Ref.TH.3692-CERN

DISCRETE SYMMETRIES IN KALUZA-KLEIN THEORIES

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ABSTRACT

We investigate discrete symmetries in theories of higher dimensional ($d > 4$) gravity and their consequences for the reduced four-dimensional theory, obtained for a ground state which is a direct product of four-dimensional Minkowski space and a compact $d-4$ dimensional internal space. If the action of pure d dimensional gravity coupled to spinors is invariant under time reversal or reflection of an odd number of spacelike co-ordinates, the reduced four-dimensional theory has a non-trivial parity or CT symmetry not consistent with observation. A non-trivial d dimensional charge conjugation results in an unwanted doubling of the four-dimensional fermion spectrum. As a consequence, realistic theories can only be obtained for Majorana-Weyl spinors in $d = 2 \bmod 8$ dimensions. The constraints are less stringent if supplementary fields are introduced in d dimensions. For $d = 11$ supergravity, for example, parity and CT invariance can be broken by a non-vanishing field strength of the totally antisymmetric three-index tensor.

A ground state invariant under reflections of "internal" co-ordinates often gives rise to a non-trivial charge conjugation in four dimensions. We find that the ground state of a realistic Kaluza-Klein theory should not be invariant under any non-trivial internal co-ordinate reflection (which cannot be obtained by a gauge transformation). We finally comment on a possible solution of the strong CP problem from Kaluza-Klein theories and discuss perspectives for finding internal spaces admitting chiral fermions.

1. INTRODUCTION

An explanation why nature violates parity and time reversal, or equivalently, CP conjugation, is one of the old puzzles in particle physics. In four dimensions, pure gravity coupled to matter fields (without gauge interactions) is not only invariant under the continuous symmetry of global Poincaré transformations (this symmetry is left invariant by Minkowski space), but also with respect to a reflection of an odd number of spacelike coordinates (parity) and time (time reversal). Parity and time reversal are discrete symmetries which cannot be obtained from unity by an integration of infinitesimal Poincaré transformations.

Electromagnetic and strong interactions conserve P (parity) and T (time reversal). The discovery of parity violation in weak interaction phenomena, and subsequently the discovery of CP violation which is equivalent to T violation by the CPT theorem, was one of the revolutions in particle physics. Up to now, our answers why P and CP are violated are only partly satisfactory. In some unified gauge theories based on SU(5) P-asymmetry can be linked to the existence of chiral fermions. This is only partly true in theories admitting a non-trivial left-right symmetry like SO(10). In this case, chiral fermions are not a sufficient condition for parity violation. No reason why CP has to be violated is provided by unified gauge theories, except phenomenological ones like our existence which requires a CP violating production of a baryon asymmetry in the early universe.

Kaluza-Klein theories of higher dimensional gravity are devised to unify gauge interactions and gravity¹⁾. One may hope that a natural explanation of the origin of P and CP violation could arise from this unification. Moreover, the existence of a four-dimensional charge conjugation is related to coordinate reflections in the supplementary dimensions and acquires a simple geometrical interpretation²⁾. In this paper we investigate discrete symmetries in higher dimensional theories and their consequences for the reduced four-dimensional theory. We study conditions on the higher dimensional theory which are necessary for a violation of parity, charge conjugation, time reversal, and CP in the reduced theory. We find that these conditions are far from being trivial. Almost all the simple

Kaluza-Klein models discussed so far have non-trivial P, C, or CP symmetries in the reduced four-dimensional theory.

In Section 2 we first derive the formalism to describe C, P, and T in an arbitrary number of dimensions. In contradistinction to four dimensions, pure gravity coupled to matter has not necessarily P or CP and T invariance. Whereas Weyl spinors coupled to gravity in $d = 4 \bmod 4$ violate parity, but conserve CP and T, Weyl spinors in $d = 2 \bmod 4$ violate P and T whereas C and PT are conserved. This difference is related to the fact that for $d = 4 \bmod 4$ Weyl spinors belong to complex representations of the Lorentz group, whereas for $d = 2 \bmod 4$ they are in real or pseudoreal representations. For Majorana spinors, d-dimensional charge conjugation is trivial. As a consequence, the only non-trivial discrete symmetry of a Majorana-Weyl spinor in $d = 2 \bmod 8$ is PT symmetry, which is the equivalent of CPT for Majorana spinors. Neither P nor T separately are symmetries of a theory of pure gravity coupled to a Majorana-Weyl spinor. This could be the fundamental origin of the observed P and CP violation in nature.

In Section 3 we discuss the consequences of the symmetries T_d (d-dimensional time reversal), P_3 (d-dimensional reflection of the first three space-like coordinates) and $\mathcal{C}_d$ (d-dimensional charge conjugation) for the reduced four-dimensional theory in detail. In pure gravity theories coupled to spinors the ground state is defined by the vacuum value of the metric (or the vielbein) and we assume that the ground state is a direct product of four-dimensional Minkowski space M^4 and some $D = d - 4$ dimensional "internal" space K^D . Such a ground state does not break T_d , P_3 or $\mathcal{C}_d$ and these symmetries reflect themselves as four-dimensional discrete symmetries. P_3 leads to some kind of four-dimensional parity, T_d induces four-dimensional CT transformations (not T transformations) and a non-trivial $\mathcal{C}_d$ generates an unobserved discrete symmetry between pairs of fermions belonging to the same representation and with the same mass. None of these symmetries is conserved in four dimensions. As a consequence, realistic Kaluza-Klein theories of pure gravity coupled to a spinor should not admit P_3 , T_d , or a non-trivial $\mathcal{C}_d$ invariance. They

must involve Majorana-Weyl spinors and this is only possible in $d = 2 \bmod 8$ dimensions. It is interesting to note that this is the only case consistent with the existence of chiral spinors³⁾ and just what is needed to explain the origin of the breaking of discrete symmetries!

In Section 4 we discuss reflections of the coordinates parametrizing the internal space K^D . Four-dimensional charge conjugation can be described by such reflections. If K^D is invariant under some coordinate reflection, this reflection corresponds to an automorphism of the algebra of the gauge group. Some abelian charges will change their sign. If the reflection can be obtained by a gauge transformation we have an inner automorphism which is harmless. In the other case, we have non-trivial internal discrete symmetries corresponding to an outer automorphism. Reflections of an odd number of coordinates are always non-trivial. The outer automorphism can either carry every complex representation in the harmonic expansion on K^D in its complex conjugate one. Then it corresponds to a non-trivial charge conjugation in four dimensions. Else, the automorphism transforms complex representations into equivalent, but different representations and results in a doubling of degrees of freedom in four dimensions (two different electrons with the same mass, etc.). None of these discrete symmetries is observed in nature. This implies that the ground state of a realistic Kaluza-Klein theory of pure gravity should not admit any non-trivial discrete symmetries.

The existence of non-trivial discrete symmetries is also very dangerous for the possible existence of chiral fermions on such spaces. In many cases specific discrete symmetries of K^D are sufficient to rule out chiral spinors in the four-dimensional spectrum. This becomes a powerful constraint on realistic Kaluza-Klein theories when combined with the fact that the number of chiral fermions is related to topological properties of K^D ⁴⁾.

In Section 5 we show how the constraints on realistic Kaluza-Klein theories derived from an investigation of discrete symmetries can be released if one introduces supplementary matter fields in higher dimensions. This is especially relevant

for supergravity theories. Supergravity in eleven dimensions is known to be T_d invariant⁵⁾ and, as a consequence of the higher dimensional generalization of the CPT theorem, has also a non-trivial invariance with respect to an odd number of coordinate reflections, in particular P_3 . Nevertheless, four-dimensional parity can be broken due to a non-vanishing vacuum value of the field strength of the totally antisymmetric three index tensor⁶⁾.

In the last section we discuss two other problems in the light of our results. The first topic concerns the problem of strong CP violation by a term $\sim \theta F_{\mu\nu} \tilde{F}^{\mu\nu}$ in the effective four-dimensional action. We argue that it is very likely that Kaluza-Klein theories provide a solution to this problem. If pure higher dimensional gravity is P_3 invariant and P_3 violation comes only from the fermionic sector, no term $\sim \theta F_{\mu\nu} \tilde{F}^{\mu\nu}$ can be induced by dimensional reduction since the ground state is invariant under P_3 . Thus the θ parameter vanishes at the compactification scale and only small calculable corrections are induced by fermion loops.

The second topic concerns some requirements on the local form of the metric of a space K^D admitting chiral fermions. We find that the possibility of obtaining unwanted discrete symmetries by continuous deformations of the internal space implies severe difficulties for finding ground states admitting chiral fermions.

2. C, P, T AND ALL THAT IN HIGHER DIMENSIONS

We start with a discussion of discrete symmetries in higher dimensions. In this section we concentrate on models of pure gravity in d dimensions coupled to a d dimensional spinor field. Models with supplementary matter fields (like $d = 11$ supergravity) will be discussed in Section 5. The action of these models is invariant under general coordinate transformations in d dimensions and under d -dimensional Lorentz rotations which form the local group $SO(d - 1, 1)$ [or, for spinors, the corresponding $Spin(d - 1, 1)$ group]. We call this combined symmetry group gen_d .

The action is of the form

$$S = - \int d^d x \hat{g}^{1/2} (f(\hat{g}_{\hat{\mu}\hat{\nu}}) - i \bar{\psi} \gamma^{\hat{m}} \hat{e}_{\hat{m}}^{\hat{\mu}} D_{\hat{\mu}} \psi) \quad (1)$$

[For investigations of spontaneous compactification it is appropriate to consider (1) as the *effective* action⁷⁾ at the length scale characteristic for the internal space. Such an effective action includes quantum fluctuations.] Here ψ is a spinorial representation of Spin $(d-1, 1)$ with anticommuting properties

$\psi_{\alpha} \psi_{\beta} = -\psi_{\beta} \psi_{\alpha}$, $\hat{e}_{\hat{\mu}}^{\hat{m}}$ is the d -dimensional vielbein, $\hat{e}_{\hat{m}}^{\hat{\mu}}$ the inverse vielbein and $\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}^{\hat{m}} \hat{e}_{\hat{\nu}}^{\hat{n}} \eta_{\hat{m}\hat{n}}$ the corresponding metric with $\eta_{\hat{m}\hat{n}} = \text{diag} (+1, -1 \dots -1)$ and $\hat{g} = |\det \hat{g}_{\hat{\mu}\hat{\nu}}|$. The covariant spinor derivative is given by

$$D_{\hat{\mu}} \psi = (\partial_{\hat{\mu}} - \frac{1}{2} \hat{\omega}_{\hat{\mu}\hat{n}\hat{p}} \Sigma^{\hat{n}\hat{p}}) \psi \quad (2)$$

with the d -dimensional spin connection $\hat{\omega}_{\hat{\mu}\hat{n}\hat{p}}$ defined by

$$\begin{aligned} \hat{\omega}_{\hat{\mu}\hat{n}\hat{p}} &= \hat{e}_{\hat{\mu}}^{\hat{m}} \hat{\omega}_{\hat{m}\hat{n}\hat{p}} \\ \hat{\omega}_{\hat{m}\hat{n}\hat{p}} &= -\hat{\Omega}_{\hat{m}\hat{n}\hat{p}} + \hat{\Omega}_{\hat{n}\hat{p}\hat{m}} - \hat{\Omega}_{\hat{p}\hat{m}\hat{n}} \\ \hat{\Omega}_{\hat{m}\hat{n}\hat{p}} &= \frac{1}{2} (\hat{e}_{\hat{m}}^{\hat{a}} \hat{e}_{\hat{n}}^{\hat{b}} - \hat{e}_{\hat{n}}^{\hat{a}} \hat{e}_{\hat{m}}^{\hat{b}}) \partial_{\hat{a}} \hat{e}_{\hat{b}\hat{p}} \end{aligned} \quad (3)$$

The d -dimensional Dirac matrices $\gamma^{\hat{m}}$ fulfil the commutation relations

$$\{ \gamma^{\hat{m}}, \gamma^{\hat{n}} \} = 2 \eta^{\hat{m}\hat{n}}, \quad \Sigma^{\hat{m}\hat{n}} = -\frac{1}{4} [\gamma^{\hat{m}}, \gamma^{\hat{n}}] \quad (4)$$

The spinor $\bar{\psi}$ is defined by

$$\begin{aligned} \bar{\psi} &= \psi^{\dagger} \gamma^0 && \text{for Dirac and Weyl spinors} \\ \bar{\psi} &= \psi^T C_d && \text{for Majorana and Majorana-Weyl spinors} \end{aligned} \quad (5)$$

with

$$(\Sigma^{\hat{m}\hat{n}})^T = -C_d \Sigma^{\hat{m}\hat{n}} C_d^{-1} \quad (6)$$

[For details on Majorana and Weyl spinors in arbitrary dimensions, see ref. 8.

