the plaquette variable⁽¹²⁾

$$F_{\mu\nu}^a(m) = (dA)_{\mu\nu}^a(m) + \frac{1}{2} f^{abc} (A^b \wedge A^c)_{\mu\nu}(m) \tag{2.26a}$$

$$\begin{aligned}
 F_{\mu\nu}^a(m) = & \Delta_{\mu} A_{\nu}^a(m) - \Delta_{\nu} A_{\mu}^a(m) \\
 & + f^{abc} s_{\nu} A_{\mu}^b(m) s_{\mu} A_{\nu}^c(m)
 \end{aligned} \tag{2.26b}$$

and for the Euclidean action we take

$$S = \frac{1}{4} \sum_{\substack{\mu, \nu \\ \mu, \nu}} [F_{\mu\nu}^a(m)]^2 \tag{2.27}$$

The relation to Wilson's action may be seen from

$$\begin{aligned}
 & \text{tr} [U_{\mu}(m) U_{\nu}(m+e_{\mu}) U_{\mu}^{\dagger}(m+e_{\nu}) U_{\nu}^{\dagger}(m)] \\
 &= \text{tr} \exp [F_{\mu\nu}(m) + O(A^3)] \\
 &= 1 - \frac{1}{2} c \sum_a (F_{\mu\nu}^a(m))^2 + O(A^3)
 \end{aligned} \tag{2.28}$$

where $U_{\mu} = \exp A_{\mu}$, $F_{\mu\nu}(m) = F_{\mu\nu}^a(m) X_a$, and $\text{tr} X_a X_b = -c \delta_{ab}$.

Our model will be completed by writing the lattice analog of continuum stochastic gauge fixing⁽⁵⁾. Let us briefly review this method which avoids the Gribov ambiguity and fermionic Faddeev-Popov ghosts, and which is a modification of the stochastic quantization method of Parisi and Wu⁽⁶⁾. These authors extended the analogy between the Euclidean distribution $P(A) =$

$\exp[-\beta S(A)]$, where $\beta = 1/g^2$ and the Boltzmann distribution in classical statistical mechanics by setting up a stochastic process described by a one-parameter family $P(A,t)$ of probability measures where the parameter t may be regarded as a kind of gauge-fixing parameter. It is required to be a solution of the diffusion equation

$$\partial P(A,t)/\partial t = \nabla \cdot [\nabla P(A,t) - \underline{K}(A)P(A,t)] \quad (2.29)$$

with drift force

$$\underline{K}(A) = -\beta \nabla S(A) ; \quad \beta = 1/g^2 \quad (2.30)$$

Here ∇ represents functional derivation with respect to $A_\mu^a(x)$. By analogy with finite dimensional diffusion equations, one might expect that any positive normalized solution $P(A,t)$ relaxes to the Boltzmann distribution, $\lim_{t \rightarrow \infty} P(A,t) = N \exp[-\beta S(A)]$, provided expectation values would be calculated from

$$\langle I \rangle = \lim_{t \rightarrow \infty} \langle I \rangle_t \quad (2.31a)$$

$$\langle I \rangle_t = \int dA I(A) P(A,t) \quad (2.31b)$$

In the continuum theory $\exp[-\beta S(A)]$ is not, in fact, normalizable because the probability diffuses freely into the infinite volume of the gauge orbits, so $\lim_{t \rightarrow \infty} P(A,t)$ does not exist. The point of the Parisi-Wu approach is that the limit (2.31) should nevertheless exist for gauge invariant quantities $I(A)$ which, by definition, are constant on gauge orbits. Stochastic gauge fixing rests on the further observation that one may modify the drift force $\underline{K} = -\beta \nabla S$ by the introduction of an additional "gauge-fixing" force

$$\underline{K} = -\beta \nabla S + \underline{K}_{gf}, \quad (2.32)$$

without changing expectation values $\langle I \rangle_t$ of gauge invariant quantities, provided that $\underline{K}_{gf}$ is tangential to gauge orbits⁽⁵⁾

$$(\tilde{K}_{gf})_{\mu}^a(x) = D_{\mu}^{ab}(A) v^b(x) \quad (2.33)$$

The point of the modification is that for suitable $v^b(x)$, the gauge-fixing force acts as a restoring force so that the equilibrium distribution

$$P_E(A) = \lim_{t \rightarrow \infty} P(A, t) \quad (2.34)$$

does exist⁽¹³⁾ and is determined by the time-independent diffusion equation

$$\tilde{\nabla} \cdot [\tilde{\nabla} P_E(A) - \tilde{K}(A) P_E(A)] = 0 \quad (2.35)$$

with

$$\langle I \rangle = \int dA I(A) P_E(A). \quad (2.36)$$

The artificial time parameter has been eliminated. The only Lorentz scalar and color vector $v^b(x)$ which gives a gauge-fixing force of the same dimension as $-\delta S / \delta A_{\mu}^a(x) = D_{\lambda}^{ab} F_{\lambda\mu}^b(x)$ is $v^b = \alpha \partial \cdot A^b$, where α is a dimensionless constant. This gives

$$(\tilde{K}_{gf})_{\mu}^a(x) = \alpha D_{\mu}^{ab}(A) \partial \cdot A^b(x) \quad (2.37a)$$

A slightly more general gauge-fixing force depending on a given fixed "background" potential $\bar{A}$, corresponding to the choice $v^b = \alpha D(\bar{A}) \cdot (A - \bar{A})$ is given by

$$(\tilde{K}_{gf})_{\mu}^a(x) = \alpha D_{\mu}^{ab}(A) D(\bar{A}) \cdot (A - \bar{A}) \quad (2.37b)$$

As $\alpha \rightarrow \infty$ these give, respectively, the Landau gauge $\partial \cdot A = 0$ or, more generally, the background gauge $D(\bar{A}) \cdot (A - \bar{A}) = 0$.

