UNIVERSALITY OF THE CONTINUUM LIMIT OF LATTICE QUANTUM THEORIES

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A B S T R A C T

The lattice approximation of the naïve continuum action in quantum mechanics or in field theory is not uniquely determined. We investigate to what extent corrections to the lattice action, which vanish in the naïve continuum limit, affect the continuum limit when taking quantum fluctuations into account. In the quantum mechanical case, modifications of the lattice action may induce non-trivial corrections to the potential of the system and thereby change the structure of the theory completely. We verify this statement analytically as well as numerically by performing a Monte Carlo simulation. In the field theoretical case we argue that the lattice corrections considered do not affect the physics of the continuum limit, at least not for asymptotically free gauge field theories. In four dimensions, one might encounter finite renormalization of CP violating terms.

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1 INTRODUCTION

The lattice regularization of field theories as introduced by Wilson^[1] provides for a non-perturbative treatment of QCD (for recent reviews see e.g. Ref. 2). In the case of the SU(2) gauge group it was observed by Creutz^[3] that the string tension with the Wilson action^[1] scales, thereby providing evidence for confinement of static quarks. The scaling behaviour of the string tension has been confirmed for the SU(2) group^[4-6] as well as for the more realistic SU(3) case^[7-12].

In these considerations the expectation value of a gauge-invariant observable, O , is given by

$$\langle O \rangle = Z^{-1} \int \prod_G dU(L) \exp(-S[U_p]) O(U(L)) , \quad (1.1)$$

where $U(L)$ are the link variables associated with the gauge group G and U_p the plaquette variables. Z is the normalizing partition function. The lattice action $S[U_p]$ is constructed in such a way that in the naive continuum limit, i.e. when the lattice unit $a \rightarrow 0$, one recovers the tree level action. Such a procedure does not determine the lattice action uniquely. As a matter of fact such a freedom in the choice of the lattice action provides means for studying the universality of the continuum limit taking quantum fluctuations into account. In principle, of course, it could be that physical quantities could depend on the actual short distance structure of the theory, i.e. on details of the lattice action, a point which has recently been stressed by 't Hooft^[13].

Modifications of the lattice action can certainly change the structure of the theory away from the continuum limit. The third order Gross-Witten phase transition^[14] in the large N limit of two-dimensional lattice QCD can e.g. be avoided by changing the lattice action^[15-16]. The question we are addressing, however, is whether such lattice modifications affect the physical continuum limit.

In the case of four-dimensional lattice gauge theories the question of universality has been studied for a class of lattice action modifications. All these have been constructed with the motivation to accelerate the approach to the continuum theory (for a recent suggestion see Ref.17). In Ref.6 the string tension was measured for the Manton action^[18] as well as for the non-Abelian generalization of the Villain action^[19], i.e.

the heat-kernel action^[16], and the results for the Λ -scale parameters extracted were compared with the exact weak-coupling results^[20]. Even though the Monte Carlo data did not agree very well with the analytical results it is believed that the inclusion of higher order terms in the lattice β -function will improve the situation (see e.g. Ref.21). Measurements of the deconfining temperature for these actions^[22] furthermore support the universality of the continuum limit.

In the case of the mixed action

$$S[U_P] = \beta_F/2 \sum_P \text{Tr} U_P + \beta_A/4 \sum_P [\text{Tr} U_P]^2, \quad (1.2)$$

studied in Ref.23 the situation is different. Here one finds roughly deviations up to 400 percent depending on β_A/β_F when comparing the analytical results with the MC-data. Recently it has been pointed out by Makeenko et al.^[24] and by Grossman and Samuel^[25] that a non-perturbative large N correction approach improves the agreement, i.e. universality seems also to be valid for the mixed action (1.2). Recently this idea was applied to Manton's action with similar result^[26]. A Langevin equation approach to the study of the large-distance correlation functions for an $SU(2)$ lattice gauge theory has provided further evidence for physical quantities being independent of the specific choice of the lattice action in the continuum limit^[27].