We use the conventions of this paper*.)] The purely gravitational part of the

*) We profit from this occasion to correct a typing error in this paper. In eq. (29) ϵ_1, ϵ_2 should be replaced by δ_1, δ_2 .

action is determined by f , a scalar function of $\hat{g}_{\hat{\mu}\hat{\nu}}$ and its derivatives. Typically, f contains invariants formed from combinations of the d -dimensional curvature tensor $\hat{R}_{\hat{m}\hat{n}\hat{p}\hat{q}}$

$$\hat{R}_{\hat{m}\hat{n}\hat{p}\hat{q}} = \hat{e}_{\hat{m}}^{\hat{\mu}} \hat{e}_{\hat{n}}^{\hat{\nu}} (\partial_{\hat{\mu}} \hat{\omega}_{\hat{\nu}\hat{p}\hat{q}} - \partial_{\hat{\nu}} \hat{\omega}_{\hat{\mu}\hat{p}\hat{q}} + \hat{\omega}_{\hat{\mu}\hat{p}\hat{s}} \hat{\omega}_{\hat{\nu}}^{\hat{s}} \hat{\omega}_{\hat{q}}^{\hat{r}} - \hat{\omega}_{\hat{\nu}\hat{p}\hat{s}} \hat{\omega}_{\hat{\mu}}^{\hat{s}} \hat{\omega}_{\hat{q}}^{\hat{r}}) \quad (7)$$

or Ricci tensor

$$\hat{R}_{\hat{m}\hat{p}} = \hat{R}_{\hat{m}\hat{n}\hat{p}\hat{q}} \gamma^{\hat{n}\hat{q}} \quad (8)$$

like the curvature scalar

$$\hat{R} = \hat{R}_{\hat{m}\hat{p}} \gamma^{\hat{m}\hat{p}} \quad (9)$$

or higher derivative terms like $\hat{R}_{\hat{m}\hat{n}} \hat{R}^{\hat{m}\hat{n}}$, but we also could include covariant derivatives of the curvature tensor or torsion. For the discussion of this and the next three sections we do not consider contractions with the d -dimensional totally antisymmetric ϵ -tensor. (See, however, the remarks in Section 6.)

This action exhibits discrete symmetries of three possible types:

- i) discrete symmetries which are contained in the general coordinate transformations;
- ii) discrete symmetries contained in $\text{Spin}(d-1, 1)$;
- iii) discrete symmetries not contained in gen_d .

Of course, combinations of the three types of discrete symmetries are also possible.

The first type of symmetries includes reflections in an arbitrary number of coordinates. We denote by $I_{\hat{\nu}}$ a reflection of the coordinate $x^{\hat{\nu}}$:

$$I_{\hat{\nu}} : x \rightarrow x'$$

with

$$\begin{aligned} x^{\hat{\nu}} &\rightarrow -x^{\hat{\nu}} \\ x^{\hat{\rho}} &\rightarrow x^{\hat{\rho}}, \quad \hat{\rho} \neq \hat{\nu} \end{aligned} \quad (10)$$

Correspondingly, all covariant and contravariant tensors are multiplied by a factor -1 for every tensor index $\hat{\nu}$:

$$\begin{aligned}
 \hat{g}_{\hat{\nu}\hat{\nu}}(x) &\rightarrow \hat{g}_{\hat{\nu}\hat{\nu}}(x') \\
 \hat{g}_{\hat{\nu}\hat{\rho}}(x) &\rightarrow -\hat{g}_{\hat{\nu}\hat{\rho}}(x') \\
 \hat{g}_{\hat{\rho}\hat{\sigma}}(x) &\rightarrow \hat{g}_{\hat{\rho}\hat{\sigma}}(x') \quad , \quad \hat{\rho} \neq \hat{\nu}, \hat{\sigma} \neq \hat{\nu}
 \end{aligned}
 \tag{11}$$

Of course, all $\partial_{\hat{\nu}}$ derivatives also change sign and the scalars (with respect to coordinate transformations) ψ , $\hat{\omega}_{\hat{m}\hat{n}\hat{p}}$ and $\hat{R}_{\hat{m}\hat{n}\hat{p}\hat{q}}$ transform as

$$\begin{aligned}
 \psi(x) &\rightarrow \psi(x') \\
 \hat{\omega}_{\hat{m}\hat{n}\hat{p}}(x) &\rightarrow \hat{\omega}_{\hat{m}\hat{n}\hat{p}}(x') \\
 \hat{R}_{\hat{m}\hat{n}\hat{p}\hat{q}}(x) &\rightarrow \hat{R}_{\hat{m}\hat{n}\hat{p}\hat{q}}(x')
 \end{aligned}
 \tag{12}$$

Special reflections of this kind are the time reflection

$$T_x = I_{\hat{0}} \tag{13}$$

and the coordinate parity

$$P_x = I_{\hat{1}} I_{\hat{2}} \dots I_{\hat{d-1}} \tag{14}$$

General coordinate invariance guarantees the invariance of the action with respect to these discrete symmetries. However, for any *ground state*, which is characterized by a fixed value of the vielbein, $\hat{e}_{\hat{\mu}}^{\hat{m}}(x)$, the discrete coordinate reflections are broken, since $\hat{e}_{\hat{\mu}}^{\hat{m}}$ transforms non-trivially with respect to these reflections. Only combinations of discrete coordinate transformations with symmetries of the type (ii) or (iii) may be conserved.

Under infinitesimal d dimensional Lorentz transformations, the spinor and the vielbein transform according to

$$\begin{aligned}
 \psi(x) &\rightarrow \psi'(x) = \left(1 - \frac{1}{2} \varepsilon_{\hat{m}\hat{n}}(x) \Sigma^{\hat{m}\hat{n}} \right) \psi(x) \\
 \hat{e}_{\hat{\mu}}^{\hat{m}}(x) &\rightarrow \hat{e}_{\hat{\mu}}^{\hat{m}'}(x) = \left(\delta_{\hat{m}}^{\hat{m}'} + \varepsilon^{\hat{m}'}_{\hat{n}}(x) \right) \hat{e}_{\hat{\mu}}^{\hat{n}}(x)
 \end{aligned}
 \tag{15}$$

with antisymmetric transformation parameters $\varepsilon_{\hat{m}\hat{n}} = -\varepsilon_{\hat{n}\hat{m}}$. A finite rotation in the $\hat{m}\hat{n}$ plane with rotation parameter $\alpha = -\varepsilon_{\hat{m}\hat{n}}$ can be written as

$$\begin{aligned}\psi'(x) &= \exp(\alpha \Sigma^{\hat{m}\hat{n}}) \psi(x) \\ &= \left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \cdot (2 \Sigma^{\hat{m}\hat{n}}) \right) \psi(x)\end{aligned}\quad (16)$$

if $\hat{m}$ and $\hat{n}$ denote both spacelike directions. In this case one has

$$\begin{aligned}\varepsilon_{\hat{m}}^{\hat{n}} &= -\varepsilon_{\hat{n}}^{\hat{m}} \\ \begin{pmatrix} \hat{e}_{\hat{m}}^{\hat{\mu}} \\ \hat{e}_{\hat{n}}^{\hat{\mu}} \end{pmatrix}' &= \exp\left(\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \begin{pmatrix} \hat{e}_{\hat{m}}^{\hat{\mu}} \\ \hat{e}_{\hat{n}}^{\hat{\mu}} \end{pmatrix} = \cos \alpha \begin{pmatrix} \hat{e}_{\hat{m}}^{\hat{\mu}} \\ \hat{e}_{\hat{n}}^{\hat{\mu}} \end{pmatrix} + \sin \alpha \begin{pmatrix} \hat{e}_{\hat{n}}^{\hat{\mu}} \\ -\hat{e}_{\hat{m}}^{\hat{\mu}} \end{pmatrix}\end{aligned}\quad (17)$$

The discrete symmetry $I_{\hat{m}\hat{n}}^{\mathcal{L}}$ is a rotation with parameter $\alpha = -\pi$:

$$\begin{aligned}I_{\hat{m}\hat{n}}^{\mathcal{L}} : \quad & \hat{e}_{\hat{m}}^{\hat{\mu}}(x) \rightarrow -\hat{e}_{\hat{m}}^{\hat{\mu}}(x) \\ & \hat{e}_{\hat{n}}^{\hat{\mu}}(x) \rightarrow -\hat{e}_{\hat{n}}^{\hat{\mu}}(x) \\ & \hat{e}_{\hat{p}}^{\hat{\mu}}(x) \rightarrow \hat{e}_{\hat{p}}^{\hat{\mu}}(x), \quad \hat{p} \neq \hat{m}, \hat{n} \\ & \psi(x) \rightarrow \gamma^{\hat{m}} \gamma^{\hat{n}} \psi(x)\end{aligned}\quad (18)$$

All tensor components with an odd number of indices $\hat{m}$ or $\hat{n}$ change sign and the action is invariant under all combinations of transformations $I_{\hat{m}\hat{n}}^{\mathcal{L}}$. If the vielbein characterizing the ground state $\hat{e}_{\hat{\mu}}^{\hat{m}}$ is independent of two coordinates $x^{\hat{0}}, x^{\hat{0}}$ and diagonal in these directions

$$\hat{e}_{\hat{\mu}_0}^{\hat{p}} \sim \delta_{\hat{\mu}_0}^{\hat{p}}, \quad \hat{e}_{\hat{\nu}_0}^{\hat{p}} \sim \delta_{\hat{\nu}_0}^{\hat{p}}, \quad \hat{e}_{\hat{\mu}}^{\hat{m}} \sim \delta_{\hat{\mu}}^{\hat{m}}, \quad \hat{e}_{\hat{\mu}}^{\hat{n}} \sim \delta_{\hat{\mu}}^{\hat{n}}$$

then the ground state is invariant under the combined reflection $I_{\hat{\nu}} I_{\hat{\mu}} I_{\hat{m}\hat{n}}^{\mathcal{L}}$. For example, Minkowski space with $\hat{e}_{\hat{\mu}}^{\hat{m}} = \delta_{\hat{\mu}}^{\hat{m}}$ is invariant under the combined reflection of two spacelike coordinates.

However, we note that a reflection of only one Lorentz-index is not contained in the Lorentz rotations, nor is the reflection of one timelike and one spacelike index a Lorentz symmetry. If such symmetries exist, they must belong to symmetries of the type (iii). Such symmetries are not contained in gen_d and they are therefore not necessarily symmetries of the action. As a consequence, reflection in the observed three spacelike coordinates or time reflection are not necessarily symmetries of the reduced four-dimensional action, even if the ground state with respect to the four observed coordinates is Minkowski space and

therefore invariant with respect to these symmetries. Since pure gravitational theories omitting spinor fields are trivially invariant with respect to a reflection of one Lorentz index $I_{\hat{m}}^{\mathcal{L}}$

$$\begin{aligned} I_{\hat{m}}^{\mathcal{L}} : \quad & \hat{e}_{\hat{m}}^{\hat{\mu}}(x) \rightarrow -\hat{e}_{\hat{m}}^{\hat{\mu}}(x) \\ & \hat{e}_{\hat{p}}^{\hat{\mu}}(x) \rightarrow \hat{e}_{\hat{p}}^{\hat{\mu}}(x) \quad , \quad \hat{p} \neq \hat{m} \end{aligned} \quad (19)$$

all violations of four-dimensional parity must therefore be due to the couplings of spinor fields.

Let us first consider symmetries of the type (iii) for one d-dimensional Dirac spinor. The reflection of one spacelike Lorentz index is then achieved by the transformation

$$\begin{aligned} I_{\hat{m}}^{\mathcal{L}} : \quad & \psi \rightarrow \eta_a \gamma^{\hat{m}} \bar{\gamma} \psi \quad \text{for d even} \\ & \psi \rightarrow \eta_a \gamma^{\hat{m}} \psi \quad \text{for d odd} \end{aligned} \quad (20)$$

where η_a is a phase and

$$\bar{\gamma} = \eta_{\gamma} \gamma^0 \gamma^1 \dots \gamma^{d-1} \quad , \quad \bar{\gamma}^2 = 1 \quad (21)$$

The invariance of the action under the transformation (19), (20) is easily checked.