Our model is defined by the lattice analog of the diffusion equations (2.29) or (2.35)

$$\frac{\partial P(A,t)}{\partial t} = \sum_{m,\mu,a} \frac{\partial}{\partial A_\mu^a(m)} \left[\frac{\partial P(A,t)}{\partial A_\mu^a(m)} - K_\mu^a(m) P(A,t) \right] \quad (2.38a)$$

$$0 = \sum_{m,\mu,a} \frac{\partial}{\partial A_\mu^a(m)} \left[\frac{\partial P_E(A)}{\partial A_\mu^a(m)} - K_\mu^a(m) P_E(A) \right] \quad (2.38b)$$

with drift force

$$K_\mu^a(m) = (K_{cl})_\mu^a(m) + (K_{gf})_\mu^a(m) \quad (2.39)$$

Here

$$(K_{cl})_\mu^a(m) = -\beta \frac{\partial S(A)}{\partial A_\mu^a(m)} \quad (2.40)$$

where $\beta = 1/g^2$, and

$$-\frac{\partial S}{\partial A_\mu^a(m)} = \sum_\lambda \left[\Delta_{-\lambda} F_{\lambda\mu}^a(m) + f^{abc} s_{-\lambda} (A_\mu^b(m) F_{\lambda\mu}^c(m)) \right] \quad (2.41)$$

with $F_{\mu\nu}^a(m)$ defined in Eq.(2.26). For the gauge-fixing force we take

$$(K_{gf})_\mu^a(m) = \alpha [D(A)v]_\mu^a(m) \quad (2.42a)$$

where $[D(A)v]_\mu^a(m)$ is the lattice analog of the continuum covariant derivative

$$\begin{aligned} [D(A)v]_\mu^a(m) &= (dv)_\mu^a(m) + f^{abc} (A_\mu^b v^c)_\mu(m) \\ &= \Delta_\mu v^a(m) + f^{abc} A_\mu^b(m) s_\mu v^c(m) \end{aligned} \quad (2.42b)$$

where

$$v^a(m) = \delta A^a(m) = \sum_\mu \Delta_{-\mu} A_\mu^a(m) \quad (2.42c)$$

or, more generally,

$$v^a(m) = \sum_{\mu} \Delta_{-\mu} [A_{\mu}^a(m) - \bar{A}_{\mu}^a(m)] + f^{abc} s_{-\mu} [\bar{A}^b(m) A^c(m)] \quad (2.42d)$$

which are the lattice analogs of Eq.(2.37).

2B. Properties of the Gauge-Fixing Force

1. We remind the reader that the diffusion Eq.(2.38a) is of the form

$$\dot{P} = - \nabla \cdot \underline{J} \quad (2.43)$$

where the current $\underline{J}$ is given by

$$\underline{J} = - \hbar \nabla P + \underline{K} P \quad (2.44)$$

and we have restored Planck's constant which plays the rôle of a diffusion constant. The second term is an analog of Ohm's law and represents the current produced in a resistive medium by an external force $\underline{K}$. The first term describes the flow from higher to lower concentration produced by the random Brownian or thermal fluctuations that are the analog of the quantum fluctuations. The drift velocity is given by

$$\underline{v} = \underline{J}/P = - \hbar \nabla \ln P + \underline{K} \quad (2.45)$$

Its curl, $\partial_i v_j - \partial_j v_i = \partial_i K_j - \partial_j K_i$ is non-vanishing because the gauge-fixing force, Eq.(2.42) is non-conservative. This implies that $\underline{v}$ and hence also $\underline{J}$ are non-vanishing, even at equilibrium, whereas if the force is conservative, $\underline{K} = -\beta \nabla S$, $P_E = N \exp[-\beta S/\hbar]$ and the current vanishes at equilibrium, $\underline{J}_E = -\hbar \nabla P_E + \underline{K} P_E = 0$. It is precisely the introduction of a non-conservative force in the diffusion equation - that is not possible in an action formalism - which eliminates the Gribov ambiguity and fermionic Faddeev-Popov ghosts. Moreover it is a geometric feature of gauge theories that no conservative force can be everywhere tangent to the gauge orbits⁽¹⁴⁾ because the Frobenius integrability condi-

tion does not hold^{(15), (16)}.

2. Theorem 1: Under the motion

$$\dot{A} = K_{gf}(A) \quad (2.46)$$

with K_{gf} given in Eqs.(2.37a-c), we have

$$\frac{d}{dt} |A|^2 = -2\alpha |v|^2 \leq 0 \quad (2.47)$$

where $v^a(m) = \delta A^a(m) = \sum_{\mu} \Delta_{\mu} A_{\mu}^a(m)$. This means that K_{gf} is a restoring force everywhere except on the hyperplane $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$ where it vanishes.

Note: Using Eq.(2.42d), one may show more generally, $\frac{d}{dt} |A - \bar{A}|^2 = -2\alpha |v|^2$

Proof: From Eq.(2.46) we have

$$\begin{aligned} \frac{d}{dt} |A|^2 &= 2 \sum_{m, \mu, a} A_{\mu}^a(m) (K_{gf})_{\mu}^a(m) \\ &= 2\alpha \sum_{m, \mu, a} A_{\mu}^a(m) \left[\Delta_{\mu} v^a(m) + f^{abc} A_{\mu}^b(m) s_{\mu} v^c(m) \right] \\ &= -2\alpha \sum_{m, \mu, a} \left[\Delta_{\mu} A_{\mu}^a(m) \right] v^a(m) \\ &= -2\alpha \sum_{m, a} \left[\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) \right]^2. \end{aligned}$$

3. The above theorem tells us that in A-space the hyperplane

$\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$, is the locus of equilibrium points of the gauge fixing force. The stability or instability of the equilibrium is described in the next theorem.

Theorem 2: Let the constants f^{abc} in the gauge-fixing force K_{gf} , Eqs.(2.42a-c), be the structure constants of a semi-simple Lie group. Then under the action of this force, a convex, bounded region including the origin of the hyperplane $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m)=0$, is an attractor for all points near it, but for points in the hyperplane outside this region the equilibrium is unstable. An obvious generalization of this theorem holds for the more general force, Eq.(2.42d).