In this paper we consider yet another type of lattice action modification which corresponds to adding to a given lattice action terms which vanish in the naive continuum limit. We shall see in the quantum mechanical case that corrections brought about by quantum fluctuations may actually change the physics of the continuum limit drastically, inducing in general changes of the potential of the system under consideration. For some lattice action modifications we show that finite renormalizations of the mass and frequency parameters are induced whereas the structure of the Hamiltonian of the system is preserved. This is verified by Monte Carlo simulations of the quantum mechanical system following the procedure of Creutz and Freedman^[28]. In the same manner we discuss an example where the spectrum of the Hamiltonian in the continuum limit is actually changed completely by the lattice action modification. In the field theoretical case we argue that these kinds of lattice action modifications do not affect the continuum limit - at least not for asymptotically free gauge field theories.

2 LATTICE QUANTUM MECHANICS

We illustrate the lattice action modifications we have in mind by considering, in Euclidean time, a lattice approximation to the partition function Z of a one-dimensional system in quantum mechanics, i.e.

$$Z = \lim_{a \rightarrow 0} \int \dots \int \prod_{\ell} dx_{\ell} \exp(-S[x]) \quad , \quad (2.1)$$

where the lattice action $S[x]$ is given by

$$S[x] \equiv \sum_{\ell} S_{\ell+1,\ell}[x] \quad ,$$

$$S_{\ell+1,\ell} = a \left\{ \frac{1}{2}[(x_{\ell+1} - x_{\ell})/a]^2 + V(x_{\ell}) \right\} \quad . \quad (2.2)$$

Here $V(x)$ is the potential and the units have been chosen such that the mass of the particle is one. Now we consider the replacement

$$S_{\ell+1,\ell}[x] \rightarrow S_{\ell+1,\ell}[x] + 2g S_{\ell+1,\ell}^2[x] \quad , \quad (2.3)$$

which clearly has the same, naive, continuum limit as the action (2.2) since one expects that $S_{\ell+1,\ell}[x] = O(a)$.

Taking into account quantum fluctuations, however, we must remember Itô's rule^[29], i.e. $x_{\ell+1} - x_{\ell} = O(\sqrt{a})$, for the Wiener measure (see also Ref.30 for an extension of the Itô calculus). Thus the additional term in the lattice action in (2.3) actually contains terms which are of the same order in a as compared to the terms in the original action (2.2). The question now is whether we may renormalize away all effects of these extra terms when we approach the continuum limit taking quantum fluctuations into account.

To answer this question let us first consider the harmonic oscillator where $V(x) = \omega^2 x^2/2$ and thus

$$S_{\ell+1,\ell}[x] = 1/2 [(x_{\ell+1}-x_{\ell})^2/a + g (x_{\ell+1}-x_{\ell})^4/a^2 + a\omega^2 x_{\ell}^2] \\ + g\omega^2 x_{\ell}^2 (x_{\ell+1}-x_{\ell})^2 + ga^2\omega^4 x_{\ell}^4/2 . \quad (2.4)$$

In the spirit of the construction of the Wiener measure^[29] we include the dominant terms in a , i.e. the $O(1)$ -terms when we respect $x_{\ell+1}-x_{\ell}=O(\sqrt{a})$, into a redefinition of the measure. The lower order terms in a will be treated in the conventional way with regard to the redefined measure (for an alternative approach see the Appendix). We therefore consider the expression

$$M[j] = \lim_{a \rightarrow 0} \int \dots \int \prod_{\ell} dx_{\ell} / N(a,g) \exp(-\sum_{\ell} (S_{\ell+1,\ell}^0[x] + ij_{\ell}(x_{\ell+1}-x_{\ell}))) , \quad (2.5)$$

where

$$S_{\ell+1,\ell}^0[x] = [(x_{\ell+1}-x_{\ell})^2/a + g (x_{\ell+1}-x_{\ell})^4/a^2]/2 . \quad (2.6)$$

and $N(a,g)$ is a normalization factor chosen in such a way that $M[0]=1$. After a change of integration variables we obtain

$$M[j] = \lim_{a \rightarrow 0} \prod_{\ell} [1 - a j_{\ell}^2 s(g)/2 + O(a^2)] \\ = \exp(- s(g)/2 \int j^2(t) dt) , \quad (2.7)$$

where we have introduced the function

$$s(g) = \int_{-\infty}^{\infty} dx x^2 \exp(-x^2/2-gx^4/2) / \int_{-\infty}^{\infty} dx \exp(-x^2/2-gx^4/2) . \quad (2.8)$$