Starting with a d-dimensional Dirac spinor and assuming a ground state $M^4 \times K^D$,

the reduced four dimensional theory is invariant under the following reflection

of the three spacelike Minkowski coordinates:

$$P_3 = I_{\hat{1}} I_{\hat{2}} I_{\hat{3}} I_{\hat{1}}^{\mathcal{L}} I_{\hat{2}}^{\mathcal{L}} I_{\hat{3}}^{\mathcal{L}} \quad (22)$$

The product of two reflections $I_{\hat{m}}^{\mathcal{L}}$ and $I_{\hat{n}}^{\mathcal{L}}$ applied on spinors is a Lorentz transformation

$$I_{\hat{m}}^{\mathcal{L}} I_{\hat{n}}^{\mathcal{L}} = \pm I_{\hat{m}\hat{n}}^{\mathcal{L}} \quad (23)$$

and we have the following commutation relations

$$\{I_{\hat{m}}^{\mathcal{L}}, I_{\hat{n}}^{\mathcal{L}}\} = 0, \quad \{I_{\hat{m}}^{\mathcal{L}}, I_{\hat{m}\hat{n}}^{\mathcal{L}}\} = 0, \quad [I_{\hat{m}}^{\mathcal{L}}, I_{\hat{n}\hat{p}}^{\mathcal{L}}] = 0, \quad \hat{m} \neq \hat{n} \neq \hat{p} \quad (24)$$

[Applied on the vielbein, all $I_{\hat{m}}^{\mathcal{L}}$ commute and $I_{\hat{m}}^{\mathcal{L}} I_{\hat{n}}^{\mathcal{L}} = I_{\hat{m}\hat{n}}^{\mathcal{L}}$, corresponding to the two to one covering of $SO(d-1, 1)$ by the group $Spin(d-1, 1)$.] The d-dimensional parity operation P_d is defined by

$$P_d = P_x P_\ell \quad (24)$$

with

$$P_\ell \psi = \eta_P \gamma^0 \psi \quad (25)$$

and

$$\prod_{\hat{m} \neq 0} I_{\hat{m}}^\ell = \pm P_\ell \quad (26)$$

For d odd, P_ℓ is a pure Lorentz transformation.

Similarly, we can define for Dirac spinors a d -dimensional time reversal operation:

$$T_d = T_x T_\ell \quad (27)$$

with

$$\begin{aligned} T_\ell \hat{e}_0^{\hat{\mu}} &= -\hat{e}_0^{\hat{\mu}}, \quad T_\ell \hat{e}_{\hat{m}}^{\hat{\mu}} = \hat{e}_{\hat{m}}^{\hat{\mu}}, \quad \hat{m} \neq 0 \\ T_\ell \psi &= \eta_T \gamma^0 \bar{\psi} B_d^{-1} \psi^* \quad \text{for } d \text{ even} \\ T_\ell \psi &= \eta_T \gamma^0 B_d^{-1} \psi^* \quad \text{for } d \text{ odd} \end{aligned} \quad (28)$$

Here η_T is again a phase and B_d is given by B_{1d} for $d = 2, 3, 4, 7 \bmod 8$ and by B_{2d} for $d = 5, 6, 8, 9 \bmod 8$ obeying

$$\begin{aligned} (\gamma^{\hat{m}})^* &= -B_{1d} \gamma^{\hat{m}} B_{1d}^{-1} \\ (\gamma^{\hat{m}})^* &= B_{2d} \gamma^{\hat{m}} B_{2d}^{-1} \end{aligned} \quad (29)$$

T_ℓ involves a complex conjugation and acts as an antilinear operator. Furthermore, T_ℓ involves a total reordering of all fields which is equivalent to a transposition:

$$T_\ell (\psi_\alpha \psi_\beta) = (T_\ell \psi_\beta) (T_\ell \psi_\alpha) = -(T_\ell \psi_\alpha) (T_\ell \psi_\beta) \quad (30)$$

Symmetries of the type (iii) contain also discrete symmetries leaving the vielbein and therefore the gravitational part of the action invariant. One such symmetry is d -dimensional charge conjugation $\mathcal{C}_d$:

$$\begin{aligned} \mathcal{C}_d \psi &= B_d^{-1} \psi^* = \psi^c \\ \mathcal{C}_d \hat{e}_{\hat{m}}^{\hat{\mu}} &= \hat{e}_{\hat{m}}^{\hat{\mu}} \end{aligned} \quad (31)$$

The charge conjugate spinor ψ^c has the same gen_d transformation properties (15) as ψ ,

$$[\mathcal{C}_d, \Sigma^{\hat{m}\hat{n}}] = (\mathcal{B}_d^{-1}(\Sigma^{\hat{m}\hat{n}})^* - \Sigma^{\hat{m}\hat{n}}\mathcal{B}_d^{-1})(*) = 0 \quad (32)$$

where the symbol $(*)$ denotes a complex conjugation of all fields on which $\mathcal{C}_d$ operates. The invariance of the action (1) under $\mathcal{C}_d$ is easily verified.

Finally, for Dirac spinors and d even, we define the discrete symmetry Γ :

$$\Gamma: \quad \psi \rightarrow \bar{\gamma}\psi, \quad \hat{e}_{\hat{m}}^{\hat{\mu}} \rightarrow \hat{e}_{\hat{m}}^{\hat{\mu}} \quad (33)$$

Again, Γ commutes with gen_d transformation

$$[\Sigma^{\hat{m}\hat{n}}, \bar{\gamma}] = 0 \quad (34)$$

and it is a symmetry of the action (1). It distinguishes the different Weyl spinors

$$\begin{aligned} \psi_+ &= \frac{1}{2}(1+\bar{\gamma})\psi, & \bar{\gamma}\psi_+ &= \psi_+ \\ \psi_- &= \frac{1}{2}(1-\bar{\gamma})\psi, & \bar{\gamma}\psi_- &= -\psi_- \end{aligned} \quad (35)$$

For $d = 4 \bmod 4$, $\mathcal{C}_d$ anticommutes with Γ and for $d = 2 \bmod 4$ $\mathcal{C}_d$ and Γ commute.

$$\begin{aligned} [\mathcal{C}_d, \bar{\gamma}] &= 0 & \text{for } d &= 2 \bmod 4 \\ \{\mathcal{C}_d, \bar{\gamma}\} &= 0 & \text{for } d &= 4 \bmod 4 \end{aligned} \quad (36)$$

This corresponds to the well-known fact that the inequivalent Weyl spinors belong to complex representations, conjugate to each other, for $d = 4 \bmod 4$, whereas for $d = 2 \bmod 4$ Weyl spinors are real or pseudoreal representations of the Lorentz group.

We finally will choose definite phases η_a, η_p, η_T . First, we require for all discrete transformations I discussed so far, that applying I twice on a spinor field results in an identity transformation up to a minus sign ($I^2 = \pm 1$). This implies $\eta_a^4 = \eta_p^4 = \eta_T^4 = 1$, consistent with the requirements (23) and (26). Second, we choose the phases so that the discrete transformations $I_{\hat{m}}^{\mathcal{L}}, P_{\mathcal{L}}$ and $T_{\mathcal{L}}$ commute with charge conjugation:

$$[\mathcal{C}_d, I_{\hat{m}}^{\mathcal{L}}] = 0, \quad [\mathcal{C}_d, P_{\mathcal{L}}] = 0, \quad [\mathcal{C}_d, T_{\mathcal{L}}] = 0 \quad (37)$$

This will allow us to use the same definition of these transformations for Dirac and for Majorana spinors. The phases are now fixed up to a minus sign and we list the operation on spinors of the discrete transformations, including the combined symmetry $(PCT)_d$, in table 1.

For Dirac spinors, the action (1) has a global $U(1)$ symmetry

$$\psi \rightarrow \exp(i\varphi) \psi \quad (38)$$

For d even, the global symmetry is $U(1) \times U(1)$ and includes chiral phase rotations

$$\psi \rightarrow \exp(i\alpha \bar{\gamma}) \psi \quad (39)$$

A combination of a discrete symmetry with a phase rotation yields of course an equivalent discrete symmetry.

Except for $d = 5, 7 \bmod 8$, Dirac spinors are not irreducible representations of $\text{Spin}(d-1, 1)$. The irreducible representations are Majorana-spinors for $d = 3, 9 \bmod 8$, Weyl spinors for $d = 4, 6, 8 \bmod 8$ (for $d = 4 \bmod 4$, Majorana and Weyl spinors are equivalent) and Majorana-Weyl spinors for $d = 2 \bmod 8$. For Majorana and Weyl spinors the symmetries $\mathcal{C}_d$ and Γ reflect the existence of more than one irreducible representation in the action. Majorana-spinors are eigenstates of charge conjugation

$$\mathcal{C}_d \psi_M = B_d^{-1} \psi_M^* = \psi_M \quad (40)$$

Since $\mathcal{C}_d$ commutes with $I_{\hat{m}}^L$, P_L and T_L , these transformations are still defined for Majorana spinors. The action is invariant under these symmetries, noting that for Majorana spinors one has the identity

$$\bar{\psi}_M = \psi_M^\dagger \gamma^0 = \psi_M^T B_d \gamma^0 = \psi^T C_d \quad (41)$$

There is no non-trivial d -dimensional charge conjugation for Majorana spinors.

Weyl spinors are eigenstates of $\bar{\gamma}$:

$$\bar{\gamma} \psi_+ = \psi_+ \quad (42)$$

The symmetry Γ is now trivial. Reflections of an odd number of Lorentz indices anticommute with $\bar{\gamma}$

$$\{I_m^\mu, \bar{\gamma}\} = 0, \quad \{P_3, \bar{\gamma}\} = 0, \quad \{P_\alpha, \bar{\gamma}\} = 0 \quad (43)$$

and these operations cannot be defined for Weyl spinors. As well, due to the commutation relations (36), the charge conjugation operation cannot be defined for Weyl spinors in $d = 4 \bmod 4$ whereas for Weyl spinors in $d = 2 \bmod 4$, $\mathcal{C}_d$ is still a symmetry of the action. For $d = 4 \bmod 4$, the combined symmetries $\mathcal{C}_d I_m^\mu$, $\mathcal{C}_d P_3$, $\mathcal{C}_d P_\alpha$ commute with $\bar{\gamma}$ and are symmetries of the action. These symmetries are not defined for Weyl spinors in $d = 2 \bmod 4$. The situation is similar for time reversal. One has

$$[\bar{\gamma}, T_\alpha] = 0 \quad \text{for } d = 4 \bmod 4 \quad (44)$$

$$\{\bar{\gamma}, T_\alpha\} = 0 \quad \text{for } d = 2 \bmod 4$$

and T_d is a symmetry for Weyl spinors only for $d = 4 \bmod 4$. Finally, Majorana-Weyl spinors are the irreducible spinor representations for $d = 2 \bmod 8$. They are eigenstates of both $\bar{\gamma}$ and $\mathcal{C}_d$:

$$\mathcal{C}_d \psi_{M+} \equiv \bar{\gamma} \psi_{M+} \equiv \psi_{M+} \quad (45)$$

The only non-trivial symmetry of type (iii) for Majorana-Weyl spinors is a combination of time reversal with a reflection of an odd number of Lorentz indices. The action is invariant under symmetries like $I_m^\mu T_d$, $P_3 T_d$, $P_\alpha T_d$. In table 2 we list the discrete symmetries for the different irreducible spinor representations of $\text{Spin}(d-1, 1)$. Of course, combinations of these symmetries are also symmetries of the action. Let us also remind at this place that the kinetic term for Weyl spinors has a global $U(1)$ invariance of phase transformations. [For the equivalent Majorana spinors in $d = 4 \bmod 4$, this corresponds to transformations $\psi_M \rightarrow \exp(i\alpha\bar{\gamma})\psi_M$.] No global $U(1)$ symmetry exists for Majorana-spinors in $d = 3, 9 \bmod 8$ or a Majorana-Weyl spinor in $d = 2 \bmod 8$.