Note: This theorem represents a sharpening of similar statements which were established for the continuum theory, in which the boundary of the attractive region is formed by the Gribov horizon, and the region itself intersects every gauge orbit and consists of all points where $\partial A^a(x)=0$ and the Faddeev-Popov operator $-\partial \cdot D(A)$ has no negative eigenvalues.

Proof: Consider the evolution of $v^a(m) = \sum_{\mu} \Delta_{\mu} A_{\mu}^a(m)$ under the motion $\dot{A} = K_{gf}(A)$. From Eq.(2.42) one has

$$\dot{v}^a(m) = \sum_{\mu} \Delta_{\mu} [\Delta_{\mu} v^a(m) + f^{abc} A_{\mu}^b(m) s_{\mu} v^c(m)] \quad (2.48)$$

To see whether a given point $A_{\mu}^{(0)b}(m)$, satisfying $\sum_{\mu} \Delta_{\mu} A_{\mu}^{(0)b}(m)=0$ belongs to an attracting set under this evolution it is sufficient to take the approximation linear in v , with $A^{(0)}$ time independent⁽¹⁷⁾,

$$\dot{v}^a(m) = \sum_{\mu} \Delta_{\mu} [\Delta_{\mu} v^a(m) + f^{abc} A_{\mu}^{(0)b}(m) s_{\mu} v^c(m)] \quad (2.49)$$

The hyperplane $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m)=0$ will be an attractor at $A^{(0)}$ if the eigenvalues of the corresponding eigenvalue equation.

$$\lambda v^a(m) = \sum_{\mu} \Delta_{\mu} [\Delta_{\mu} v^a(m) + f^{abc} A_{\mu}^{(0)b}(m) s_{\mu} v^c(m)] \quad (2.50)$$

all have negative real parts, and the equilibrium will be unstable at $A^{(0)}$ if any λ has a positive real part, for v of the form $\Delta_{\mu} A_{\mu}^a(m)$. Call h the matrix of the operator on the right-hand side. On a finite lattice it is finite dimensional. We shall show that it is symmetric. For this purpose the condensed notation of Sec. 2A is convenient. We have

$$(hv)^a = \delta_d v^a + f^{abc} \delta(A^{(0)b} v^c) \quad (2.51)$$

By Eq.(2.24b), with $\delta A^{(0)}(m)^b = \sum_{\mu} \Delta_{\mu} A^{(0)b}(m) = 0$ this gives

$$(hv)^a = \delta d v^a + f^{abc} A^{(0)b} \cdot dv^c \quad (2.52)$$

To see that h is symmetric use

$$(w^a, (hv)^a) = - (dw^a, dv^a) - f^{abc} (A^{(0)b} w^c, dv^a)$$

by Eqs.(2.7) and (2.22), and so

$$(w^a, hv^a) = (\delta dw^a, v^a) + f^{abc} (\delta (A^{(0)b} w^c), v^a)$$

which gives $(w^a, hv^a) = (hw^a, v^a)$, as asserted. Thus, all the eigenvalues λ are real.

From Eq.(2.52) we see that the constant color vectors $v^a(m) = v^a$, independent of m, form a null space L_0 of h (recall that $(dv)^a_{\mu}(m) = \Delta_{\mu} v^a(m)$). However, because $v^a(m) = \sum_{\mu} \Delta_{\mu} A^a_{\mu}(m)$, we have $\sum_m v^a(m) = 0$, so the $v^a(m)$ of interest fall into the space $L_{\perp}$ which is orthogonal to subspace L_0 of constant color vectors. Henceforth we consider h as an operator on L . To indicate the dependence of h on A^0 we write $h(A^0)$. The lattice Laplacian δd is negative definite on $L_{\perp}$, so $h(0) = \delta d$ is negative definite and we conclude that the hyperplane $\sum_{\mu} \Delta_{\mu} A^a_{\mu}(m) = 0$ is an attractor for all points in a neighborhood of the point $A^{(0)} = 0$.

Call Ω the region of the hyperplane $\sum_{\mu} \Delta_{\mu} A^a_{\mu}(m) = 0$ where h(A) is negative definite or, in other words, where all eigenvalues $\lambda_i(A)$ are negative, $\lambda_i(A) < 0$. Because the eigenvalues are continuous in A, this is an open set of the hyperplane. From what has been said, Ω is an attractor for all points in its neighborhood that are off the plane. We shall show that Ω is convex, namely, that if A_1 and A_2 are in Ω , then so also is $\alpha A_1 + (1-\alpha) A_2$ for all α in the unit interval $0 < \alpha < 1$. Because h(A) is linear in A, we have, for all v, $(v, h(\alpha A_1 + (1-\alpha) A_2) v) = \alpha (v, h(A_1) v) + (1-\alpha) (v, h(A_2) v)$. The right-hand side is, by assumption, negative for all $v \neq 0$ in $L_{\perp}$ and so, consequently, is the left-hand side. Thus Ω is convex.

We next show that Ω is bounded by showing that for any $A \neq 0$, satisfy-

ing $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$, $h(vA)$ has a positive eigenvalue for sufficiently large positive v . This means that starting from the origin, which is a point of the convex, open set Ω , and travelling along any ray vA , one eventually leaves Ω . From Eqs.(2.7) and (2.52) we have

$$(v, h(vA)v) = - (dv, dv) + v f^{abc} (A^{(0)b} v^a, dv^c)$$

or, more explicitly,

$$(v, h(vA)v) = - \sum (\Delta_{\mu} v, \Delta_{\mu} v) + v \sum_{m, \mu} v^a(m) f^{abc} A_{\mu}^{(0)b}(m) v^c(m + e_{\mu}) \quad (2.53)$$

where we have used the anti-symmetry of f^{abc} . The first term is negative definite on $L_{\perp}$, but the second term has no particular sign and it is sufficient to exhibit some $v^a(m)$ which makes the second term positive. Then

$(v, h(vA)v)$ will be positive for sufficiently large positive v . Let $A_{\mu}^b(m)$ be a non-vanishing color vector for some m_0 and μ_0 and choose $v^a(m) = 0$ for all m except $m = m_0$ and $m = m_0 + e_{\mu_0}$, so the second term in Eq.(2.53) is simply

$$v v^a(m_0) f^{abc} A_{\mu_0}^b(m_0) v^c(m_0 + e_{\mu_0}) \quad (2.54)$$

(without any summation on m or μ).