The equations (2.4) and (2.6) together with Itô's rule then yield for the lattice action (2.3) the following effective form

$$S[x] = \sum_{\ell} [(x_{\ell+1} - x_{\ell})^2 / 2as(g) + a\omega^2(1+2gs(g))x_{\ell}^2 / 2] , \quad (2.9)$$

to be used in the partition function (2.1). Clearly (2.9) corresponds to finite mass and frequency renormalizations. As a matter of fact, this holds for arbitrary potentials (see the appendix).

In the example above the effective action of the system has the same form as the original action. As the following example shows, in general the form of the action is changed. Instead of (2.3) we consider the lattice action

$$S_{\ell+1,\ell}[x] \rightarrow S_{\ell+1,\ell}[x] + g (x_{\ell+1} - x_{\ell})^4 x_{\ell}^4 / a , \quad (2.10)$$

where the additional terms in the action also vanishes in the naive continuum limit. An extension^[30] of Itô's rule mentioned above then leads to the following effective action (see also the appendix)

$$S[x] = \sum_{\ell} [(x_{\ell+1} - x_{\ell})^2 / 2a + a\omega^2 x_{\ell}^2 / 2 + 3agx_{\ell}^4] . \quad (2.11)$$

Here an anharmonic term is induced; part of the action that formally vanishes gives rise to a non-vanishing potential (such a phenomenon has been observed in the case of singular perturbations^[31]). In other words, the path integral representation of quantum theory as defined by a lattice discretization is, apart from the conventional operator ordering problems, in general ambiguous (for a related discussion see Ref.32).

We have applied the Monte Carlo simulation technique, as described e.g. by Creutz and Freedman^[28], in order to see how local deformations of the lattice action (eqs.(2.3) and (2.10)) numerically exhibit themselves as we approach the continuum limit.

Let us first consider the harmonic oscillator (2.4). Let $\omega(g)$ denote the renormalized frequency as derived from the effective action (2.9), then (for $a \rightarrow 0$)

$$\omega(g) / \omega \rightarrow [s(g)(1+2gs(g))]^{1/2} = 0.940 \text{ for } g=0.05, \quad (2.12)$$

where, of course, $\omega(0)=\omega$. With $\omega=1$ and $g=0.05$ we have found it possible, with reliable accuracy, to extract from a Monte Carlo simulation $\omega(g)/\omega$. For lattice unit values $a = 1.0, 0.5$ and 0.25 we use time lattice sizes 200, 400 and 800 respectively. The frequency $\omega(g)$ is obtained by extracting the slope of the correlation function $\ln \langle x_i x_{i+j} \rangle$ which, in the case of the harmonic oscillator, should behave as $-aj\omega(g)$. The mean values are extracted from 4000 Monte Carlo updates of all sites at each value of a .

In fig. 1 we exhibit the ratio $\rho(\omega) \equiv \omega(g)/\omega$ and its approach to the expected result for decreasing a . We also show how the expectation values $\langle |x| \rangle$, $\langle x^2 \rangle$ and $\langle x^4 \rangle$ approach their expected values as the lattice spacing goes to zero, i.e.

$$\begin{aligned} \langle |x| \rangle(g) / \langle |x| \rangle &\rightarrow [s(g)/(1+2gs(g))]^{1/4} = 0.932 \text{ for } g=0.05, \\ \langle x^2 \rangle(g) / \langle x^2 \rangle &\rightarrow [s(g)/(1+2gs(g))]^{1/2} = 0.869 \text{ for } g=0.05, \\ \langle x^4 \rangle(g) / \langle x^4 \rangle &\rightarrow s(g)/(1+2gs(g)) = 0.756 \text{ for } g=0.05. \end{aligned} \quad (2.13)$$