3. DIMENSIONAL REDUCTION OF d-DIMENSIONAL DISCRETE SYMMETRIES: REFLECTIONS OF MINKOWSKI COORDINATES AND d-DIMENSIONAL CHARGE CONJUGATION

For a ground state $M^k \times K^D$, the d -dimensional spinor (as well as the vielbein) decomposes into infinitely many four-dimensional spinors $\phi^i(x)$:

$$\psi(y, x) = \sum_i \psi_i(y) \varphi_i(x) \quad (46)$$

(We denote the coordinates of K^D by y^α or ϕ^α and the coordinates of Minkowski space by x^μ . For details of the decomposition and notations see ref. 3.) The reduced four-dimensional theory is a gauge theory of spinors coupled to scalars, spin one and spin two bosons. Therefore, the four-dimensional parity $\bar{P}$, time reversal $\bar{T}$ and charge conjugation $\bar{C}$ involve also non-trivial transformations of the scalar and vector fields. (Transformation properties of spinor bilinears can be found in Section 5.) We note that $\bar{P}$, $\bar{T}$, and $\bar{C}$ act only on Minkowski coordinates and $\phi_i(x)$, but not on $\psi_i(y)$:

$$\begin{aligned} \bar{P}: \quad x^\mu &= (x^0, x^i) \rightarrow (x^\mu)' = (x^0, -x^i) \\ \bar{P} \varphi_i(x) &= i \gamma^0 \varphi_i(x') \end{aligned} \quad (47)$$

$$\begin{aligned} \bar{T}: \quad x^\mu &\rightarrow -(x^\mu)' \\ \bar{T} \varphi_i(x) &= \gamma^0 \gamma^5 B_4^{-1} \varphi_i^*(-x') = -\gamma^1 \gamma^3 \varphi_i^*(-x') \end{aligned} \quad (48)$$

$$\begin{aligned} \bar{C}: \quad x^\mu &\rightarrow x^\mu \\ \bar{C} \varphi_i(x) &= B_4^{-1} \varphi_i^*(x) = i \gamma^2 \varphi_i^*(x) \end{aligned} \quad (49)$$

Using

$$\begin{aligned} \bar{\gamma} &= \bar{\gamma}^5 \bar{\Gamma}_D \\ \bar{\gamma}^5 &= i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\ \bar{\Gamma}_D &= \eta_D \gamma^4 \gamma^5 \dots \gamma^{d-1} = \eta_D \Gamma^1 \Gamma^2 \dots \Gamma^D \end{aligned} \quad (50)$$

with D anticommuting Dirac matrices Γ^a acting on $\psi^i(y)$, one finds that the d -dimensional Weyl constraint implies a Weyl constraint on the spinors ϕ_i : Left-handed spinors ϕ_{iL} are obtained from the expansion of $\psi^+(y) = \frac{1}{2}(1 + \bar{\Gamma}_D)\psi(y)$, whereas right-handed spinors ϕ_{iR} derive from $\psi^-(y) = \frac{1}{2}(1 - \bar{\Gamma}_D)\psi(y)$. The Weyl spinor $\psi_+(y, x) \equiv \bar{\gamma}\psi_+(y, x)$ contains both left-handed and right-handed four-dimensional spinors. For Weyl spinors ϕ_{iL} the transformations $\bar{P}$ and $\bar{C}$ are not defined separately; only $\bar{T}$, $\overline{CP}$ and $\overline{CPT}$ are consistent with the Weyl constraint. However, a left-handed and a right-handed Weyl spinor belonging to the same representation of the gauge group may be combined into a Dirac spinor. (This is the case for all massive spinors belonging to complex representations.) Weyl spinors belonging

to real or pseudoreal representations are equivalent to Majorana spinors, for which $\bar{P}$ is defined (and equivalent to $\overline{CP}$). If there are left-handed Weyl spinors ϕ_{CL} belonging to a complex representation C together with other left-handed spinors $\phi_{\bar{C}L}$ belonging to the complex conjugate representation $\bar{C}$, a non-trivial charge conjugation $\bar{C}$ can be defined as an exchange of these spinors. (It may be used to combine them into a Dirac spinor.) Furthermore, it may be possible to combine a reflection of the three spacelike coordinates with an outer automorphism of the gauge symmetry. [A typical example is left-right-symmetry in $SO(10)$ or $SU(2)_L \times SU(2)_R \times U(1) \times SU(3)_C$.]

Since all these complexions can occur in dimensional reduction, we simplify our discussion by defining *classes* of discrete transformations for the effective four-dimensional theory. The class $\tilde{P}$ contains all symmetries containing a reflection of the three spacelike coordinates plus all transformations which can be obtained from such a symmetry by a four-dimensional global Lorentz-rotation or translation. Similarly, the symmetries of the class $\tilde{T}$ contain a reversal of time. Then all symmetries belong to one of the four classes $\tilde{I}$ (no coordinate reflection), $\tilde{P}$, $\tilde{T}$, and $\tilde{PT}$ (reflection of all Minkowski coordinates). Some elements of these symmetry classes are given in table 3. We note that there exists a group structure with respect to a multiplication of these classes, with $\tilde{I}$ the identity element and $\tilde{P}^2 = \tilde{T}^2 = (\tilde{PT})^2 = \tilde{I}$.

After these preparations we now address the question of how the d -dimensional discrete symmetries are related to four-dimensional transformations. First we note that the definition of the classes $\tilde{I}$, $\tilde{P}$, $\tilde{T}$, and $\tilde{PT}$ does not refer to dimensional reduction. A symmetry reversing the first three spacelike coordinates in d dimensions does the same for the reduced four-dimensional theory. The d -dimensional symmetries P_3 , T_d , C_d , Γ transform the internal components of the vielbein and the metric trivially. They cannot be broken by spontaneous compactification. [The vacuum value $e^a_\alpha(y)$ is invariant.] If the symmetry P_3 can be defined in d -dimensions, the reduced four-dimensional theory necessarily has a conserved symmetry of the class $\tilde{P}$. If our model of a spinor and pure gravity

admits the symmetry T_d , some symmetry of the class $\tilde{T}$ leaves the four-dimensional action invariant.

This gives severe constraints on acceptable d-dimensional models of pure gravity coupled to spinors. Experimentally, we know that the weak interactions neither conserve $\bar{P}$ nor $\overline{CP}$. The only conserved discrete symmetry for spinors which is not contained in Lorentz or gauge transformations is $\overline{PCT}$. No conserved symmetry of the class $\tilde{P}$ or $\tilde{T}$ is consistent with experiment. There is no symmetry of the class $\tilde{I}$ transforming the known spinors non-trivially. (This would imply a non-trivial charge conjugation or a degeneracy of the fermionic mass spectrum.) Reading table 2 one finds: gravity theories in $d = 3, 4, 5, 7, 8, 9$ reduce to four-dimensional theories with discrete symmetries of the type $\tilde{P}$ and $\tilde{T}$. For $d = 5, 6, 7$ the transformation $\mathcal{C}_d$ is a symmetry of the class $\tilde{I}$ transforming all four-dimensional spinors non-trivially. The only acceptable possibility in this context is a Majorana-Weyl spinor for $d = 2 \bmod 8$. It is interesting to remember that $d = 2 \bmod 4$ is the only case where the existence of chiral four-dimensional fermions cannot be excluded by using merely the properties of the Clifford algebra³⁾. A Majorana-Weyl spinor in $d = 2 \bmod 8$ induces a non-trivial four-dimensional symmetry of the class $\tilde{PT}$. We will show that this corresponds to the transformation $\overline{PCT}$.

Let us now look at the symmetries $\mathcal{C}_d$, P_3 and T_d in detail. The transformation $\mathcal{C}_d$ induces a four-dimensional transformation of the class $\tilde{I}$:

$$\begin{aligned}\mathcal{C}_d \psi_i(y) \varphi_j(x) &= B_D^{-1} \psi_i(y)^* B_+^{-1} \varphi_j(x)^* = \psi_i(y) B_{ij} B_+^{-1} \varphi_j(x)^* \\ \mathcal{C}_d \varphi_i(x) &= B_{ij} B_+^{-1} \varphi_j(x)^*\end{aligned}\tag{51}$$

(For $d = 5, 6, 8, 9 \bmod 8$, B_+^{-1} has to be replaced by $\bar{\gamma}^5 B_+^{-1}$.) If $\psi_i(y)$ corresponds to a D-dimensional Majorana spinor, one has $B_{ij} = \delta_{ij}$ and $\mathcal{C}_d$ corresponds either to $\bar{C}$ or to $\bar{\gamma}^5 \bar{C}$. Otherwise, B_{ij} involves an exchange of different four-dimensional spinors ϕ_i and ϕ_j with equal mass and belonging to representations of the gauge group which are complex conjugate to each other. $\mathcal{C}_d$ transforms ϕ_i into the charge conjugate of ϕ_j which belongs to the same representation as ϕ_i . Therefore $\mathcal{C}_d$ has

not the properties of usual four-dimensional charge conjugation which changes the sign of all abelian charges and transforms every representation into its complex conjugate. $\mathcal{C}_d$ can be interpreted as a four-dimensional charge conjugation only for real and pseudoreal representations. For complex representations $\mathcal{C}_d$ exchanges two different spinors with degenerate mass. Clearly, such a symmetry does not exist in nature and we must restrict acceptable models to d-dimensional Majorana spinors for which $\mathcal{C}_d$ is trivial. In this case we have four-dimensional Majorana spinors for the case $B_{ij} = \delta_{ij}$. Otherwise, the spinor $B_{ij} B_4^{-1} \phi_j(x)$ is identified with $\phi_i(x)$ and $\phi_j(x)$ is not an independent spinor anymore.

A reflection of the first three spacelike coordinates reads in four dimensions:

$$\begin{aligned}
 P_3 \psi_i(y) \varphi_i(x) &= \eta_3 \bar{\Gamma}_3 \psi_i(y) \gamma^0 \varphi_i(x') = \eta_3 \psi_i(y) \Gamma_3 \gamma^0 \varphi_i(x') \\
 P_3 \varphi_i(x) &= \eta_3 \Gamma_3 \gamma^0 \varphi_i(x') \quad \text{for } d \text{ even} \\
 P_3 \psi_i(y) \varphi_i(x) &= -\eta_3 \psi_i(y) \bar{\gamma}^5 \gamma^0 \varphi_i(x') \\
 P_3 \varphi_i(x) &= -\eta_3 \bar{\gamma}^5 \gamma^0 \varphi_i(x') \quad \text{for } d \text{ odd}
 \end{aligned} \tag{52}$$

The phase η_3 can be found in table 1. For massless four-dimensional spinors Γ_{ij} can be chosen diagonal: $\Gamma_{ij} = \delta_{ij}$. For massive spinors Γ_{ij} transforms between two different spinors with equal mass (up to an irrelevant minus sign). Clearly, a symmetry of the type P_3 is not observed in nature and we have to start with a Weyl spinor in d dimensions for which P_3 is not defined. Since for Weyl spinors in $d = 4 \bmod 4$ the symmetry $\mathcal{C}_d P_3$ is defined, we again end with an unacceptable symmetry of the class $\tilde{P}$ in four dimensions. Only Weyl spinors in $d = 2 \bmod 4$ are left as possible candidates for theories of pure gravity without some kind of parity conservation in four dimensions.

Time reversal in d dimensions reduces to a symmetry of class $\tilde{T}$ in four dimensions:

$$\begin{aligned}
 T_d \psi_i(y) \varphi_i(x) &= \eta_T \bar{\Gamma}_5 B_d^{-1} \psi_i(y)^* \gamma^0 \bar{\gamma}^5 B_4^{-1} \varphi_i^*(-x) = \eta_T \psi_i(y) B_{ij} \Gamma_{j5} \bar{T} \varphi_j(x) \\
 T_d \varphi_i(x) &= \eta_T B_{ij} \Gamma_{j5} \bar{T} \varphi_j(x) \quad \text{for } d \text{ even} \\
 T_d \psi_i(y) \varphi_i(x) &= \eta_T B_d^{-1} \psi_i(y)^* \gamma^0 \bar{\gamma}^5 B_4^{-1} \varphi_i^*(-x) = \eta_T \psi_i(y) B_{ij} \bar{T} \varphi_j(x) \quad (53) \\
 T_d \varphi_i(x) &= \eta_T B_{ij} \bar{T} \varphi_j(x) \quad \text{for } d \text{ odd}
 \end{aligned}$$

For $d = 3, 6, 7, 8 \bmod 8$ one should replace in eq. (53) B_4^{-1} by $\bar{\gamma}^5 B_4^{-1}$ and $\bar{T}$ by $\bar{\gamma}^5 \bar{T}$.

We note that T_d can correspond to a usual four-dimensional time reversal only for real or pseudoreal representations when B_{ij} is diagonal. For complex representations B_{ij} transforms a spinor into an other spinor belonging to the complex conjugate representation. T_d behaves rather like a four-dimensional $\overline{CT}$ transformation.

