Ref.TH.3697-CERN

POSSIBLE SYMMETRY BREAKING PATTERNS WITH TOTALLY SYMMETRIC
AND ANTISYMMETRIC REPRESENTATIONS

P. Jetzer

Theoretical Physics
ETH - Honggerberg
CH - 8093 Zürich

and

J.-M. Gérard and D. Wyler
CERN - Geneva

A B S T R A C T

Possible symmetry breaking patterns for $SU(N)$ gauge theories with Higgs scalars in totally antisymmetric and symmetric representations of degree k are discussed by solving the extremum conditions. In all cases considered, the scalar potential has only one local minimum, the other extremal solutions being saddle points. We conjecture that this holds for all k, N such that the symmetry breaking pattern is uniquely determined. According to the sign of the scalar self-interacting couplings, we find $SU(N) \rightarrow SU(k) \times SU(N-k)$ or $SU(k)^{[N/k]}$ (where $[]$ means the integer part) for an antisymmetric tensor ($k > 2$) and $SU(N) \rightarrow SU(N-1)$ or nothing for a symmetric tensor ($k > 2$).

Ref.TH.3697-CERN

September 1983

1. INTRODUCTION

At present, it is widely assumed that the fundamental forces are described by (partly) broken gauge theories. The breaking of original invariance group G is usually achieved by introducing one or several scalar fields (Higgs fields) transforming in a certain way under G . Their self-interaction (potential), non-derivative polynomial terms up to degree four (renormalizability), is chosen in such a way that at its minima, some fields have a non-zero value. This non-zero vacuum expectation value renders some of the gauge fields massive, the remaining massless ones correspond to the unbroken symmetry group. We note that the renormalizability condition (only up to fourth power in fields) leads often to considerable simplifications; and the result cannot be generally used to treat all other potentials.

In constructing models, it is clearly desirable to have rules which prescribe how a given set of scalars break G . But also in itself, this can be a nice mathematical problem.

In an early paper, Li ¹⁾ investigated the breaking due to one irreducible representation of $SU(N)$ and $SO(N)$. He considered the fundamental, the adjoint, and the symmetric or antisymmetric tensors of rank 2. The analysis of Magg and Shafi ²⁾ concerning one adjoint with one fundamental representation in $SU(5)$ has been generalized for $SU(N)$ by Buccella, Ruegg and Savoy ³⁾ and Ruegg ⁴⁾ [see also Ref. 5)], while Mitra ⁶⁾ looked at several adjoints. Buccella, Ruegg and Savoy ⁷⁾ and Buccella and Ruegg ⁸⁾ treated the spinor in $O(10)$ and $O(14)$. Wu ⁹⁾ reconsidered some of the representations in Ref. 1) and extended the analysis to some cases with two different representations. Kim ¹⁰⁾ gave a preliminary treatment of the antisymmetric tensor of degree k . Frampton, Nandi and Scanio ¹¹⁾ and Kalyniak and Ng ¹²⁾ looked at the breaking of $SU(5)$ by one adjoint and one 45-dimensional representation; this latter was considered in detail by Eckert, Gérard, Ruegg and Schücker ¹³⁾. Some of the results in these papers are incomplete; for example not all extrema are considered and not always is the nature of an extremum (minimum / saddle point) determined.

In addition, Michel and Radicati ¹⁴⁾ gave a general theorem of the structure of the minima, while Michel ¹⁵⁾ conjectured a very general criterion for minima or maxima. These results are often useful guides in finding the solutions, but do not replace some tedious calculations. We note here that none of our results contradict Michel's conjecture.

Motivated by extended grand unified groups¹⁶⁾ and also by composite models, we consider in this paper the totally antisymmetric tensor of rank k in $SU(N)$, represented by a one-column Young tableau of length k . These representations have the property that they do not give exotics (color $SU(3)$ sextets, etc.), and give often automatic solutions to anomaly constraints¹⁶⁾. Besides these important physical properties, the relatively simple mathematical structure makes these representations more easily treatable. In this spirit, we have also considered the k symmetric tensor to some extent.