We have also studied the lattice action (2.10) using the Monte Carlo method. Here we compare the values of $\langle |x| \rangle$, $\langle x^2 \rangle$ and $\langle x^4 \rangle$ for the action (2.10) with the corresponding values for the effective anharmonic action (2.11). Both systems were studied for lattice spacings $a = 1.0, 0.5, 0.25$ and 0.125 for corresponding lattice sizes 200, 400, 800 and 1600. The mean values were measured over 2000 iterations of the whole lattice. The results of this calculation are given in fig.2. The expectation values for the two actions (2.10) and (2.11) clearly approach each other as the lattice spacing goes to zero.

3 FIELD THEORETICAL CONSIDERATIONS.

In order to investigate possible effects on the physical continuum limit of the lattice action modifications considered above, in the field theoretical case, we proceed as follows. We consider the effective action taking the lattice action modification into account and, by dimensional arguments, we extract those primitively divergent one-particle irreducible Feynman diagrams which may survive when we remove the cutoff.

Let us illustrate the procedure by considering a scalar ϕ^4 -coupling lattice action in d-dimensions

$$S_L = \sum_n [a^d S_n + g a^{2d} S_n^2] , \quad (3.1)$$

where we have introduced the notation

$$S_n = \sum_{\mu} [(\phi_{n+\mu} - \phi_n)^2 / 2a^2] + m^2 \phi^2 / 2 + \lambda \phi_n^4 / 4! , \quad (3.2)$$

and n labels the lattice sites and the μ -sum runs over the appropriate directions on the lattice. In order to exhibit those Feynman diagrams which may survive in the continuum limit, the precise form of the cutoff is not essential. Instead of (3.1) we therefore consider an analogous continuum action, S_C , with $\Lambda = 1/a$ being a momentum cutoff i.e.

$$S_C = \int_x [\mathcal{L}_C + g / \Lambda^d \mathcal{L}_C^2] , \quad (3.3)$$

where

$$\mathcal{L}_C = (\partial_{\mu} \phi)^2 / 2 + m^2 \phi^2 / 2 + \lambda \phi^4 / 4! , \quad (3.4)$$

and $\int_x$ stands for the space-time integration. In passing we notice that the Λ -dependence in (3.3) is chosen in such a way that g is a dimensionless parameter. Such a procedure of holding dimensionless coupling constants, as $\Lambda \rightarrow \infty$, has been used by K.G. Wilson^[33] to show that, e.g., a ϕ^4 -coupling leads to anomalous critical exponents if $2 < d < 4$. For a recent MC- study of a scalar ϕ^4 -coupling using Wilson's renormalization scheme see Ref. 34.

In order to investigate possible effects of the g -correction in (3.3) on the effective action we notice that a compensation of the Λ^{-d} factors may arise due to momentum integrations in closed loops i.e. we consider primitively divergent one-particle irreducible diagrams only. The degree of divergence will then be computed by a dimensional analysis, as stated above.

The dimension of a term in the effective action is, at most, equal to the sum of the dimensions of the contributing vertices. This is so because a contraction does not increase the dimension of a given vertex. We therefore consider a vertex Γ_{1234} , constructed out of the following (number of) vertices: $(n_1) \lambda \phi^4$, $(n_2) g \Lambda^{-d} (\phi_{,\mu}^2)^2$, $(n_3) g \lambda \Lambda^{-d} \phi_{,\mu}^2 \phi^4$, $(n_4) g \Lambda^{-d} \lambda^2 (\phi^4)^2$. Other possible vertices as derived from the action (3.3), which involve the bare mass m , will not be considered since they do not affect our analysis. Since the dimension of scalar field in d dimensions is equal to $\dim[\phi] = d/2 - 1$, we find that $\dim[\Gamma_{1234}] = (d-4) \times (n_1 + n_3 + 2n_4)$. Here we apply Wilson's renormalization scheme^[33] i.e. we choose the bare parameters in such a way that $\dim[m] = \dim[\lambda] = 0$. Schematically, we can also write the vertex Γ_{1234} in the following form