Indeed, if a gauge group admits complex representations [this is the case of electromagnetism or $SU(3)$], we can find another classification of discrete symmetries in four dimensions: symmetries which transform every field belonging to a complex representation into a field belonging to the complex conjugate representation are in the class $\overset{\circ}{C}$. This class includes four-dimensional charge conjugation, $\overline{CP}$ and $\overline{PCT}$. Symmetries which do not change the representation with respect to the gauge group belong to the class $\overset{\circ}{I}$. This classification can be combined with the classes $\tilde{I}$, $\tilde{P}$, $\tilde{T}$, and $\tilde{PT}$. For example, $\overline{CP}$ transformations belong to the classes $\tilde{P}$ and $\tilde{C}$. The symmetry T_d is in the class $\tilde{T}\overset{\circ}{C}$, whereas P_3 and $\mathcal{C}_d$ belong to $\tilde{P}\overset{\circ}{I}$ and $\tilde{I}\overset{\circ}{I}$.

The transformation $P_3 T_d$ belongs to the class $\tilde{PT}\overset{\circ}{C}$ which is the class of $\overline{PCT}$ transformations. We note that this transformation is a symmetry of the action for all $\text{Spin}(d-1, 1)$ representations in arbitrary dimensions. (For Weyl spinors in $d = 4 \bmod 4$ P_3 should be replaced by $\mathcal{C}_d P_3$.) One has

$$\begin{aligned}
 P_3 T_d \psi_i(y) \varphi_i(x) &= \eta_3 \eta_T B_d^{-1} \psi_i(y)^* \bar{\gamma}^5 B_4^{-1} \varphi_i^*(-x) \\
 &= -i \eta_3 \eta_T \psi_i(y) B_{ij} \bar{P} \bar{T} \varphi_j(x) \quad (54)
 \end{aligned}$$

$$P_3 T_d \varphi_i(x) = -i \eta_3 \eta_T B_{ij} \bar{P} \bar{T} \varphi_j(x)$$

where B_4^{-1} and $\bar{T}$ should be replaced by $\bar{\gamma}^5 B_4^{-1}$ and $\bar{\gamma}^5 \bar{T}$ for $d = 5, 6, 8, 9 \bmod 8$. For d -dimensional Majorana spinors one has $B_{ij} \phi_j \equiv B_4^{-1} \phi_i = \bar{C} \phi_i$ (with $\bar{\gamma}^5$ multiplication for $d = 5, 6, 8, 9 \bmod 8$) and therefore

$$P_3 T_d \phi_i(x) = -i \gamma_3 \gamma_T \bar{P} \bar{C} \bar{T} \phi_i(x) \quad (55)$$

which identifies $P_3 T_d$ with the only allowed non-trivial discrete symmetry for four-dimensional spinors.

4. REFLECTIONS OF INTERNAL COORDINATES

In this section we will discuss reflections of coordinates of the internal space. Consider a reflection in one of the internal coordinates which we will denote by ϕ :

$$I_\phi : \quad \begin{aligned} \varphi &\rightarrow -\varphi \\ y^\alpha &\rightarrow y^\alpha, \quad y^\alpha \neq \varphi \end{aligned} \quad (56)$$

I_ϕ is always a symmetry of the action. Suppose for a moment that we can find a corresponding reflection of Lorentz indices I_1^ℓ and that the ground state is invariant with respect to the combined transformation

$$I_1 = I_1^\ell I_\phi \quad (57)$$

Then I_1 will result in a non-trivial transformation of the class $\tilde{T}$ in four dimensions. Furthermore, the transformation $P_3 I_1$ is always a symmetry of the action since it is a combination of symmetries contained in gen_d . An I_1 symmetric ground state induces a four-dimensional theory with a $\tilde{P}$ symmetry. Thus even theories with Weyl spinors in $2 \bmod 4$ dimensions, where neither I_1^ℓ nor $C_d I_1^\ell$ can be defined, lead to unacceptable four-dimensional theories if the ground state is symmetric under I_1 . (Remember that the action of I_1 on the vielbein can always be defined.) Similar arguments hold for the transformation $I_1 T_d$ or $I_1 C_d T_d$. The reduced theory corresponding to an I_1 symmetric ground state has a conserved symmetry of the class $\tilde{T}$.

However, there is an important difference between symmetries of the type I_1 and symmetries discussed in Section 3: symmetries like P_3 , T_d and C_d are never broken by a ground state of the form $M^4 \times K^D$. On the contrary, reflections of

internal coordinates may be symmetries of the action, but not symmetries of the ground state K^D . They can be broken by spontaneous compactification! And we will show that only internal spaces which break all possible non-trivial internal coordinate reflections can lead to realistic models without unobserved non-trivial symmetries of the classes $\tilde{P}$, $\tilde{T}$, or $\tilde{I}$ in four dimensions!

The action of internal coordinate reflections is closely related to the gauge group of K^D , since the gauge group is a subgroup of general coordinate and Lorentz transformations in D dimensions (gen_D). Before proceeding to a more detailed description of symmetries of the type I_1 , we briefly have to review how gauge transformations act on the vielbein and the spinor. An infinitesimal gauge transformation reads (with a, b "internal" Lorentz indices)

$$\delta_G \hat{e}_\alpha^a = -\theta^z (S_z)_{\alpha b}^{\beta a} \hat{e}_\beta^b$$

$$(S_z)_{\alpha b}^{\beta a} = \delta_b^a \partial_\alpha K_z^\beta + \delta_b^\beta \delta_\alpha^\gamma K_z^\gamma + \delta_\alpha^{\beta\gamma} \hat{e}_{z\gamma}^a \quad (58)$$

$$\delta_G \psi = -\theta^z S_z \psi$$

$$S_z = K_z^\alpha \partial_\alpha - \frac{1}{2} \hat{e}_{zab} \Sigma^{ab} \quad (59)$$

$\theta_z(x)$ are the infinitesimal transformation parameters (their number is equal to the dimension of the gauge group G) and $K_z^\alpha(y)$ are the corresponding Killing vectors. The antisymmetric functions $\hat{e}_{zab}(y) \equiv -\hat{e}_{zba}$ are given by

$$\hat{e}_{zab}(y) = \hat{e}_a^\alpha \hat{e}_{\beta b} \partial_\alpha K_z^\beta + \hat{e}_a^\alpha K_z^\beta \partial_\beta \hat{e}_{\alpha b} \quad (60)$$

and by definition of the Killing vectors the vielbein $\hat{e}_\alpha^a(y)$ and the metric $\hat{g}_{\alpha\beta}(y)$ of the ground state K^D are left invariant by the transformations (58). If the D -dimensional spinors $\psi_{nH}(y)$ are the components of a representation H of G , they transform according to

$$S_z \psi_{nH}(y) = \psi_{n'H}(y) (T_z)_{n'n} \quad (61)$$

with constant matrices T_z . Both S_z and T_z fulfil the commutation rules of the group

$$[S_x, S_y] = f_{xyz} S_z \quad (62)$$

$$[T_x, T_y] = f_{xyz} T_z$$

The generators T_z describe the action of the gauge group on the four-dimensional spinors $\phi_{nH}(x)$

$$\begin{aligned} \delta_\theta \psi(y,x) &= \delta_\theta \psi_{nH\epsilon}(y) \phi_{nH\epsilon}(x) \\ &= -\theta^z S_z \psi_{nH\epsilon}(y) \phi_{nH\epsilon}(x) = \psi_{nH\epsilon}(y) \left\{ -\theta^z (T_z)_{nn'} \phi_{n'H\epsilon}(x) \right\} \end{aligned} \quad (63)$$

[Summation over all representations H and all components n is implied. The index k counts how many times a representation H is contained in $\psi(y,x)$.]

Suppose that K^D admits a continuous symmetry group G and consider an arbitrary U(1) subgroup of G. We parametrize the orbit of this U(1) subgroup by a coordinate ϕ . [This is always possible except for the points where U(1) acts trivially.] Since this orbit is closed, ϕ is periodic $-\pi \leq \phi < \pi$ with $-\pi$ and π identified. We note that one U(1) rotation may induce several rotations in ϕ (but at least one) and if ϵ parametrizes the U(1) transformations, one has $\phi = n\epsilon$ with n integer. The Killing vector K_1 corresponding to this transformation is given by

$$\begin{aligned} K_1^\phi &= n \\ K_1^\alpha &= 0, \quad \alpha \neq \phi \end{aligned} \quad (64)$$

The U(1) invariance of the ground state metric

$$\delta_{U(1)} \overset{\circ}{g}_{\mu\nu} = -\epsilon n \partial_\phi \overset{\circ}{g}_{\mu\nu} = 0 \quad (65)$$

implies that $\overset{\circ}{g}_{\alpha\beta}$ is ϕ independent. As well, we can always find a choice of $\overset{\circ}{e}_\alpha^a$ which is ϕ independent. The U(1) invariance of $\overset{\circ}{e}_\alpha^a$ implies that every ϕ dependence of $\overset{\circ}{e}_\alpha^a$ is a pure Lorentz transformation and it can therefore be absorbed by a suitable chosen ϕ dependent Lorentz transformation, corresponding to a special choice of the veilbein basis. For this choice the function $\overset{\circ}{e}_{1ab}$ [eq. (60)] vanishes identically. Supplementing the reflection $I_\phi(\phi \rightarrow -\phi)$ by a reflection of say, the first internal Lorentz index I_1^ρ the internal veilbein components transform under $I_1 = I_\phi I_1^\rho$ as

$$\begin{aligned} I_1: \quad \hat{e}_1^\phi(\phi, y^\mu, x) &\rightarrow \hat{e}_1^\phi(-\phi, y^\mu, x) \\ \hat{e}_1^\alpha(\phi, y^\mu, x) &\rightarrow -\hat{e}_1^\alpha(-\phi, y^\mu, x) \\ \hat{e}_a^\phi(\phi, y^\mu, x) &\rightarrow -\hat{e}_a^\phi(-\phi, y^\mu, x) \\ \hat{e}_a^\alpha(\phi, y^\mu, x) &\rightarrow \hat{e}_a^\alpha(-\phi, y^\mu, x), \quad \alpha \neq \phi, \alpha \neq 1 \end{aligned} \quad (66)$$

The ground state conserves the symmetry I_1 if $\tilde{e}_1^\alpha = 0, \tilde{e}_a^\phi = 0$ and correspondingly I_1 symmetry is broken only by non-zero non-diagonal elements $g_{\alpha\phi}$ of the metric.

Next we have to investigate the action of I_1 on the generators of the gauge group. We split the generators iS_z into the ℓ hermitian generators S_i^Q of the Cartan-subalgebra (ℓ is the rank of the group) and the remaining ladder operators $S_k^\pm$. The commutation relations in this basis have the standard form:

$$[S_i^Q, S_j^Q] = 0 \quad (67)$$

$$[S_i^Q, S_\pm^\pm] = \alpha_i^{(\pm)} S_\pm^\pm, \quad [S_i^Q, S_\pm^\mp] = -\alpha_i^{(\mp)} S_\pm^\mp \quad (68)$$

[The vectors $(\alpha_1 \dots \alpha_\ell)$ are called the roots of the algebra.] We take the generator $iS_1 = i\partial_\phi$ to be one of the generators of the Cartan subalgebra

$$iS_1 = i\partial_\phi \equiv S_{i_0}^Q \quad (69)$$

Using coordinates on the orbit of the abelian group corresponding to the Cartan subalgebra, the other generators $S_j^Q, j \neq i_0$ can be written in the form

$$S_j^Q = i(K_j^\alpha \partial_\alpha + K_j^\varphi \partial_\varphi), \quad \alpha \neq \varphi, j \neq i_0 \quad (70)$$

where $[S_{i_0}^Q, S_j^Q] = 0$ implies

$$\partial_\varphi K_j^\alpha = 0, \quad \partial_\varphi K_j^\varphi = 0 \quad (71)$$

Furthermore, we choose the S_i^Q to be orthogonal with $K_j^\phi = 0$. One immediately finds that I_1 reverses the sign of $S_{i_0}^Q$ and leaves all other S_j^Q invariant:

$$\begin{aligned} I_1^{-1} S_{i_0}^Q I_1 &= -S_{i_0}^Q \\ I_1^{-1} S_j^Q I_1 &= S_j^Q, \quad j \neq i_0 \end{aligned} \quad (72)$$

Thus for any given system of weights, I_1 reverses the sign of the i_0 components of all weights.