Our method will be very straightforward. We first find a "normal" form for the tensor, e.g. such a form in which as many components as possible are made zero by a suitable unitary transformation. Then we calculate all the possible solutions to the extremum conditions and determine the unbroken gauge group. As it turns out, there are many extrema, but we will see that only a few are actually minima; most are saddle points. In this way, we conjecture the complete solution of the symmetry breaking pattern for any N and k ^{*)}.

In the next section we treat the antisymmetric tensor of rank k of $SU(N)$ completely up to $N = 6$, $k = 3$. The higher N are treated at some length, and a more concise discussion is made for larger N and k ; e.g., we give only some of the possible extrema; but we will argue that the omitted ones are all saddle points. The third section contains a brief treatment of the symmetric tensor; the last section, a summary and a discussion of the results. Three appendices explain some results used in the main text.

2. ANTISYMMETRIC REPRESENTATION OF DEGREE k OF $SU(N)$

We consider the symmetry breaking due to one irreducible antisymmetric tensor of rank k , $k < N$ ($k = N$ is the trivial representation), the completely antisymmetric tensor product of k fundamental representations (a Young tableau with k boxes arranged in one column) with dimension $C_k^N \equiv N! / [(N-k)!k!]$. We denote the elements of this tensor by

$$H_{a_1, a_2, \dots, a_k} \quad a_i = 1 \dots N \quad (1)$$

$$a_i \neq a_j \quad \forall i, j$$

*) We consider always only one irreducible representation.

which satisfy

$$H_{P(a_1 \dots a_k)} = \pi(P) H_{a_1 \dots a_k} \quad (2)$$

where $P(a_1 \dots a_k)$ is a permutation of $(a_1 \dots a_k)$ and $\pi(P)$, its sign.

The complex conjugated fields are

$$(H_{a_1 \dots a_k})^* = H^{a_1 \dots a_k} \quad (3)$$

They transform like an antisymmetric tensor of rank $(N-k)$. Therefore, one can restrict k to be less than or equal to $N/2$, since H and H^* should yield the same symmetry breaking (the potential is Hermitian). The most general renormalizable (e.g., up to fourth power in the fields) $SU(N)$ invariant Higgs potential of the fields $H_{a_1 \dots a_k}$ can be written as

$$\begin{aligned} V = & -\frac{\mu^2}{2} [H^{a_1 \dots a_k} H_{a_1 \dots a_k}] \\ & + \sum_{i=1}^{[k/2]+1} \frac{\lambda_i}{4} [H^{a_1 \dots a_{k-i+1} a_{k-i+2} \dots a_k} H_{a_1 \dots a_{k-i+1} b_{k-i+2} \dots b_k} \\ & H^{b_1 \dots b_{k-i+1} b_{k-i+2} \dots b_k} H_{b_1 \dots b_{k-i+1} a_{k-i+2} \dots a_k}] \\ & + \tilde{V} \end{aligned} \quad (4.a)$$

with μ and λ_i real ($\lambda_1 > 0$). Here

$$\begin{aligned} \tilde{V} = & g_2 \varepsilon^{a_1 \dots a_k a_{k+1} \dots a_{2k}} H_{a_1 \dots a_k} H_{a_{k+1} \dots a_{2k}} \\ & + g_3 \varepsilon^{a_1 \dots a_k a_{k+1} \dots a_{2k} a_{2k+1} \dots a_{3k}} H_{a_1 \dots a_k} H_{a_{k+1} \dots a_{2k}} H_{a_{2k+1} \dots a_{3k}} \\ & + g_4 \varepsilon^{a_1 \dots a_k a_{k+1} \dots a_{2k} a_{2k+1} \dots a_{3k} a_{3k+1} \dots a_{4k}} H_{a_1 \dots a_k} H_{a_{k+1} \dots a_{2k}} H_{a_{2k+1} \dots a_{3k}} H_{a_{3k+1} \dots a_{4k}} \\ & + \text{c.c.} \end{aligned} \quad (4.b)$$

It is clear that the g_i ($i=2,3,4$) vanish unless $N = ik$. Furthermore, $\tilde{V}$ is non-zero only if k is even.