$$\Gamma_{1234} = \int_x \Lambda^a (\partial_\mu \phi)^{2b} \phi^{2c-2b} f(m^2/\Lambda^2, \partial^2/\Lambda^2) , \quad (3.5)$$

where a is the largest possible exponent of Λ i.e. $f(0,0) \neq 0$. By a dimensional argument (3.5) leads to the following relation

$$a = (d-4) \times (n_1 + n_3 + 2n_4) + d - 2b - (d-2)c , \quad (3.6)$$

Here we make the observation that if the function f contains $\ln(\partial^2/\Lambda^2)$ -terms we may get a momentum dependence in these renormalizations and hence a possible contribution to, say, the

β -function that does not vanish in the limit $\Lambda \rightarrow \infty$. In the lattice gauge theory case, to be discussed below, this would lead to a change of the scaling behaviour and consequently it could affect the physical string tension. Such a situation would endanger the entire meaning of e.g. lattice QCD. A logarithmic momentum dependence in f could arise when we encounter several-loop integrals and when a subdiagram, without g -vertices, is logarithmically divergent. On the one-loop level this cannot happen. We therefore reach the conclusion that for asymptotically free field theories the β -function is not changed when we approach the continuum limit. For gauge theories the string tension will therefore be unaffected by the lattice modification of the form (3.3). [As we see, there is a close, formal, relationship with the lattice action modification we consider in the present paper and "irrelevant terms", such as next to nearest neighbour interactions, in analogous spin systems (cf e.g. Ref. 35)].

If $d=4$ we then have that $a=2(2-b-c)$ which leads to the following renormalizations. If $a=1$ and $b=0$, $c=1$ we get a mass renormalization, $a=0$ and $b=1$, $c=1$ leads to a ϕ -renormalization and if $a=0$ and $b=0$, $c=2$ we get a λ -renormalization.

In three dimensions we derive, using the Equation (3.6), the same set of induced renormalizations as in the four-dimensional case discussed above.

In two dimensions we make use of the fact that the $\phi^{2(c-b)}$ factor in (3.5) must arise from a n_1 , n_3 or a n_4 vertex. At least two of the lines of a vertex are used to form loops and hence $0 \leq c-b \leq n_1+2n_3+3n_4$ and (3.6) lead to

$$a = 2[1 - b - (n_1 + n_3 + 2n_4)] \quad , \quad (3.7)$$

If $n_1=n_3=n_4=0$ and $c=b=1$ we obtain a ϕ -renormalization. Next we consider the case when $n_1+n_3+2n_4=1$ and hence $b=0$. If $c=2$ then $n_3=1$ and $n_1=n_4=0$ and we obtain a ϕ -renormalization. If $c=1$ then $n_1=1$ or 2 or $n_3=1$, which leads to ϕ -renormalization.

In one dimension we use the inequality $0 \leq c-b \leq n_1+n_3+3n_4$ together with (3.6) to obtain

$$a = 1 - 2b + c - 3(n_1 + n_3 + 2n_4) \leq 1 - b - 2n_1 - n_3 - 3n_4, \quad (3.8)$$

