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BLOCK-SPIN RENORMALIZATION GROUP IMPROVED
NON - LINEAR σ MODEL

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A B S T R A C T

Block-spin renormalization group transformation is used at the weak coupling regime to find an improved action. MC calculations show that for this action scaling sets in at small lattice correlation length ($\xi \approx 2.5$). The results are compared with the results obtained using the Symanzik improved and the standard model.

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Regularizing a field theory on a lattice is one of the most promising approaches to the extraction of numerical information. Different lattice actions belonging to the same universality class give the same continuum limit when the cut-off is removed, but they may behave differently at finite cut-off (finite lattice correlation length): they start to show weak coupling scaling behaviour from different lattice correlation lengths.

One usually investigates these models by MC or other numerical methods. It is easier to perform MC simulations with actions which show scaling behaviour from smaller lattice correlation lengths (improved actions). Symanzik's programme for constructing an improved action is based on a systematic perturbative calculation¹⁾. Another possibility, proposed by K. Wilson, is to find an action which is close to the renormalized trajectory of a block-spin renormalization group (BSRG) transformation²⁾. If we consider the points in the multidimensional action space, which has the same finite correlation length ξ , the points lying on the renormalized trajectory of an RG transformation are absolutely the best points: they give exactly the Euclidean invariant continuum results, because they are connected to the fixed point by an exact RG transformation.

To find the renormalized trajectory of a given RG transformation is beyond our capability. However, for asymptotically free models, such as the two-dimensional $O(N)$ spin models or the four-dimensional $SU(N)$ gauge theories, the fixed point can be determined by perturbative calculation. If we take an action which starts in the weak coupling close to the fixed point, we may hope that it will not be far from the renormalized trajectory when the scaling behaviour sets in.

For the two-dimensional $O(3)$ model, Symanzik's improved action was considered in Refs. 3) and 4). In the MC calculation for the one-loop improved action, early scaling and a wide scaling window were found, while the tree level improved action was not much better than the standard model⁵⁾⁻⁷⁾.

In this paper we investigate the BSRG improved action. This action has a similar scaling behaviour to the one-loop improved action, but its structure is much simpler.

Our aim with the BSRG approach is to find an improved action which is closer to the renormalized trajectory of the given block-spin transformation than the standard action. Then, the improved action describes the same continuum physics as the standard one on a smaller length scale, i.e., there is weak coupling scaling behaviour at smaller lattice correlation length.

Shenker and Tobochnik⁸⁾ used spin-wave approximation to determine a BSRG improved action in the weak coupling limit. We use the same method to investigate the BSRG properties of the $O(3)$ model in a three-parameter action space. We considered an action containing nearest and next-to-nearest neighbour interactions

$$\mathcal{X} = K_1 \sum_{n,\mu} s_n s_{n+\mu} + K_2 \sum_n s_n (s_{n+x+y} + s_{n+x-y}) + K_3 \sum_{n,\mu} s_n s_{n+2\mu}$$

where s_n is a three-component unit vector. For the weak coupling calculation, it is convenient to rewrite this action in the form

$$\mathcal{X} = - \left\{ \alpha_1 \sum_{n,\mu} (\nabla_\mu s_n)^2 + \alpha_2 \sum_{n,\mu,\nu} (\nabla_\mu \nabla_\nu s_n)^2 + \alpha_3 \sum_{n,\mu} (\nabla_\mu^2 s_n)^2 \right\}$$

where $\nabla_\mu s_n = s_{n+\mu} - s_n$, and

$$K_1 = 2\alpha_1 + 16\alpha_2 + 8\alpha_3, \quad K_2 = -4\alpha_2, \quad K_3 = -2\alpha_2 - 2\alpha_3$$

In order to get the proper continuum limit, we choose $\alpha_1 = 0.5$. The partition function is defined

$$Z = \int [ds] e^{\beta \mathcal{X}},$$

where β is the inverse temperature, $\beta = 1/t$.

To define the block-spin variables, we group the spins of the original lattice into 2×2 blocks. The block variable w is defined as the normalized average of the original spins in the given block:

$$w_{n'} = \sum_{j \in n'} s_j / \left\| \sum_{j \in n'} s_j \right\|$$

where n' denotes the block sites, and j the original sites on the lattice. We can repeat this procedure by grouping the w variables into new blocks and so on.

To examine how close we are to the weak coupling fixed point, we calculated several block spin-spin expectation values in spin-wave approximation. If we were at the fixed point, the expectation values would not change after a block transformation. Therefore, from the changes of the expectation values we can estimate how far the action is from the fixed point. We found that there is an irrelevant operator combination on this three-parameter space at $\beta = \infty$. The actions with approximately $1.52\alpha_2 + \alpha_3 = \text{const.}$ parameters are equivalent in the weak coupling limit (Fig. 1).

The action $\alpha_2 = 0.1, \alpha_3 = 0.0$, which was proposed in Ref. 8), is one of the closest actions to the fixed point: it approaches the fixed point within one-two block transformations. The action $\alpha_2 = 0.0, \alpha_3 = 1/24$, which corresponds to the tree level improved Symanzik's action, approaches the fixed point about one-two block transformations later than the $(\alpha_2 = 0.1, \alpha_3 = 0.0)$ action, while the standard $(\alpha_2 = \alpha_3 = 0)$ one needs even more blocking steps. Of course, it is not necessary that these weak coupling properties remain valid at the finite coupling regime. But if they remain approximately valid, it would mean that for the $(\alpha_2 = 0.1, \alpha_3 = 0.0)$ model one would need a lattice correlation length a few times smaller to find weak coupling scaling behaviour than for the standard model. Later we shall see that this indeed the case.