In equation (4.a) $[k/2]$ means the integer part of $k/2$. Indices which occur twice are summed over from 1 to N.

In the following, we will neglect $\tilde{V}^*$). In order to find the minimum of the potential (4.a), we must first solve the extremum condition $\delta V / \delta H_{a_1 \dots a_k} = 0$. The $H_{a_1 \dots a_k}$ are not all independent (e.g., $H_{a_1 a_2 \dots a_k} = -H_{a_2 a_1 \dots a_k}$ etc.); we take this into account by replacing above condition by

$$\sum_{\text{permutations } P} \frac{\partial V}{\partial H_{P(a_1 \dots a_k)}} \pi(P) = 0 \quad (5)$$

We give here the equations for the extremum conditions only for the case $k = 3$ and $k = 4$; generalization to higher k is obvious. For our purposes we found it useful to write the above equation in solving it respectively for the tensor components H^{abc} and H^{abcd}

$$\begin{aligned} \underline{k=3} \quad H^{abc} = \frac{2\lambda_2}{3![\mu^2 - \lambda_1 H^{ijk} H_{ijk}]} & \left[H^{abd} H^{ejc} H_{efd} + H^{cad} H^{ejb} H_{efd} + \right. \\ & \left. + H^{bcd} H^{efa} H_{efd} \right] \end{aligned} \quad (6)$$

$$\begin{aligned} \underline{k=4} \quad H^{abcd} = \frac{1}{4![\mu^2 - \lambda_1 H^{ijk} H_{ijk}]} & \left[3! \lambda_2 (H^{abce} H^{fghd} H_{fghc} + H^{bcde} H^{fgha} H_{fghc} + \right. \\ & + H^{cdae} H^{fghb} H_{fghc} + H^{dabe} H^{fghe} H_{fghc}) + \\ & \left. + 8\lambda_3 (H^{abet} H^{ghcd} H_{ghet} + H^{bcef} H^{ghad} H_{ghet} + H^{caet} H^{ghbd} H_{ghet}) \right] \end{aligned} \quad (7)$$

Again, summation over repeated indices is implied. Here, and in the following, H^{abc} will denote the vacuum expectation value of the corresponding field.

To proceed, we must find first the solutions to (5). After having found them, we determine the little group for each possible solution. For this purpose, we will calculate the mass matrix of the gauge bosons and see which ones remain massless. This mass matrix can be calculated from the (minimal) coupling of the scalar multiplet to the vector bosons, given by

*) We note that setting $\tilde{V} = 0$ is equivalent to require the potential V to be invariant under an additional $U(1)$ global symmetry ($H^{a_1 \dots a_k} \rightarrow e^{i\phi} H^{a_1 \dots a_k}$, $H_{a_1 \dots a_k} \rightarrow e^{-i\phi} H_{a_1 \dots a_k}$). The potential has then the symmetry $SU(N) \times U(1)$. A Goldstone boson will appear when the $U(1)$ symmetry is broken. It will get a mass through radiative corrections, if the Yukawa couplings do not respect this $U(1)$ symmetry.