Choose $v^c(m_0 + e_{\mu_0})$ so the color vector $u^a = f^{abc} A_{\mu_0}^b(m_0) v^c(m_0 + e_{\mu_0})$ is non-zero, which can always be done if the f^{abc} are the structure constants of a semi-simple Lie group. Choose $v^a(m_0) = u^a$ and the second term becomes

$$v \sum_a (u^a)^2$$

For sufficiently large v $h(vA)$ is positive. Hence it has a positive eigenvector which lies in $L_{\perp}$ because L_0 is a null space. Thus for any A the point vA lies outside Ω , and Ω is bounded.

We next show that the equilibrium is unstable outside of Ω . Let A be a point on the boundary of the open bounded set Ω . Then $h(A)$ has at least one eigenvalue $\lambda_0(A)$ that is not negative corresponding to an eigenvector v_0 in $L_{\perp}$, for otherwise A would be an interior point of Ω . Moreover, this eigenvalue $\lambda_0(A)$ must be zero because the eigenvalues $\lambda_i(A)$

are continuous functions of A . Because Ω is a bounded convex open set containing the origin, every point outside it, in the hyperplane $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$, is of the form νA where A is a boundary point of Ω and $\nu > 1$.

Let v_0 be the null eigenvector in $L_{\perp}$ of $h(A)$, $h(A)v_0 = 0$. We have

$$\begin{aligned} (v_0, h(\nu A)v_0) &= \nu (v_0, h(A)v_0) + (1-\nu)(v_0, \delta d v_0) \\ &= (\nu-1)(dv_0, dv_0) \end{aligned}$$

which is positive for $\nu > 1$ and v_0 in $L_{\perp}$. Hence $h(\nu A)$ has a positive eigenvalue and the equilibrium is unstable for a generic point in the hyperplane $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$ exterior to the bounded convex region Ω . QED.

4. As a corollary we obtain a bound on the size of the region Ω . For the choice of $v^a(m)$ used in Eq.(2.54) we have

$$\begin{aligned} (v, h(A)v) &= -8(\vec{v}_1^2 + \vec{v}_2^2) + f^{abc} v_1^a A_{\mu_0}^b(m_0) v_2^c \\ \vec{v}_1 &= (v_1^a) = v^a(m_0) \\ \vec{v}_2 &= v^a(m_0 + e_{\mu_0}) \end{aligned}$$

and we have made the further choice $\vec{v}_1 \cdot \vec{v}_2 = 0$. The factor 8 comes from the number of links entering a site in 4 dimensions. For $A_{\mu}^b(m)$ to be in Ω this expression must be negative for all $\vec{v}_1$ and $\vec{v}_2$ with $\vec{v}_1 \cdot \vec{v}_2 = 0$. In the group $SU(2)$ this gives the condition

$$-8(\vec{v}_1^2 + \vec{v}_2^2) + (\vec{v}_2 \times \vec{v}_1) \cdot \vec{A} < 0$$

where $\vec{A} = \vec{A}_{\mu_0}(m_0)$. If we choose $|\vec{v}_1| = |\vec{v}_2| = 1$ and $\vec{v}_1$ and $\vec{v}_2$ orthogonal to $\vec{A}$ (and each other) we obtain $|\vec{A}| < 16$. Thus for all configurations A satisfying $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$ that lie in Ω each link variable $A_{\mu}(m)$ is less than 16 in magnitude

$$|\vec{A}_{\mu}(m)| < 16 \quad (2.55)$$

This defines a polyphere of radius 16.

Moreover, for slowly varying $A_\mu^a(m)$ the boundary is quite close to the origin. If $A_\mu^a(m)$ is a constant independent of m , $A_\mu^a(m) = A_\mu^a$ then h is diagonalized by choosing momentum eigenstates:

$$(hv)^a(k) = \sum_\mu \left[-4 \sin^2(k_\mu/2) v^a(k) + 2 \sin(k_\mu/2) \cos(k_\mu/2) i f^{abc} A_\mu^b v^c(k) \right]$$

where $k_\mu = 2\pi n_\mu/N$, $n_\mu = 1, 2, \dots, N$ and N is the number of lattice sites on an edge of the lattice. The first positive eigenvalue develops when A_μ^b is of order $2n/N$. For example, let $\vec{A} = (A_1^b)$ be a non-zero color vector and let $A_2^b = A_3^b = A_0^b = 0$. Let $n_1 = 1$ and let $n_2 = n_3 = n_0 = 0$ so

$$(hv)^a = \left[-4 \sin^2(\pi/N) v^a + 2 \sin(\pi/N) \cos(\pi/N) i f^{abc} A^b v^c \right]$$

If the group is $SU(2)$ the eigenvalues are $-4 \sin^2(\pi/N)$ and $-4 \sin^2(\pi/N) \pm 2 \sin(\pi/N) \cos(\pi/N) |\vec{A}|$. One of them is non-negative unless $|\vec{A}| < 2 \tan(\pi/N)$. Thus, for $A_1^b(m) = A_1^b$ independent of m and $A_2^b(m) = A_3^b(m) = A_0^b(m) = 0$, the boundary of Ω is at $|\vec{A}^b(m)| = 2 \tan(\pi/N)$. As $N \rightarrow \infty$ the boundary approaches $A = 0$, as was found for the continuum theory⁽¹⁰⁾. The non-zero constant potentials have zero probability in the Landau gauge.