If $n_1=n_3=n_4=0$, and hence $c=b=1$, we obtain a ϕ -renormalization. If $n_3=1$, and $n_1=n_4=0$ and $b=0$ then $c=2$ leads to a λ -renormalization. In this case $c=1$ leads to $a=-1$ and hence there is no contribution in the limit when $\Lambda \rightarrow \infty$. For $d=1$ these considerations, of course, agree with the explicit calculations presented in section 2. In $d=1$ we can also consider lattice action modifications of the form (2.10). In this case g has, however, a positive dimension and the dimensional arguments above do not apply. Depending on the dimensions of g we can obtain new interactions, as was shown in Section 2. In quantum mechanics we therefore do not have universality of the continuum limit as defined above. In dimensions equal to or larger than two we may consider an analogous action modification with a vertex of the form $\int_x g [\phi_{,\mu}^2 \phi^2] m^4$, where we choose g in such a way that $\dim[g] = 0$. Following the line of reasoning above one can then verify that no renormalizations are induced by presence of the additional vertex on the lattice when approaching the continuum limit. If $\dim[g]$ is negative we may encounter new couplings but such terms would spoil the perturbative renormalizability of the theory if they survive in the continuum limit.

Let us, finally, turn to a brief discussion of the gauge field theoretical case. If the gauge coupling, g , is kept dimensionless the Wilson action^[1] takes the form

$$S_L[U_P] = \sum_P 1/g^2 \text{Tr}(2 - U_P - U_P^\dagger), \quad (3.9)$$

which in the naive continuum limit corresponds to the action

$$S_C = \int_x 1/g^2 \Lambda^{d-4} \text{Tr} F_{\mu\nu}^2, \quad (3.10)$$

suppressing group indices. The gauge invariant form of the lattice action modification we have considered above then corresponds to adding a term

$$\Delta S_L = \lambda \sum_P [1/g^2 \text{Tr}(2 - U_P - U_P^\dagger)]^2, \quad (3.11)$$

to (3.9) or, equivalently,

$$\Delta S_C = \lambda \int_x \Lambda^{d-8} 1/g^4 [\text{Tr} F_{\mu\nu}^2]^2, \quad (3.12)$$

to the naive continuum action from (3.10). In (3.11) and (3.12) we have chosen $\dim[\lambda]=0$. We notice that since $\dim[F_{\mu\nu}^2]=4$ all terms in $S_C + \Delta S_C$ (or $S_L + \Delta S_L$) are dimensionless. After a rescaling, $A_\mu \rightarrow g\Lambda^{2-d/2} A_\mu$, we obtain for the Lagrangian density

$$\Delta \mathcal{L}_C = \lambda \Lambda^{-d} [\text{Tr} F_{\mu\nu}^2]^2, \quad (3.13)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g\Lambda^{2-d/2} [A_\mu, A_\nu]. \quad (3.14)$$

After the rescaling, $\dim[A_\mu] = d/2-1$. Schematically, and due to gauge invariance, we consider the effective action

$$\Delta S_C = \int_x \Lambda^a (F_{\mu\nu}^2)^b f(\partial^2/\Lambda^2), \quad (3.15)$$

which should be dimensionless i.e.

$$a = d(1 - b), \quad (3.16)$$

In any dimension we therefore obtain $a = 0$ if $b = 1$. As we have argued above we do not encounter any logarithmic functional dependence in $f(\partial^2/\Lambda^2)$ on the one-loop level. Due to asymptotic freedom we can therefore conclude that the β -function is unchanged in the continuum limit and hence also the string tension.

By inspection, we realize the close connection between $S_L + \Delta S_L$ and the mixed action (1.1). By making use of the background field method^[36] one can then verify^[20,23] that one encounters at most a finite coupling constant renormalization for the lattice action modification (3.11) due to asymptotic freedom.

Finally, we notice that when $d = 4$ we may also consider lattice action modifications which resemble the structure considered in the Equation (2.10) namely

$$\Delta \mathcal{L}_C = \lambda \Lambda^{-m} [\text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}]^n, \quad (3.17)$$

where $\tilde{F}_{\mu\nu}$ is the dual field strength and $4n - m = 4$. The corresponding effective action would have the following schematic form

$$\Delta S_C = \int_x \Lambda^a [\text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}]^2 g(\theta^2/\Lambda^2). \quad (3.18)$$

As above, we would then obtain $a = 4(1-b) = 0$ for $b = 1$. This would then lead to a strong CP-violation which was not present in the original action unless n is odd. Hence, in the presence of a strong CP-violation we can also allow for a finite renormalization of the CP-violation Θ -angle on the lattice^[37].