$$g \sum_{n=1}^N \sum_{l=1}^N W_{\mu a_n}^l H_{a_1 \dots l \dots a_k} \quad (8)$$

where, for fixed r , the summation index l is inserted in place of a_r in H . The $W_{\mu a_r}^l$ are defined as :

$$W_{\mu a_r}^l \equiv \sum_{\alpha=1}^{N^2-1} A_{\mu}^{\alpha} (\lambda_{\alpha})_{a_r}^l \quad (9)$$

where A_{μ}^{α} are the gauge fields (g is the gauge coupling) and λ_{α} are the generators of $SU(N)$ for the fundamental representation. The squared mass matrix is obtained by taking the absolute square of (8) and identifying the coefficients of $A_{\mu}^{\alpha} A_{\mu}^{\beta}$. From Eqs.(8) and (9) one can see that the unbroken generators $(\lambda_{\alpha})_{a_r}^l$ which characterize the little group, satisfy the following equations :

$$\sum_{n=1}^N \sum_{l=1}^N (\lambda_{\alpha})_{a_n}^l H_{a_1 \dots l \dots a_k} = 0 \quad (10)$$

$$\forall a_1 \dots a_k \in (1 \dots N)$$

We return now to Eqs.(6) and (7) to find all possible extremal solutions. We could in principle take the various components of the $H_{a_1 \dots a_k}$ to be non-zero in all possible ways, check whether it is a solution, and then study the resulting breaking. This is clearly a tedious way (feasible possibly by computer); furthermore it would give many equivalent solutions. We must therefore find a simple "normal" form for $H_{a_1 \dots a_k}$. It is well known that for $k = 1$, we can always bring H_{a_1} into the form $H_{a_1} = c \delta_{a_1, N}$. Equally, since $H_{a_1 a_2}$ ($k = 2$) can be represented as an antisymmetric $N \times N$ matrix, it can be brought in a quasi diagonal form such that only the elements H_{12}, H_{34}, H_{56} , etc. are non-zero in Eq.(5) *) 1). These procedures allow to find quickly all inequivalent solutions. For $k > 2$ **) , we can also give a rule which reduces the number of possible solutions substantially :

*) We notice, however, that this procedure is only valid if $\tilde{V} = 0$.

**) In Ref. 10), Kim has given a "diagonal" form for arbitrary k ; however, his form is not the most general one !

Rule : For any non-vanishing component of the tensor, all other ones obtained by changing only one index can be put equal to zero by a gauge transformation. Equivalently, all the components that take a non-vanishing vacuum expectation value have at least two indices different between each other.

We checked it in many examples and in fact, a proof of this rule is given in appendix A. This rule is a necessary condition in the sense that it discards equivalent solutions, but, on the contrary, not all solutions to the extremum equations which satisfy the rule give necessarily different symmetry breaking patterns. In appendix B, we show that for $k = 3$, the maximal number ℓ of components of the tensor satisfying the above rule is

$$N \text{ odd} \quad \ell \leq \left[\frac{N(N-1)}{6} \right] \quad (11.a)$$

$$N \text{ even} \quad \ell \leq \left[\frac{N(N-2)}{6} \right] \quad (11.b)$$

($[\]$ means again the integer part).

Explicit calculations show that up to $N = 10$, the above bounds are saturated, with the exception of $N = 5$ (two solutions with a bound of three). A similar formula might be derived for larger k . For $k = 2$, the rule is equivalent to bringing H^{ab} into "diagonal" form ¹⁾: we can set $H^{12} = a$, $H^{34} = b$, etc., and put all other elements equal to zero. Every number of components different from zero up to the maximal number can be an extremal solution. For a given number of non-zero components there are in general different choices of these components. It is important to note that they can lead to different symmetry breaking patterns (see below).

Although the case $k = 2$ has been treated before ^{1),9) and 10)}, we will give here, for completeness, a list of the patterns of symmetry breaking.

a) $k = 2$

For $SU(N)$, the number j of components that can be different from zero satisfies : $0 \leq j \leq [N/2]$. From the extremum equations, one obtains that all the non-vanishing components take the same value a , with

$$|a|^2 = \frac{\mu^2}{2j\lambda_1 + \lambda_2} \quad *) \quad (12)$$

and the value of the potential at the extremum (12) is then

$$\langle V \rangle = -\frac{\mu^4}{2} \frac{j}{2j\lambda_1 + \lambda_2} \quad (13)$$