In the following we shall denote the action

$\alpha_2 = 0.1,$	$\alpha_3 = 0.0$	b_y	I_1
$\alpha_2 = 0.0,$	$\alpha_3 = 0.1517$	b_y	I_2
$\alpha_2 = 0.0,$	$\alpha_3 = 1/24$	b_y	I_2
$\alpha_2 = 0.0,$	$\alpha_3 = 0.0$	b_y	S

In the weak coupling regime the action I_2 is equivalent to I_1 within the given RG spin-wave approach (see Fig. 1).

To connect the physical results obtained from the different improved actions, we need their scales⁹⁾. We calculated the relations between the weak coupling Λ parameters at the one-loop level. Since the value of Λ does not depend on the irrelevant operator combination given above, it is sufficient to give its value along the $\alpha_2 = 0$ axis (see the table).

We have carried out MC calculations for the I_1, I_2 and I_3 models. We used the heat bath method to update the spins. The lattice size was 50×50 with periodic boundary conditions. For the I_1 and I_3 models at each value of β we made two independent runs, each consisting of 4000 iterations, while for the I_2 model we made four independent runs with 6000 iterations at each value of β .

We measured the magnetic susceptibility and the mass gap from the zero momentum correlation length. We found that the I_1 and I_2 models, which are equivalent in the weak coupling spin wave approximation, remain equivalent within statistical errors at the scaling regime. In Fig. 2 the mass gap is shown

for the I_2 , I_3 , S and for the one-loop improved Symanzik's action. The data for the one-loop improved model was taken from Ref. 3) and for the standard model from Ref. 6)*). The latter was obtained on a 200×200 lattice. In Fig. 3 the magnetic susceptibility scaling defect

$$\delta_\chi = 2 \cdot 10^3 \cdot e^{-4\pi/\beta} \cdot \beta^4 \cdot \chi$$

is plotted for the I_2 , I_3 and S models. The data for the S model are from Ref. 7).

From the figures, one can see that the BSRG improved I_2 action shows similar scaling behaviour to the Symanzyk's one-level improved action. It begins to scale at lattice correlation length $\xi \geq 2.5$. The tree-level improved action begins to scale at $\xi \geq 4$, while the standard model begins at $\xi \geq 10$ 4),5).

Using the two-loop weak coupling formula for the Λ parameter, we get for the mass gap and for the magnetic susceptibility:

$$\text{model } I_1, I_2 \quad m = (10.0 \pm 0.5) \Lambda_{I_2}$$

$$\delta_\chi = 0.91 \pm 0.05$$

$$\text{model } I_3 \quad m = 37.8 \Lambda_{I_3}$$

$$\delta_\chi = 0.097$$

one-loop improved
model

$$m = 16 \Lambda_{1-e}$$

standard model

$$m = 95.7 \Lambda_S$$

Using the numbers of the table for the rates of the Λ parameters, we can express these results in terms of Λ_S :

*) There may be a systematic error of about 10% in the mass gap of the standard model, because in Ref. 6) not the zero momentum correlation function was used.

model I_1, I_2	$m = (88.6 \pm 4) \mu_s$
model I_3	$m = 84 \mu_s$
one-loop improved m.	$m = 35.5 \mu_s$
standard m.	$m = 95.7 \mu_s$

While the I_1, I_2 models are not evidently inconsistent with the standard model, there is a large discrepancy between the one-loop improved and the standard models. This may indicate that there are large $O(t)$ corrections to the Λ parameter of the one-loop improved model.

Finally, we conclude that the weak coupling BSRG properties of the improved actions remain approximately valid at the scaling region, thus these actions show early scaling behaviour. Similar weak coupling analysis for the four-dimensional $SU(2)$ gauge theory is in progress. A careful analysis of the $SU(3)$ improved actions may help to solve the scaling problem of the string tension^{10),11)}.

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Table

The ratio of the Λ parameters of different lattice actions

α_2	α_3	$\Lambda_{(\alpha_2, \alpha_3)} / \Lambda_S$
0.0	0.02	1.51
0.0	0.04	2.16
0.0	0.06	2.96
0.0	0.08	3.92
0.0	0.10	5.03
0.0	0.12	6.37
0.0	0.14	7.88
0.0	0.16	9.60
0.0	0.18	11.53
0.0	0.20	13.70
0.0	1/24	2.22
0.0	0.1517	8.86
0.1	0.0	8.86

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FIGURE CAPTIONS

- Fig. 1 : The equivalent actions in the given BSRG spin-wave approximation are shown. The dashed line corresponds to the "best" actions.
- Fig. 2 : The mass gap for the I_2 (dots), I_3 [filled triangles; the crosses are from Ref. 3), the squares are from Ref. 4)], the one-loop improved action [open circles, Ref. 3)] and for the standard model [triangles, Ref. 6)].
- Fig. 3 : The magnetic scaling defect for the I_2 (dots), I_3 [filled triangles, squares from Ref. 4)] and for the standard model model [triangles, Ref. 7)].

Fig. 1

Fig. 2

Fig. 3