We define the reflected generators

$$\begin{aligned} \tilde{S}_i^Q &= I_1^{-1} S_i^Q I_1 \\ \tilde{S}_\pm^\pm &= I_1^{-1} S_\pm^\pm I_1 \end{aligned} \quad (73)$$

If all reflected generators $\tilde{S}_k^\pm$ leave the ground state K^D invariant, the reflection I_1 corresponds to an automorphism of the algebra:

$$\begin{aligned} \tilde{S}_{i_0}^\alpha &= -S_{i_0}^\alpha, & \tilde{S}_f^\alpha &= \tilde{S}_f^\alpha, & f \neq i_0 \\ \tilde{S}_z^\pm &= a_{zz'}^\pm S_{z'}^\pm + a_{\bar{z}z'}^\pm S_{\bar{z}'}^\pm \end{aligned} \quad (74)$$

If K^D is I_1 -invariant this is necessarily the case, but I_1 could be an automorphism even for spaces which are not I_1 invariant. A necessary condition for I_1 being an automorphism can be derived from the commutation relations:

$$[\tilde{S}_i^\alpha, \tilde{S}_z^\pm] = \pm \alpha_i^{(\pm)} \tilde{S}_z^\pm \quad (75)$$

$$[S_i^\alpha, \tilde{S}_z^\pm] = \pm \tilde{\alpha}_i^{(\pm)} \tilde{S}_z^\pm \quad (76)$$

The vectors $\tilde{\alpha}_i^{(k)}$ are obtained from $\alpha_i^{(k)}$ by reversing the sign of all i_0 components [compare eq. (72)]. If I_1 corresponds to an automorphism of the algebra, all $\tilde{\alpha}_i^{(k)}$ must again be roots of the algebra. The automorphism induced by I_1 is then a reflection of the root system on the i_0 -axis.

An automorphism I_1 transforms any irreducible representation of the gauge group into an irreducible representation of the same dimension. In particular, if the spinors $\psi_{nH}(y)$ are the components of an irreducible representation H , then the reflected spinors

$$\psi_{nH}^I(\varphi, y^\alpha) = I_1 \psi_{nH}(\varphi, y^\alpha) = \eta_a \Gamma^a(\bar{D}) \psi_{nH}(-\varphi, y^\alpha) \quad (77)$$

are components of an irreducible representation of the same dimension:

$$S_z \psi_{nH}^I = \psi_{n'H}^I (T_z^I)_{n'n} \quad (78)$$

For real or pseudoreal representations, the automorphism can be written in the form

$$T_z^I = A T_z A^{-1} \quad (79)$$

where the matrix A represents the reflection on the i_0 axis of the weight system. I_1 is called an inner automorphism if it can be represented as an element of the gauge group. In this case I_1 transforms every representation into itself. Otherwise, I_1 is an outer automorphism which transforms complex representations into the complex conjugate ones or exchanges inequivalent real or pseudoreal representations with the same dimension or else transforms some representations into equivalent, but different representations.

We can define similar coordinate reflections for the coordinates corresponding to the orbits of the other generators of the Cartan subalgebra. Consider the system of the resulting ℓ coordinate reflections $I_1, I_2, \dots, I_\ell$. Every two of these reflections anticommute for spinors and commute when applied on tensors. Every product of these transformations is again a discrete transformation. In particular, we define the reflection of all coordinates corresponding to orbits of the Cartan subalgebra

$$J_\ell = I_1 I_2 \dots I_\ell \quad (80)$$

The answer to the question which one of these discrete symmetries can correspond to an automorphism of the algebra is well known in mathematics (compare table 17 in ref. 9). The transformation J_ℓ can be an automorphism for all groups. It changes the sign of all weights and transforms S_k^+ into S_k^- . J_ℓ transforms every complex representation into its complex conjugate and every real or pseudoreal representation into itself. Thus J_ℓ results in a four-dimensional transformation in the class $\overset{\circ}{C}$. Reflection of only one coordinate cannot be an automorphism for the groups G_2, E_6, E_7 , and E_8 . The weight system obtained from the action of I_1 does not correspond to a representation of the group in this case. In particular, this implies that spaces admitting a symmetry G_2, E_6, E_7 , or E_8 cannot be symmetric with respect to a reflection of one coordinate. For $SU(n)$ or $SO(2n+1)$ the reflection I_1 can be an automorphism which transforms every representation into an equivalent representation (not necessarily the same). The four-dimensional symmetry belongs to the class $\overset{\circ}{I}$. For $SO(2n)$ symmetric spaces I_1 exchanges inequivalent real or pseudoreal representations for $2n = 4 \bmod 4$ (it belongs to the class $\overset{\circ}{I}$) and transforms complex representations into complex conjugate ones for $2n = 2 \bmod 4$. In the second case I_1 corresponds again to a four-dimensional charge conjugation of the class $\overset{\circ}{C}$. For $SO(n)$ symmetric spaces all products of reflections I_i can correspond to automorphisms of the algebra. An even number of reflections transforms all representations into equivalent ones. An odd number of reflections belongs always to the class $\overset{\circ}{C}$ for $SO(4n+2)$ symmetric spaces. $SO(n)$ symmetric internal spaces invariant with respect to an odd number of reflections

are in fact $O(n)$ symmetric. We remember that an odd number of reflections is not contained in gen_D . Since the gauge group is a subgroup of gen_D , an odd number of reflections can only correspond to an outer automorphism. Conversely, an even number of reflections may be an inner automorphism and therefore harmless.

As a consequence, no internal space admitting a symmetry of the type J_ℓ can yield a realistic theory since it induces a non-trivial conserved four-dimensional charge conjugation. Similarly, internal spaces admitting a symmetry I_1 or an odd number of coordinate reflections of this type induce a non-trivial symmetry of the class $\tilde{II}^0$ or $\tilde{IC}^0$ in four dimensions which has not been observed in nature. If I_1 or J_ℓ are not symmetries of the action, the transformations $I_1 P_3$ or $J_\ell P_3$ are and they are equally unacceptable. The vielbein and the metric of the ground state of a realistic Kalura-Klein theory must break both I_1 and J_ℓ ! There is, however, a difference between exceptional and orthogonal or unitary groups: the I_1 transform of an E_6 symmetric ground state metric is of course invariant under E_6 transformations corresponding to the generators $\tilde{S}$. However, it is not invariant with respect to the original E_6 transformations obtained by applying $S_k^\pm$. On the contrary, if I_1 is an automorphism, the I_1 transform of an $SO(n)$ or $SU(n)$ symmetric metric is again invariant under the transformation group constructed with the $S_k^\pm$. Two different ground states with the same $SO(n)$ symmetry may be transformed into each other by the action of I_1 . Nevertheless, any given ground state breaks I_1 in this case. To resume, the internal space K^D should not be invariant under an odd number of reflections of internal coordinates. This leads to four-dimensional symmetries of the class $\tilde{P}$ and $\tilde{T}$. A symmetry under an even number of internal coordinate reflections is only acceptable if this is a gauge transformation. Otherwise, unobserved non-trivial symmetries of the class $\tilde{II}^0$ or $\tilde{IC}^0$ are induced in four dimensions.

Let us look closer at the action of the reflections $I_1 P_3$, J_ℓ (for ℓ even) or $J_\ell P_3$ (for ℓ odd) within the reduced four-dimensional theory. These transformations are always defined, even if the symmetry is broken by spontaneous compactification. We will assume from now on that J_ℓ and I_1 correspond to automorphisms of the algebra. One has

$$\begin{aligned}
 I_1 \psi_{nH\bar{z}}(\varphi, y^d) \varphi_{nH\bar{z}}(x) &= \gamma_a \Gamma^a \bar{\Gamma}_D \psi_{nH\bar{z}}(-\varphi, y^d) \varphi_{nH\bar{z}}(x) \\
 &= \psi_{nH\bar{z}}(\varphi, y^d) I_{nn'HH'\bar{z}\bar{z}'} \varphi_{n'H'\bar{z}'}(x) \quad \text{for } d \text{ even} \\
 I_1 \psi_{nH\bar{z}}(\varphi, y^d) \varphi_{nH\bar{z}}(x) &= \gamma_a \Gamma^a \psi_{nH\bar{z}}(-\varphi, y^d) \bar{\gamma}^5 \varphi_{nH\bar{z}}(x) \\
 &= \psi_{nH\bar{z}}(\varphi, y^d) I_{nn'HH'\bar{z}\bar{z}'} \bar{\gamma}^5 \varphi_{n'H'\bar{z}'}(x) \quad \text{for } d \text{ odd}
 \end{aligned} \tag{81}$$

The constant coefficients $I_{nn'HH'kk'}$ characterize the four-dimensional action of I_1 and I is proportional to $\delta_{HH'}$, if the automorphism carries every representation into an equivalent one.

We may characterize the ψ_{nHk} as eigenvalues of the mass matrix $\psi_{nHk} = \psi_{nHmj}$

$$\hat{P}^{\alpha} \hat{\delta}_\alpha \psi_{nHmj}(y) = M \psi_{nHmj}(y) \tag{82}$$

The mass operator $\hat{P}^{\alpha} \hat{\delta}_\alpha$ is the Dirac operator on the internal manifold corresponding to the ground state vielbein $\hat{e}_\alpha^a$. Since I_1 (or $I_1 P_3$) leaves the kinetic term of the higher dimensional spinor invariant, it commutes with $\hat{P}^{\alpha} \hat{\delta}_\alpha$ for D even and anticommutes for D odd. If the ground state does not break I_1 , these commutation relations can be transferred to the mass operator. The transformation I_1 carries every spinor into a spinor with equal mass². For D odd, the sign of the mass (which has no physical relevance) changes and I_1 necessarily transforms two different massive spinors into each other.

If the ground state is not I_1 invariant, one finds

$$[I_1, \hat{P}^{\alpha} \hat{\delta}_\alpha] = (\Gamma^a \Sigma^{ab} \hat{\omega}_{1ab} + 2\Gamma^a \Sigma^{1a} \hat{\omega}_{a1a} - 2\Gamma^1 \hat{e}_1^a \hat{\delta}_a - 2\Gamma^a \hat{e}_a^1 \hat{\delta}_1) I_1 \tag{83}$$

with $a, b \neq 1$ and $\hat{\omega}$ the spin connection formed with the ground state vielbein.

For d odd, one has to replace the commutator on the left-hand side by an anti-commutator and to multiply the right-hand side with a minus sign. Obviously,

the right-hand side of eq. (83) vanishes for $\hat{e}_1^\alpha = 0$, $\hat{e}_a^\phi = 0$. If spontaneous

compactification breaks I_1 , this reflection transforms a spinor into a spinor of different mass. We note that the I_1 transform of a mass eigenstate is not

necessarily an eigenstate of the mass operator again. Nevertheless, it would be interesting to investigate cases where mass eigenstates are transformed into each other and I_1 has some features of a ladder operator within the mass spectrum! Of course, all the discussion can be applied similarly to the reflection J_ℓ and to other similar reflections.

To conclude this subsection, we show the strength of the constraints on acceptable internal spaces K^D by illustrating the consequences of internal coordinate reflections on the possible existence of four-dimensional chiral fermions. Here we define chiral spinors with respect to the unification group. Any realistic theory must have such chiral spinors. [The existence of chiral spinors is connected with topological properties of K^D ⁴⁾.] We could imagine that the unified theory has a non-trivial $\bar{P}$, $\overline{CP}$, or $\bar{T}$ invariance which could be spontaneously broken. Spontaneous breakdown of symmetries can equivalently be described by deformations of the internal space. Consider the space $\overset{\circ}{K}^D$ which is symmetric under the unification group. $\overset{\circ}{K}^D$ is not the ground state, but it is related to the ground state K^D by a continuous deformation. (This means that K^D and $\overset{\circ}{K}^D$ have the same topology and differentiable structure.) Then we can perform dimensional reduction on $\overset{\circ}{K}^D$ as well as on K^D and the fact that $\overset{\circ}{K}^D$ is not the ground state will show up in the feature that the four-dimensional theory obtained from dimensional reduction on $\overset{\circ}{K}^D$ admits spontaneous symmetry breakdown. The vacuum expectation values of scalars reflect the difference in the metric of K^D and $\overset{\circ}{K}^D$. Now we could imagine that $\overset{\circ}{K}^D$ has some non-trivial discrete symmetries. Of course, all non-trivial reflection symmetries of $\overset{\circ}{K}^D$ must finally be broken by the deformations to K^D . This corresponds to spontaneous breakdown of four-dimensional discrete symmetries. The restrictions on $\overset{\circ}{K}^D$ from discrete symmetries could be less severe than the restrictions on the ground state K^D . However, $\overset{\circ}{K}^D$ should admit chiral four-dimensional spinors. Chiral fermions can *never* be produced by spontaneous symmetry breakdown ⁴⁾ and must exist in the unified theory. This will give us severe restrictions on possible reflection symmetries of $\overset{\circ}{K}^D$.