5. The above theorems lead to the following picture of the motion of a Brownian particle in A -space whose motion, when averaged over long times, gives the equilibrium probability distribution $P_E(A)$. In a first orientation we suppose that the coefficient of the gauge-fixing force is large and that β , the coefficient of the classical force $-\beta \nabla S$ is zero. Let the Brownian particle start in the convex, bounded region Ω of the hyperplane $\sum_\mu \Delta_\mu A_\mu^a(m) = 0$. It cannot move far in a direction perpendicular to this hyperplane because Ω is an attractor. However, it can diffuse freely in Ω , and out across its boundary because the equilibrium is neutral with respect to motion in the hyperplane. Once outside, by thermal fluctuation, it develops a component in the unstable mode. Whereupon it is forcibly driven further off the hyperplane by the strong gauge-fixing force K_{gf} , and consequently arrives at a region of A -space where the diverg-

ence $V^a(m) = \sum_{\mu} \Delta_{\mu} A_{\mu}^a(m)$ is considerable. At this point the general restoring force found in theorem 1 drives it back to Ω . This type of cycle is illustrated in Fig. 2 and corresponds to the non-zero equilibrium current J_E found in point 1 above. Thus, by a kind of magnetic confinement, the Brownian particle in A-space is prevented by the unstable modes outside of Ω from freely diffusing to infinity along the hyperplane $\sum \Delta_{\mu} A_{\mu}^a(m) = 0$, and remains confined to a neighborhood of the compact region $\bar{\Omega}$.

We conclude that, in marked contrast to the Abelian case, the gauge-fixing force of the non-Abelian theory supports a normalized equilibrium probability distribution $P_E(A)$ concentrated around the compact region $\bar{\Omega}$, even in the absence of the classical force $-\beta \nabla S$. Moreover, for β finite in the Landau gauge, which is the limit $\alpha \rightarrow \infty$, the above description leads us to believe that the probability becomes strictly confined to the region Ω . Finally, we observe that, with α fixed, a strong coupling expansion in $\beta = 1/g^2$ should exist in principle (though it might be hard to evaluate analytically). In this respect the lattice version of the non-compact theory resembles Wilson's compact theory and may be said to be self-compactifying.

3. MEASUREMENTS

In this section we present measurements of various physical quantities, mostly taken at $\alpha = \beta$. For this choice the gauge fixing force and the "classical" force $\frac{\partial S}{\partial A_{\mu}^a(m)}$ are of the same order of magnitude. This means that the optimal step size η which should be at the same time small enough not to introduce a large systematic error and large enough to obtain a reasonable speed of convergence can be chosen in a consistent way; for $\alpha \gg \beta$ or $\alpha \ll \beta$ the two forces demand different step sizes and this leads to much slower convergence if we do not want large systematic errors. Of course, if the model really were a gauge theory, expectation values of gauge invariant quantities would not depend on α . It turns out that at not too small values of β ($\beta \gtrsim .1$) the variation of physical quantities with α is reasonably small in the region $\alpha \sim \beta$.

Nevertheless there is some interest in studying the more extreme regions $\alpha \gg \beta$, $\beta \gg \alpha$. In fact, in (18) the model with $\alpha = 0$ is studied (remarkably, it does not show any instability) and in (19) we study the case $\beta = 0$ which

essentially amounts to a study of the gauge-fixing force, the Gribov horizon and the like; it is even more noteworthy that the system remains stable for this case ($\beta = 0$), an effect that may be called self-compactification (see Refs.(9) and (10)).

To obtain useful data we have both to minimize the statistical errors and to get rid of the step size dependence; the general procedure was described and its success illustrated in Ref.(11). The quality of the extrapolation to zero step size η depends of course on the quality of the data at small η . For some sensitive (and typically small) quantities like correlations over three or more sites, the step-size dependence is completely blurred by the statistical errors.

As already mentioned in (11), in order to improve the speed and thus the statistics, we introduced a parallel updating of all three colors of a link while using the same (old) configuration to calculate the force in Eq. (2.39). Comparing data obtained from parallel and sequential updating shows a difference in the step size dependence which, as a rule, is steeper in the sequential case. The extrapolation to $\eta = 0$ should not, however, feel the difference. This is indeed seen for quantities like Creutz's force $\chi(I, J)$, Polyakov loop and correlations where the parallel and sequential extrapolations agree within their errors. A discrepancy exists only for the action and the plaquette average, probably indicating the limits of our extrapolation accuracy. In Fig. 3 we present some data with good statistics where such fine effects can be seen. In the remainder of the paper the error bars try to account for both the statistical fluctuations and the systematic uncertainties. We mostly combine sequential and parallel data, except for $\beta \geq .5$ where there are practically no uncertainties introduced by the difference and therefore we only use parallel runs.

Fig. 4a shows the results of the Polyakov loop for the "quark", i.e. isospin 1/2 representation

$$\langle P_{1/2} \rangle \equiv \frac{1}{2} \left\langle \text{Tr} \prod_{m_0=1}^{L_0} e^{i A^a(m+e_0) \tau^a / 2} \right\rangle$$

Note that for $\alpha = \beta$ the data stay very close to the "Abelian" result

$$\langle P_{1/2} \rangle = \exp \left[- \frac{3g^2}{8} L_0 V_C(0) \right]$$

where $V_C(x-y) = (-\Delta^{-1})(x-y)$ is the lattice Coulomb potential, $(-\Delta^{-1})(0) = V_C(0) = 0.2527$ (see next section). For $\beta \geq .5$ $\langle P_{1/2} \rangle$ does not seem to depend on α . Below $\beta \simeq .5$, for $\alpha < \beta$ $\langle P_{1/2} \rangle$ becomes larger, for $\alpha > \beta$ it becomes smaller. This might seem at first surprising since increasing α seems to reduce fluctuations. In fact, it produces more disorder through the "Gribov cycles" described and discussed in (19). In any case, for $\alpha \leq \beta$ clearly $\langle P_{1/2} \rangle$ does not show any sudden drop to zero as β is reduced, the way it does in Wilson's model⁽²⁰⁻²¹⁾. This shows that the non-compact model we are studying does not confine quarks even at rather strong coupling (as strong as we can measure reliably, and for the range of α considered⁽²²⁾). Fig. 4b finally presents our measurements of $\langle P_1 \rangle$ on the 4^4 lattice. We include these results (that measure some "gluon self-energy") mainly in order to check agreement with some theoretical calculations for $\beta = 0$ given in the next section. Notice the departure of $\langle P_1 \rangle$ from the weak coupling prediction for $g^2 \gtrsim 6$ and the approach to the strong coupling prediction. This indicates that beyond $g^2 \simeq 6$ we see the strong coupling régime of this model.