APPENDIX

Here we present another derivation of (2.9) by considering the Schrödinger equation derived from the path integral (see e.g. Feynman and Hibbs^[38]). Let us work in Euclidean time and consider the wave function development over one lattice unit a , i.e.

$$\psi_{\ell+1}(x_{\ell+1}) = N \int dx_{\ell} \exp(-S_{\ell+1,\ell}) \psi_{\ell}(x_{\ell}) , \quad (\text{A.1})$$

where N is a normalization constant and $S_{\ell+1,\ell}$ is the contribution to the lattice action at the site considered and is defined in (2.2). From (A.1) the Schrödinger equation can be derived in a standard manner^[38].

Let us discuss the effect of a local deformation (2.3) of the lattice action. We first consider the free particle and we always work in one space dimension. Introducing the variable y by $x_{\ell} = x_{\ell+1} + y\sqrt{a}$, we can write

$$\psi_{\ell+1}(x) = N\sqrt{a} \int dy \exp(-y^2/2 - gy^4/2) \psi_{\ell}(x + y\sqrt{a}) . \quad (\text{A.2})$$

If we choose the normalization constant in (A.2) such that for a given g

$$N\sqrt{a} \int dy \exp(-y^2/2 - gy^4/2) \equiv 1 , \quad (\text{A.3})$$

the equation (A.2) leads to

$$\psi_{\ell+1}(x) = \psi_{\ell}(x) + as(g)\psi_{\ell}''(x)/2 + O(a^2) , \quad (\text{A.4})$$

since the term containing the first derivative of the wave function vanishes identically due to the symmetry of the integrand in (A.2). The function $s(g)$ is defined in (2.8). In the limit $a \rightarrow 0$ we obtain a Schrödinger-like equation where the mass now, however, has been renormalized by a factor determined by the function $s(g)$.

We now consider the case with an arbitrary potential $V(x)$, i.e. (A.2) is replaced by

$$\begin{aligned} \psi_{\ell+1}(x) = N\sqrt{a} \int dy \exp(-y^2/2 - aV(x+y\sqrt{a})) \\ \times \exp(-2g(y^2/2 + aV(x+y\sqrt{a}))^2) \psi_{\ell}(x+y\sqrt{a}) . \end{aligned} \quad (A.5)$$

Neglecting terms of order a^2 we then obtain

$$\begin{aligned} \psi_{\ell+1}(x) = \psi_{\ell}(x) + a [s(g)\psi_{\ell}''(x) \\ + (1+2gs(g))V(x)\psi_{\ell}(x)] , \end{aligned} \quad (A.6)$$

from which we now obtain a Schrödinger equation in the limit when the lattice spacing a goes to zero. The $s(g)$ dependent factors in (A.6) may be renormalized away by a change of the time scale and a finite mass renormalization

$$t \rightarrow t/(1+2gs(g)) , \quad (A.7)$$

and

$$m \rightarrow m/(1/s(g)+2g) . \quad (A.8)$$

The effective action (2.11) given in the text due to the spectrum changing modification (2.10) can be derived applying the same technique and the details are left to the reader.

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FIGURE CAPTIONS

- Figure 1: The ratios $\rho(f)=\langle f \rangle(g=0.05)/\langle f \rangle(g=0)$ for $f = \omega, |x|, x^2, x^4$ for different values of $a = 1.0, 0.5, 0.25$ and $\omega = 1$. The arrows indicate the results of an analytical evaluation using the effective action (2.9) given in the text.
- Figure 2: Ratios $\rho(f)=\langle f(a) \rangle/\langle f(b) \rangle$ for $f = |x|, x^2, x^4$ determined for different values of $a = 1.0, 0.5, 0.25, 0.125$ using the action of (a) eq.(2.9) and (b) eq.(2.10) for $g = 0.05$ and $\omega = 1$. The ratios are all expected to approach 1.0 for $a \rightarrow 0$.

FIG. 1

FIG. 2