We remember that D-dimensional versions of the symmetries I_1 , J_ℓ , etc. can always be defined and reflect themselves in the mass spectrum coming from the expansion of a D-dimensional *Dirac* spinor. Let us assume for a moment that I_1 and J_ℓ are symmetries of the internal space $\overset{\circ}{K}^D$ corresponding to the unbroken unified theory. For D even, we have the commutation relations

$$\begin{aligned} \{I_1, \bar{P}\} &= 0 \\ J_\ell \bar{P} &= (-1)^\ell \bar{P} J_\ell \end{aligned} \tag{84}$$

An odd number of reflections transforms a representation contained in $\psi^+(y)$ into a representation of the same dimension contained in $\psi^-(y)$. For d-dimensional Weyl spinors this corresponds to a flip in four-dimensional helicity. An even number of internal coordinate reflections does not change four-dimensional helicity. For ℓ even, J_ℓ is a non-trivial four-dimensional charge conjugation and chiral spinors are excluded. For ℓ odd, $P_3 J_\ell$ is a four-dimensional CP conjugation which does not exclude chiral spinors. If I_1 is in the class $\overset{\circ}{C}$, which is the case for a reflection of coordinates in the orbit of orthogonal groups $SO(4n+2)$, $P_3 I_1$ is again a four-dimensional CP transformation consistent with chiral spinors. For all other cases I_1 is in the class $\overset{\circ}{I}$ and the $\bar{P}$ transformation corresponding to $P_3 I_1$ forbids the existence of chiral spinors for a ground state $\overset{\circ}{K}^D$. We note that D-dimensional charge conjugation also transforms all complex representations into complex conjugate ones, but may have different commutation properties with $\bar{P}_D$ compared to J_ℓ ($e_D \bar{P}_D = (-1)^{D/2} \bar{P}_D e_D$). We illustrate the action of J_ℓ and e_D on complex spinors graphically in fig. 1. Only fig. 1d is consistent with chiral spinors. Chiral fermions are consistent with an odd number of internal reflections only if such a transformation belongs to the class $\overset{\circ}{C}$. This may be relevant if the symmetry group of $\overset{\circ}{K}^D$ is a direct product of $SO(10)$ [or $SU(5)$] and a generation group G_F . If G_F has only real or pseudoreal representations like $SO(3)$, $SO(4)$, or $SO(5)$, a reflection of one of the coordinates characterizing the orbit of G_F carries a left-handed four-dimensional spinor into a right-handed one of the same representation. Chiral spinors are not consistent with such a reflection symmetry of $\overset{\circ}{K}^D$!

5. SUPPLEMENTARY FIELDS IN HIGHER DIMENSIONS

In this short section we discuss possible modifications of our investigation of discrete symmetries if one introduces supplementary fields in d dimensions. This is typically the case for higher dimensional supergravity theories. For example, supergravity in eleven dimensions needs an antisymmetric third rank tensor $\hat{A}_{\mu\nu\rho}$. The ground state is now characterized by vacuum expectation values of the metric and four-dimensional Lorentz scalars. We therefore can restrict our discussion to the effects of supplementary tensor fields in higher dimensions.

Two modifications of our previous discussion for pure gravity are possible: first, some of the d -dimensional discrete symmetries (compare table 2) may be violated by the couplings of d -dimensional tensors. An arbitrary tensor $\hat{A}_{\mu\nu\dots\sigma}$ has the following transformation properties under symmetries of the type T_d , P_3 , I_1 :

$$\begin{aligned} T_d \hat{A}_{\mu\nu\dots\sigma}(x^0, x^i, \varphi, y^\alpha) &= \epsilon_T \hat{A}_{\mu\nu\dots\sigma}^T(-x^0, x^i, \varphi, y^\alpha) \\ P_3 \hat{A}_{\mu\nu\dots\sigma}(x^0, x^i, \varphi, y^\alpha) &= \epsilon_{P_3} \hat{A}_{\mu\nu\dots\sigma}^{P_3}(x^0, -x^i, \varphi, y^\alpha) \\ I_1 \hat{A}_{\mu\nu\dots\sigma}(x^0, x^i, \varphi, y^\alpha) &= \epsilon_{I_1} \hat{A}_{\mu\nu\dots\sigma}^{I_1}(x^0, x^i, -\varphi, y^\alpha) \end{aligned} \quad (85)$$

where $\hat{A}^T$ is obtained from $\hat{A}$ by multiplication with $(-1)^{J_0}$ with J_0 the number of indices taking the value zero, and similar rules for $\hat{A}^{P_3}$ and $\hat{A}^{I_1}$. A possible strategy is to determine the values ϵ_T , ϵ_{P_3} and ϵ_{I_1} , which can take the values plus or minus one, from the coupling of the tensor to spinors and then to check if the action is invariant. Just as in four dimensions, scalars and pseudoscalars, for example, should not be mixed and couple to different fermion bilinears. We note that I_1 can be obtained from P_3 by a gen_d transformation fixing $\epsilon_{I_1} = \epsilon_{P_3}$. The transformation properties of some spinor bilinears under the symmetries T_d , P_3 , $\mathcal{C}_d$ and PCT are given in table 4 for arbitrary dimension d . With our phase conventions of Section 2 all bilinears of the type $\tilde{\psi}\Gamma^{(k)}\psi$ with $\tilde{\psi} = \psi^T C_d$ have the same transformation properties as bilinears of the type $\overline{\psi}\Gamma^{(k)}\psi$.

Similar as in four dimensions, the symmetry $P_d \mathcal{C}_d T_d$ is always a symmetry of the action for d even. For Majorana spinors $\mathcal{C}_d$ is trivial and $P_d \mathcal{C}_d T_d$ reduces to a $P_d T_d$ symmetry. The symmetry $P_3 T_d$ is then obtained by a gen_d transformation and is again a symmetry of the action. It corresponds to four-dimensional $\overline{PCT}$ symmetry. It cannot be broken by spontaneous compactification. For odd dimensions, $P_3 \mathcal{C}_d T_d$ is always a symmetry of the action (this is not true for $P_d \mathcal{C}_d T_d$) and yields the usual four-dimensional $\overline{PCT}$ transformation if $\mathcal{C}_d$ is trivial.

A second modification is related to further possibilities of breakdown of discrete symmetries by spontaneous compactification. Four-dimensional pseudo-scalars may acquire vacuum expectation values and break $\overline{P}$. For example, there could be vacuum expectation values of some antisymmetric four-index tensor proportional to the totally antisymmetric four-dimensional ϵ -tensor. Such an expectation value is Lorentz invariant but breaks $\overline{T}$ and $\overline{P}$ invariance. Actually, the action of eleven dimensional supergravity is T_d invariant. The corresponding four-dimensional $\overline{CT}$ invariance can only be broken by a non-zero vacuum expectation value of the Minkowski components of the field strength corresponding to the three-index tensor $\hat{A}_{\mu\nu\rho}$:

$$\begin{aligned} \hat{F}_{\mu\nu\rho\sigma} &= D_{[\mu} \hat{A}_{\nu\rho\sigma]} \\ \langle \hat{F}_{\mu\nu\rho\sigma} \rangle &\sim \epsilon_{\mu\nu\rho\sigma} \end{aligned} \quad (86)$$

which characterizes certain solutions⁶⁾ of the field equations.

6. CONCLUSIONS AND DISCUSSION

We have seen that discrete symmetries put many constraints on possible realistic Kaluza-Klein theories. The first class of constraints limits the possible d -dimensional theories to start with. These come from symmetries like d -dimensional time reversal, reflection of an odd number of coordinates and charge conjugation. T_d , P_3 , and $\mathcal{C}_d$ cannot be broken by the ground state of pure gravity theories. Therefore theories of pure higher dimensional gravity should not be invariant under T_d , P_3 , $\mathcal{C}_d P_3$, or a non-trivial $\mathcal{C}_d$. Otherwise, unobserved discrete symmetries are induced in four dimensions. This limits the possible theories of pure gravity coupled to a spinor to the case of a Majorana-Weyl spinor in $d = 2 \bmod 8$ dimensions.

The constraints on the d-dimensional theory become less stringent if supplementary fields are introduced. This can allow the breaking of T_d and P_3 invariance in the purely bosonic part of the effective d-dimensional action. If this is not the case, like in $d = 11$ supergravity which has a T_d and P_3 invariance of the action, there is still a way out by breaking T_d and P_3 through spontaneous compactification. This requires a totally antisymmetric tensor of rank four or the field strength of a totally antisymmetric tensor of rank three to acquire a vacuum expectation value $\sim \epsilon_{\mu\nu\rho\sigma}$ for its Minkowski indices. This is exactly what happens for some solutions of spontaneous compactification in $d = 11$ supergravity and parity and CT are therefore broken without a d-dimensional Weyl constraint.

For the derivation of these results we have assumed that the pure gravitational part of the d-dimensional action (omitting the spinors) is invariant under time reversal and reflection of an odd number of spacelike coordinates. This requires that the function $f(\hat{g}_{\mu\nu})$ in the *effective* action (including gen_d invariant contributions from quantum fluctuations) contains only scalars but no pseudoscalars. However, in four dimensions a pseudoscalar can be constructed from the metric tensor and its derivatives:

$$R_{\mu\nu\sigma\lambda} \tilde{R}^{\mu\nu\sigma\lambda} = \frac{1}{2} R_{\mu\nu\sigma\lambda} R_{\rho\tau}{}^{\sigma\lambda} \epsilon^{\mu\nu\rho\tau} (-g)^{-\frac{1}{2}} \quad (87)$$

It violates $\bar{P}$, $\bar{CP}$, and $\bar{T}$. Its space time integral corresponds to a topological charge, the Pontryagin number

$$P = \frac{1}{16\pi^2} \int d^4x (-g)^{\frac{1}{2}} R_{\mu\nu\sigma\lambda} \tilde{R}^{\mu\nu\sigma\lambda} \quad (88)$$

This is very similar to a possible $\bar{CP}$ violating pseudoscalar contribution to the effective action from gauge fields

$$\theta F_{\mu\nu} \tilde{F}^{\mu\nu} = \theta F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} (-g)^{-\frac{1}{2}} \quad (89)$$

which is at the origin of the strong CP problem. Similarly, a term $\epsilon R_{\mu\nu\sigma\lambda} \tilde{R}^{\mu\nu\sigma\lambda}$ in the *effective* action in four dimensions gives rise to a gravitational CP violation.

We do not know in which dimensions a pseudoscalar can be constructed from the metric and its derivatives. It would require a non-vanishing contraction of products of the curvature tensor and Ricci tensor and their covariant derivatives with an odd number of totally antisymmetric d-dimensional ε -tensors. The generalized construction

$$\varepsilon^{\mu_1 \mu_2 \mu_3 \dots \mu_d} \hat{R}^{\mu_1 \mu_2} \hat{R}^{\mu_3 \mu_4} \dots \hat{R}^{\mu_{d-1} \mu_d} (-\hat{g})^{-\frac{1}{2}} \quad (90)$$

is only possible in $d = 4 \bmod 4$ dimensions. The absence of a gravitational pseudoscalar in two dimensions suggests that gravitational pseudoscalars could be absent for $d = 2 \bmod 4$ -- this remains to be shown (or found in the mathematical literature). Even if gravitational pseudoscalars are possible, they involve many derivatives if the number of dimensions is sufficiently high and their contribution to the reduced four-dimensional theory may be very small.

The question of gravitational pseudoscalars in higher dimensions is strongly related to the strong CP problem in four dimensions. Suppose that there exists no such gravitational pseudoscalar in d dimensions. The gen_d invariant part of the effective action without spinors is then T_d and P_3 invariant. If the ground state is a direct product of Minkowski space and an internal space K^D these symmetries cannot be broken by spontaneous compactification in purely gravitational theories. The purely bosonic part of the effective action in four dimensions must then be P and CP invariant up to gen_d violating contributions from quantum fluctuations involving length scales larger than the scale of spontaneous compactification. No $F_{\mu\nu} \tilde{F}^{\mu\nu}$ term can be obtained from dimensional reduction in this case. This means that the θ parameter vanishes at the scale of spontaneous compactification. A non-zero value at present energies can only be induced by quantum fluctuations, and it must be due to the CP violation in the *fermionic* sector. This is calculable and turns out¹⁰⁾ to be typically very small. We also have to check that the determinant of the quark mass matrix obtained from dimensional reduction is real. This can be done by applying the general results of ref. 3 to the case where the isometry group of the ground state is the unbroken gauge group $SU(3)_C \times U(1)_{em}$. It follows that the quark mass matrix is hermitean and the determinant is therefore real. CP violation in the fermionic sector occurs at most

in weak interactions, but not through the couplings to the massless fields. As a consequence, no strong CP problem exists in those theories of pure gravity coupled to spinors which either do not admit gravitational pseudoscalars at all or where the contribution of such pseudoscalars to the effective action in four dimensions is sufficiently small. Kaluza-Klein theories may be an elegant new way to get rid of one of our long-standing problems!