Fig. 5 shows our data for Creutz's parameter

$$\chi(I, J) \equiv - \ln \frac{\langle W(I, J) \rangle \langle W(I-1, J-1) \rangle}{\langle W(I-1, J) \rangle \langle W(I, J-1) \rangle}$$

measured at $\alpha = \beta$. Note that for square loops χ always comes out lower than for non-square ($I \neq J$) ones. This is typical for Coulomb-like forces; it would not be so were there a strict area law.

Fig. 6 shows the correlation functions $\langle P_{1/2}(x) P_{1/2}(y) \rangle$ for a pair of Polyakov loops, measured at distances 0, 1, 2, 3, 4 on a 8^4 lattice for $\alpha = \beta$. Note that the values obtained for $x-y = 4$ are consistent with the square of $\langle P_{1/2} \rangle$ and are also different from zero wherever they are measurable reliably.

Fig. 7 shows some results concerning asymptotic freedom. We have measured

$$F(1/g^2) \equiv \ln \frac{\langle W(4,4) \rangle \langle W(2,2) \rangle}{\langle W(4,2) \rangle^2}$$

on an 8^4 lattice (empty circles) and

$$G(1/g^2) \equiv \ln \frac{\langle W(2,2) \rangle \langle W(1,1) \rangle}{\langle W(2,1) \rangle^2}$$

on a 4^4 lattice (small dots). According to asymptotic freedom (using the 1-loop continuum β function) we should have

$$F(1/g^2) \simeq G(1/g^2 - .064)$$

The right-hand side is given by full circles and the agreement is reasonable even though the errors are too large to make this a really convincing test.

4. INTERPRETATION AND ANALYSIS

Our results show first of all that the random walk method works. A more difficult question is the physical interpretation of the results. The outcome is rather different from Wilson's model; qualitatively it resembles rather the non-compact model in the temporal gauge studied in Ref. (3). Even at very strong coupling we do not obtain any clear indication of confinement.

For our non-compact model a strong coupling expansion is not easy to produce but at least we can compute the limit $\beta \rightarrow 0$ ($g^2 \rightarrow \infty$) and compare analytical and numerical results. It turns out that even in this limit there is a big difference between Wilson's model and ours.

Let us compute the expectation values of arbitrary Polyakov loops P_j , $j = 0, \frac{1}{2}, 1, \dots$ for $\beta \rightarrow 0$. In this limit all links become independent (just as in Wilson's model), but instead of averaging each link variable over the group $G = SU(2)$ with Haar measure, we have to average over the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ with Lebesgue measure. Let $U_j(\exp i \frac{\vec{\tau} \cdot \vec{A}}{2})$ be the unitary matrix representing $\exp(i \frac{\vec{\tau} \cdot \vec{A}}{2})$ in the representation with isospin j . We have to compute

$$B_j \equiv \lim_{R \rightarrow \infty} \left(\frac{4\pi}{3} R^3 \right)^{-1} \int_{|\vec{A}| \leq R} U_j(e^{i\vec{\tau} \cdot \vec{A}/2}) d^3A$$

This is a multiple of the unit matrix because it does not change if we conjugate it by $V_j(g)$, $g \in \text{SU}(2)$ since this induces a rotation of the A 's and the integration measure is rotationally invariant; furthermore the representation $U_j(\cdot)$ is irreducible. So

$$B_j = \frac{1}{2j+1} \mathbb{1} \text{Tr } B_j$$

$\text{Tr} B_j$ is computed easily:

$$\begin{aligned} \text{Tr } B_j &= \lim_{R \rightarrow \infty} \left(\frac{4\pi}{3} R^3 \right)^{-1} \int_{|\vec{A}| \leq R} d^3A \frac{\sin(j+\frac{1}{2})|\vec{A}|}{\sin \frac{1}{2}|\vec{A}|} \\ &= \lim_{R \rightarrow \infty} \left(\frac{1}{3} R^3 \right)^{-1} \int_0^R dt t^2 \frac{\sin(j+\frac{1}{2})t}{\sin \frac{1}{2}t} \\ &= \lim_{R \rightarrow \infty} \left(\frac{1}{3} R^3 \right)^{-1} \int_0^R dt t^2 \sum_{m=-j}^j e^{imt} \end{aligned}$$

Only the term with $m = 0$ survives (if present) and gives

$$\text{Tr} B_j = 1$$

provided j is an integer, otherwise

$$\text{Tr} B_j = 0.$$

So we obtain for

$$\langle P_j \rangle \equiv \frac{1}{2j+1} \text{Tr} \left\langle \prod_{m_0=1}^{L_0} U_j \right\rangle$$

at $\beta = 0$:

$$\langle P_j \rangle \Big|_{\beta=0} = \left[\frac{1}{2j+1} \text{Tr } B_j \right]^{L_0} = (2j+1)^{-L_0}$$

for integer j and

$$\langle P_j \rangle \Big|_{\beta=0} = 0$$

for half-integer j .

In particular

$$\langle P_1 \rangle \Big|_{\beta=0} = 3^{-L_0}$$

and

$$\begin{aligned} \langle P_{1/2}^2 \rangle \Big|_{\beta=0} &= \frac{1}{4} (1 + 3 \langle P_1 \rangle \Big|_{\beta=0}) \\ &= \frac{1}{4} + \frac{3}{4} 3^{-L_0} \end{aligned}$$

On a 4^4 lattice this is

$$\begin{aligned} \langle P_1 \rangle \Big|_{\beta=0} &\simeq .0123457 \\ \langle P_{1/2}^2 \rangle \Big|_{\beta=0} &\simeq .25926 \end{aligned}$$

which agrees reasonably well with our measurements for β around 0.1 to 0.15 (see Fig. 4b) where they depart clearly from the weak coupling extrapolation

$$\langle P_1 \rangle \simeq \exp[-.38g^2 L_0] ; \quad \langle P_{1/2}^2 \rangle \Big|_{\beta \simeq .15} \simeq .25004$$

It is also instructive to look at the perturbative regime $\beta \rightarrow \infty$, $g^2 \rightarrow 0$.