The second class of constraints obtained from discrete symmetries concerns the structure of the internal space K^D . Non-trivial symmetries from reflections of internal coordinates (which cannot be obtained by gauge transformations) lead to unacceptable discrete symmetries in four dimensions. Some of these internal reflections lead to four-dimensional charge conjugation. As has been noted before, this gives a simple physical (geometrical) interpretation to the four-dimensional charge conjugation. The internal space K^D of the ground state of a realistic Kaluza-Klein theory should not have any non-trivial reflection symmetry!

Reflection symmetries of K^D and of the space K^{2D} corresponding to the unbroken unified gauge symmetry are closely related to the possible existence of chiral spinors. Chiral spinors are perhaps the most serious problem for realistic Kaluza-Klein theories at this time. Possible candidates of theories admitting chiral spinors get more and more restricted -- which gives some hope of finding them if they exist.

Pure considerations of the Clifford algebra in d and D dimensions restrict D to be $2 \bmod 4$ for all theories where the gauge group is a subgroup of $\text{gen}_D^{3)}$. [This is the case for the typical Kaluza-Klein theories where the gauge group is related to isometries of the metric of K^D . On the contrary, hidden local symmetries like $SU(8)$ in $d = 11$ supergravity are not of this type.] Indeed, a Weyl constraint is needed in d dimensions in order to make possible a difference between left-handed and right-handed spinors in four dimensions. This difference is related to possible differences in the mass and representation spectrum of the inequivalent D dimensional Weyl spinors. Chiral four-dimensional fermions require the D dimensional Weyl spinors to belong to complex representations of the D -dimensional Lorentz group $SO(D)$. This restricts D to $2 \bmod 4$.

The existence of a non-trivial d -dimensional charge conjugation leads to an unwanted doubling of the four-dimensional mass spectrum for fermions. $\mathcal{C}_d$ cannot be broken by any ground state of pure gravity. This requires d -dimensional Majorana spinors and restricts D to be $6 \bmod 8$ for realistic theories.

Due to a theorem by Lichnerowicz¹¹⁾ we know that no massless spinors exist if the curvature scalar of K^D is negative or zero everywhere on K^D but not identically zero (in conventions where the curvature scalar for spheres is negative). Doing dimensional reduction by steps³⁾ this can be generalized to a space K^D which is a direct product of several internal spaces $K^D = K_1 \times K_2 \times \dots \times K_n$. None of the spaces K_i should have negative scalar curvature in order for K^D to admit chiral fermions. The argument can be extended to the case where one of the K_i is a one dimensional circle S^1 . Starting with an even dimensional theory and first reducing to a $d-1$ dimensional theory by integrating out the S^1 coordinate leaves us with a massless fermion in $d-1$ dimensions. However, this fermion is neutral with respect to the $U(1)$ group of the circle. Further dimensional reduction to an effective four dimensional theory cannot give chiral fermions since we now start with a vector-like theory in odd dimensions. An internal space $K^D = K^{D-1} \times S^1$ admits no chiral fermions.

These arguments become very powerful constraints when combined with topological considerations⁴⁾. The number of chiral fermions is not changed by continuous deformations of K^D leaving the gauge group invariant. Chiral fermions must be in complex representations with respect to at least one $U(1)$ subgroup of the isometries of K^D . (In a realistic theory this $U(1)$ subgroup can be identified with hypercharge.) This requires an internal space K^D which cannot be changed continuously, while keeping the $U(1)$ subgroup unbroken, into one of the spaces discussed in the precedent paragraph. Especially, the $U(1)$ group must have fixpoints [points in K^D where the orbit of this $U(1)$ is shrunk to a point]. Otherwise, K^D has topology $K^{D-1} \times S^1$ and does not admit chiral fermions. The number of chiral fermions becomes to a large extent a property of the topology and differentiable structure of K^D !

As we have shown in this paper, another class of constraints is related to possible discrete symmetries of K^{OD} . Chiral fermions are excluded if K^{OD} admits a symmetry of one of the following two types

- i) an odd number of coordinate reflections corresponding to an automorphism of the algebra which does not transform complex representations of the gauge group into complex conjugate ones;
- ii) an even number of coordinate reflections which corresponds to an automorphism exchanging complex representations with their conjugates.

Combined with index considerations, this argument is again very restrictive. In realistic gauge theories there are often vector-like $U(1)$ subgroups commuting with the chiral $U(1)$ subgroup. [In the standard model, the third component of weak isospin generates always a vector-like $U(1)$ group since $SU(2)$ has not complex representations and it commutes with the chiral hypercharge generator.] If K^{OD} has a reflection symmetry with respect to the coordinate ϕ_v parametrizing the orbit of the vector-like $U(1)$ group, it cannot exhibit chiral fermions.

The metric of any internal space which is invariant under both a vector-like and a commuting "chiral" $U(1)$ group must be independent of both ϕ_v and ϕ_c . [ϕ_c is the coordinate parametrizing the orbit of the "chiral" $U(1)$ group]. As discussed in section 4, breaking of the discrete symmetry $\phi_v \rightarrow -\phi_v$ is then only possible by non-diagonal terms in the metric. However, if it is possible to make these non-diagonal terms vanish by a continuous deformation not breaking the "chiral" $U(1)$, then such a space does again not admit chiral fermions. In pure gravity coupled to spinors it seems to be rather difficult to avoid this possibility and to construct spaces which actually admit chiral fermions.

Note added

After completing this work, I learned from E. Witten that there exists a proof by Atiyah and Hirzebruch¹²⁾ that the relevant index counting the number of chiral zero modes of the Dirac operator on the internal space vanishes for *all* compact

internal spaces with a positive (or negative) definite signature of the metric. Thus, no chiral fermions can be obtained from dimensional reduction of standard gravity in higher dimensions coupled to a generalized spin $\frac{1}{2}$ fermion, if the ground state has the usual direct product form $M^4 \times K^D$.

I am very grateful to E. Witten for his hospitality in Princeton and for explaining me his own proof of the above theorem.

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Table 1

Discrete transformations of spinors in arbitrary dimensions d
 $\psi(x) \rightarrow U\psi(x')$ or $\psi(x) \rightarrow U\psi^*(x')$: complex conjugation is symbolized by (*)

$d \pmod{8}$	P_d	C_d	T_d	$(PCT)_d$	I_a	P_3
2	$i\gamma^0$	$B_1^{-1} (*)$	$-i\gamma^0 \bar{\gamma} B_1^{-1} (*)$	$\bar{\gamma}$	$i\gamma^a \bar{\gamma}$	$\gamma^0 \bar{\Gamma}_D$
3	$i\gamma^0$	$B_1^{-1} (*)$	$-i\gamma^0 B_1^{-1} (*)$	1	$i\gamma^a$	$-\gamma^0 \bar{\gamma}^5$
4	$i\gamma^0$	$B_1^{-1} (*)$	$\gamma^0 \bar{\gamma} B_1^{-1} (*)$	$i\bar{\gamma}$	$\gamma^a \bar{\gamma}$	$i\gamma^0 \bar{\Gamma}_D$
5	γ^0	$B_2^{-1} (*)$	$\gamma^0 B_2^{-1} (*)$	-1	γ^a	$-i\gamma^0 \bar{\gamma}^5$
6	γ^0	$B_2^{-1} (*)$	$\gamma^0 \bar{\gamma} B_2^{-1} (*)$	$-\bar{\gamma}$	$\gamma^a \bar{\gamma}$	$i\gamma^0 \bar{\Gamma}_D$
7	$i\gamma^0$	$B_1^{-1} (*)$	$-i\gamma^0 B_1^{-1} (*)$	-1	$i\gamma^a$	$-\gamma^0 \bar{\gamma}^5$
8	γ^0	$B_2^{-1} (*)$	$-i\gamma^0 \bar{\gamma} B_2^{-1} (*)$	$-i\bar{\gamma}$	$i\gamma^a \bar{\gamma}$	$\gamma^0 \bar{\Gamma}_D$
9	γ^0	$B_2^{-1} (*)$	$\gamma^0 B_2^{-1} (*)$	1	γ^a	$-i\gamma^0 \bar{\gamma}^5$

Table 2

Possible discrete symmetries consistent with
Majorana and/or Weyl constraints

$d \pmod{8}$	Type of spinor	Discrete symmetries
2	M-W	PT
3	M	P, T, I_a , P_3
4	W	CP, T, CI_a , CP_3
	M	P, T, I_a , P_3
5	D	C, P, T, I_a , P_3
6	W	C, PT
7	D	C, P, T, I_a , P_3
8	W	CP, T, CI_a , CP_3
	M	P, T, I_a , P_3
9	M	P, T, I_a , P_3

D: Dirac spinor, M: Majorana spinor, W: Weyl spinor,
M-W: Majorana-Weyl spinor.

Table 3

Some elements of the four-dimensional symmetry classes $\tilde{I}$, $\tilde{P}$, $\tilde{T}$, and $\tilde{PT}$

$\tilde{I}$:	$(1, \mathcal{C}_d, I_a, \Gamma, \bar{C}, \dots)$
$\tilde{P}$:	$(P_3, \mathcal{C}_d P_3, I_x, P_3 I_a, P_d, \bar{P}, \gamma^5 \bar{P} \dots)$
$\tilde{T}$:	$(T_d, \mathcal{C}_d T_d, I_a T_d, \bar{T}, \overline{CT}, \dots)$
$\tilde{PT}$:	$(P_3 T_d, P_d T_d, (PCT)_d, \overline{PT}, \overline{PCT}, \dots)$

Table 4

Transformation properties of fermion bilinears under discrete transformations in arbitrary dimensions
(for example $\mathcal{C}_d: \bar{\psi}_\gamma \hat{m} \psi \rightarrow \epsilon_c \bar{\psi}_\gamma \hat{m} \psi = -\bar{\psi}_\gamma \hat{m} \psi$)

Bilinear	ϵ_P	ϵ_T	ϵ_c	ϵ_{PCT}
$\bar{\psi} \psi$	1	$(-1)^{d+1} \alpha_B$	$-\alpha_B$	$(-1)^d$
$\bar{\psi}_\gamma \hat{m} \psi$	1	-1	-1	1
$\bar{\psi} \Sigma \hat{m} \psi$	1	$(-1)^d \alpha_B$	α_B	$(-1)^d$
$\bar{\psi}_\gamma [\hat{m} \hat{n} \hat{p}] \psi$	1	1	1	1
$\bar{\psi} \gamma \psi$	-1	$(-1)^{d/2} \alpha_B$	$(-1)^{(d-2)/2} \alpha_B$	1
$\bar{\psi}_\gamma \hat{m} \gamma \psi$	-1	$(-1)^{(d-2)/2}$	$(-1)^{d/2}$	1

$$\alpha_B = \begin{cases} -1 & \text{for } B_d = B_1 \\ +1 & \text{for } B_d = B_2. \end{cases}$$

ℓ even

$$\underline{D = 4 \bmod 4}$$

$$\begin{array}{ccc} & J_\ell, e_D & \\ C_+ & \longleftrightarrow & \bar{C}_+ \\ & J_\ell, e_D & \\ \bar{C}_- & \longleftrightarrow & C_- \end{array}$$

Fig. 1a

ℓ even

$$\underline{D = 2 \bmod 4}$$

$$\begin{array}{ccccc} & & J_\ell & & \\ C_+ & \longleftrightarrow & \bar{C}_+ & & \\ e_D \downarrow & & J_\ell & & \downarrow e_D \\ \bar{C}_- & \longleftrightarrow & C_- & & \end{array}$$

Fig. 1b

ℓ odd

$$\underline{D = 4 \bmod 4}$$

$$\begin{array}{ccc} & e_D & \\ C_+ & \longleftrightarrow & \bar{C}_+ \\ J_\ell \uparrow & & \uparrow J_\ell \\ & e_D & \\ \bar{C}_- & \longleftrightarrow & C_- \end{array}$$

Fig. 1c

ℓ odd

$$\underline{D = 2 \bmod 4}$$

$$\begin{array}{ccc} C_+ & & \bar{C}_+ \\ J_\ell, e_D \uparrow & & \uparrow J_\ell, e_D \\ \bar{C}_- & & C_- \end{array}$$

Fig. 1d

Fig. 1 Action of e_D and J_ℓ on D -dimensional Weyl spinors belonging to a complex representation of the isometry group. C_+ denotes a complex representation in $\psi^+(y)$.