In Gaussian approximation we obtain for a single Polyakov loop

$$\langle P_{1/2} \rangle \simeq \frac{\frac{1}{2} \text{Tr} \int \exp \left[-\frac{1}{2g^2} \sum_{p,a} (F_p^a)^2 \right] \prod_{m_0=0}^{L_0} e^{i \vec{A}_0 \cdot \vec{\tau}/2}}{\int \exp \left[-\frac{1}{2g^2} \sum_{p,a} (F_p^a)^2 \right]}$$

where F_p^a is computed without the commutator term and the gauge-fixing is not displayed explicitly. For $g^2 \rightarrow 0$ only very small A_0^a will contribute, and we may ignore the commutators between the τ^a (cf. Ref.(23) for a detailed discussion of such an integral). The result is

$$\langle P_{1/2} \rangle \approx \exp \left[-\frac{3g^2}{8} L_0 V_C(0) \right]$$

where $V_C(x-y) = (-\Delta^{-1})(x-y)$ is the lattice Coulomb potential in 3 dimensions, $V_C(0) = 0.2527$ ⁽²⁴⁾. The data show that this is a very good approximation for $\alpha \approx \beta$ down to β as small as 0.25. As α increases from $\alpha \approx \beta$ to $\alpha > \beta$ the self-energy of a quark (which is measured by $-\frac{1}{L_0} \log \langle P_{1/2} \rangle$) increases, resulting in lower values of $\langle P_{1/2} \rangle$ whereas for $\alpha < \beta$ $\langle P_{1/2} \rangle$ comes out larger.

So we obtain good agreement with these simple analytic calculations. This, of course, does not settle the central question: Does our model have a continuum limit that is a gauge theory? The lattice model, even though formally derived from a gauge theory, is not a gauge theory in any obvious sense. A direct reflection of the violation of the gauge structure is the observed dependence of the results on the strength α of the "gauge-fixing force". The crucial question is whether this dependence will go away in the continuum limit which involves both sending β to infinity and rescaling observables (Wilson loops, etc.) appropriately. Formally all gauge violating terms in the classical Lagrangian are $O(a)$ just like the terms breaking Euclidean invariance and the extra higher vertices present in the perturbation theory of compact models ^(25,26). The question is whether the non-gauge-invariant operators multiplying this $O(a)$ are soft enough not to give a contribution in the limit $a \rightarrow 0$. ⁽²⁷⁾

If this is not the case, explicitly non-gauge-invariant counterterms (like $m^2 A_\mu^2$) will have to be added to the classical Lagrangian. A hopeful sign is that asymptotic freedom does not seem to be violated. It should also be noted that addition of a mass term $m^2 A_\mu^2$ with a negative sign would make the system unstable (just as it was the case in Ref.(3)); the fields A_μ would drift to infinity along Abelian directions. So the fact that the system acts more or less as if a dynamical Higgs mechanism was operating cannot be blamed on the violation of gauge invariance.

Certainly it would be desirable to have more analytical insight into the question of restoration of gauge invariance in the continuum limit. For

this purpose, a perturbative computation of the β -function of the present lattice model would be valuable.

But so far the fact remains that non-compact lattice QCD does not seem to show confinement (even though in this case it is in some sense self-compactifying). We cannot answer the question whether this is an indication that non-compact lattice QCD defines a different, non-confining, though gauge invariant continuum field theory. So far we cannot exclude the possibility that in the (properly scaled) continuum limit confinement would be restored. It is conceivable that the potential first rises linearly and then levels off at a value that tends to ∞ as the lattice spacing a goes to zero.

Which of these two possibilities is realized can only be determined by much more detailed investigations which at present are not within the reach of our numerical or analytical capabilities.

REFERENCES

- (1) F.J. Wegner: J. Math. Phys. 12, 2259 (1971);
K. Wilson: Phys. Rev. D10, 2445 (1974).
- (2) M. Creutz: Phys. Rev. D21, 2308 (1980).
- (3) A. Patrascioiu, E. Seiler, I.O. Stamatescu: Phys. Letts. 107B, 364 (1981).
- (4) B. Baaquie: Phys. Rev. D16, 2612 (1977).
- (5) D. Zwanziger: Nucl. Phys. 192, 259 (1981).
- (6) G. Parisi and Y.-S. Wu: Scientia Sinica 24, 483 (1981).
- (7) K. Drühl and H. Wagner: Ann. Phys. (N.Y.) 141, 225 (1982);
A. Guth: Phys. Rev. D21, 2291 (1980);
J. Fröhlich and I. Spencer: Comm. Math. Phys. 83, 41 (1982);
E. Seiler: Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics (Lecture Notes in Physics 159), Springer Verlag, Berlin 1982.
- (8) V. Gribov: Nucl. Phys. B139, 1 (1978).
- (9) D. Zwanziger: Phys. Letts. 114B, 337 (1982).
- (10) D. Zwanziger: Nucl. Phys. B209, 336 (1982).
- (11) I.O. Stamatescu, U. Wolff and D. Zwanziger: Simulation of Euclidean Quantum Field Theories by the Random Walk Process, Max Planck-Inst. preprint MPI-PAE/PTH 14/83.
- (12) It is amusing to identify precisely why this lattice theory fails to be a gauge theory. If one poses as the infinitesimal gauge transformation $\delta A_\mu^a = (d\Lambda^a)_\mu + f^{abc}(A^b \wedge \Lambda^c)_\mu$ then F fails to transform covariantly. The reason is that the exterior products (2.13), (2.15) and (2.17) are not associative:

$$[M \wedge (N \vee)]_{\mu\nu} - [(M \wedge N) \vee]_{\mu\nu} = \frac{1}{4} s_\nu M_\mu \Delta_\mu N_\nu s_\nu \Delta_\mu \vee - \frac{1}{4} s_\mu M_\nu \Delta_\nu N_\mu s_\mu \Delta_\nu \vee$$
- (13) Relaxation to an equilibrium distribution is a very general feature of stochastic processes (for a discrete parameter process and a finite dimensional state space it is an immediate consequence of Frobenius's theorem; see, for example, F.R. Gantmacher: The Theory of Matrices, Chelsea Publishing Co., N.Y. (1959), Vol.2, Chapter 13, pgs. 2 and 7). This property is sometimes called "ergodic". However, it would seem preferable to reserve this adjective for another property on which the validity of the Monte Carlo calculation rests. Namely averages with respect to the equilibrium distribution are approached by long time averages over the configurations successively occupied by a single system (or Brownian particle) that is governed by given

transition probabilities or "updatings". In the present calculation by random walk simulation, the configuration space is discretized so the updating process defines a discrete parameter Markov chain. The relevant ergodic theorem may be found in K.L. Chung, Markov Chains with Stationary Transition Probabilities, Springer Verlag, Berlin (1960), chapter 1, para. 15, theorem 2.

- (14) M. Creutz, I. Muzinich, I. Tudron: Phys. Rev. D19, 531 (1979), Sec. V;
A. Chodos and V. Moncrief: J. Math. Phys. 21, 364 (1980).
- (15) H. Flanders: Differential Forms, Academic Press, New York (1963), p. 92 ff.
- (16) Y. Choquet-Bruhat, C. De Witt-Morette, M. Dillard-Bleick: Analysis, Manifolds and Physics, North Holland, Amsterdam (1982), p. 243 ff.
- (17) To a point A with $v = \delta A$ small, a neighboring point $A^{(o)}$ with $\delta A^{(o)} = 0$ may be assigned, for example, by orthogonal projection (i.e. choose that $A^{(o)}$ which minimizes $|A - A^{(o)}|^2$ as $A^{(o)}$ varies over all transverse potentials $\delta A^{(o)} = 0$). Other reasonable choices of $A^{(o)}$ will differ by terms of order v , and thus not change materially Eqs.(2.49) and (2.50) which are correct only to first order in v . The proof shows for which $A^{(o)}$ v decreases, and thus, for which $A^{(o)}$, the nearby A approach the hyperplane $\delta A = 0$, but they do not, in general, approach the point $A^{(o)}$.
- (18) P. Rossi and D. Zwanziger: in preparation.
- (19) E. Seiler, I.O. Stamatescu and D. Zwanziger: Ref. TH 3622-CERN.
- (20) J. Kuti, P. Polónyi and K. Szlachanyi: Phys. Letts. 98B, 199 (1981).
- (21) L. McLerran and B. Svetitsky: Phys. Letts. 98B, 195 (1981).
- (22) It is worth noting from Fig. 4a that as α is increased, at fixed β , from $\alpha = \beta$ to $\alpha > \beta$ then $-\ln\langle P \rangle / L_0$, which represents the self-mass of an external quark, increases. Thus the model with $\alpha \gg \beta$ (corresponding to stricter gauge-fixing, $\text{div } A$ closer to zero) is closer to a confining theory than with $\alpha = \beta$. Unfortunately the present method of computation does not allow us to explore this region reliably.
- (23) C. Borgs and E. Seiler: Lattice Yang-Mills Theory at Nonzero Temperature and the Confinement Problem, Max Planck-Inst. preprint MPI-PAE/PTh 16/83, Comm. Math. Phys., in print.
- (24) A. Ukawa, P. Windey, A. Guth: Phys. Rev. D21, 1013 (1980).
- (25) H.S. Sharatchandra: Phys. Rev. D18, 2042 (1978).
- (26) L.H. Karsten and J. Smit: Nucl. Phys. B183, 103 (1981).
- (27) B. M. McCoy and Tai Tsun Wu, in vol. Bifurcation Phenomena in Mathematical Physics, p.69, C.Bardos and D.Bessis eds., D.Reidel 1980

FIGURE CAPTIONS

Fig. 1: The lattice curl.

Fig. 2: The hyperplane in A-space, $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$, is represented by the infinite line through a and b, and the stable region by the segment (a,b). For large values of the gauge-fixing parameter, a representative point in A-space may be expected to display cycles of the type shown. The vertical arrows symbolize the direction and magnitude of the gauge-fixing force near the hyperplane $\sum_{\mu} \Delta_{\mu} A_{\mu}^a(m) = 0$.

Fig. 3: Step size dependence for "parallel" and "sequential" data

(a) Action per degree of freedom, average plaquette

(b) $\chi(2,2)$ and $\chi(3,2)$

Fig. 4: Average Polyakov loops (the dashed lines are the weak coupling predictions)

(a) $\ln \langle P_{1/2} \rangle / L_0$ on the 8^4 and 4^4 lattices

(b) $\ln \langle P_1 \rangle / L_0$ on the 4^4 lattice
($\alpha = \beta$)

Fig. 5: Creutz's parameter $\chi(I, J)$

(a) $1/g^2 = 0.2$

(b) $1/g^2 = 0.25$

(c) $1/g^2 = 0.3$

(d) $1/g^2 = 0.5$ ($\alpha = 1.0$)

Fig. 6:

Polyakov loop correlation functions

$$\langle P_{1/2}(0) P_{1/2}(R) \rangle$$

- (a) $1/g^2 = 0.2$
- (b) $1/g^2 = 0.25$
- (c) $1/g^2 = 0.3$
- (d) $1/g^2 = 0.5$ ($\alpha = 1.0$)

Fig. 7:

Asymptotic freedom test: Note that the large full circles and the empty circles appear to lie on the same curve.

Fig. 1

Fig. 2

Fig. 3a

Fig. 3b

Fig. 4a

Fig. 4b

Fig. 5

Fig. 6

Fig. 7