The residual symmetry is 1), 9) and 10)

$$SU(N) \longrightarrow Sp(2j) \times SU(N-2j) \quad (14)$$

For $SU(4)$, one can also find the solutions in another way, considering the possible symmetry breaking patterns of $SO(6)$ [isomorphic to $SU(4)$]. The $k = 2$ antisymmetric representation, which has six elements, corresponds then to the vector representation of $SO(6)$, but due to the fact that the elements are now complex, one has to take two vectors with real components. At the minimum, one obtains with two vector representations ¹⁾ the symmetries $SO(5)$, isomorphic to $Sp(4)$, or $SO(4)$, isomorphic to $SU(2) \times SU(2)$, depending on the values of the parameters of the Higgs potential.

In order to see which one of the above solutions can be a minimum (and not only an extremum), we require the potential to be bounded from below. A necessary condition is

$$2j\lambda_1 + \lambda_2 > 0 \quad \forall j \quad (15)$$

One sees from (13) that $j = 1$ is the lowest extremum (and thus clearly a minimum) for $-2\lambda_1 < \lambda_2 < 0$ and equally $j = \lceil N/2 \rceil$ for $\lambda_2 > 0$ ¹⁾. One can, however, go further and calculate the scalar squared mass matrix for all the other j 's (for $2j\lambda_1 + \lambda_2 > 0$).

*) The fact that the phase remains undetermined is a direct consequence of the extra global $U(1)$ symmetry of the potential V (see footnote on page 4).

We have found that the corresponding extrema are all saddle points and not local minima (see appendix C). This result will be seen to hold also for larger k . Thus it will be possible to give all minima, although the number of solutions to the extremum equations grows.

For $N = 6$, one can also introduce the cubic term $\tilde{V}$ and for $N = 8$ the quartic one. For $N = 6$, the presence of the cubic term only allows $j = 1, 3$ as solutions to the extremum equations. This actually holds for any k, N if $N = 3k$, such that cubic couplings given by Eq.(4b) are allowed; there, only $j = 1, 3$ are solutions.

b) $k = 3$

We first consider the possible symmetry breaking patterns for $SU(4)$, $SU(5)$ and $SU(6)$, since for these cases we are able to give the complete list of solutions to the extremum equations [see Eq.(6)].

- $SU(4)$: For $SU(4)$, there is only one solution [see (B.2)] beside the trivial one where all components take the value zero : only one component of H^{abc} is non-vanishing. We can freely choose a, b, c to be respectively 1, 2, 3. Then

$$H^{123} = a \quad \text{with} \quad |a|^2 = -\frac{\mu^2}{6\lambda_1 + 2\lambda_2} \quad (16)$$

and

$$\langle V \rangle = -\frac{3}{2} \frac{\mu^4}{6\lambda_1 + 2\lambda_2} \quad (17)$$

The remaining symmetry is then $SU(3)$. This result can be immediately checked because it must be the same as if one breaks $SU(4)$ with a vector representation ($k = 1$), which is in fact the case.

- $SU(5)$: For $SU(5)$, there are two solutions :

- 1) $H^{123} = a$ with $|a|^2$ and $\langle V \rangle$ given in Eqs.(16) and (17) and the remaining symmetry is $SU(3) \times SU(2)$;
- 2) $H^{123} = H^{145} = a$ with

$$|a|^2 = -\frac{\mu^2}{12\lambda_1 + 8\lambda_2} \quad (18)$$

$$\langle V \rangle = -\frac{9}{4} \frac{\mu^4}{9\lambda_1 + 2\lambda_2} \quad (19)$$

and the remaining symmetry is $Sp(4)$.

As expected, this gives the same breaking patterns as $k = 2$. One sees clearly that solution 2) cannot be obtained from the form of Ref. 10).

- SU(6) : For $SU(6)$, the following solutions are possible (the components which are not mentioned are zero) :

- 1) $H^{123} = a$ with $|a|^2$ and $\langle V \rangle$ given by Eqs.(16) and (17) and the remaining symmetry is $SU(3) \times SU(3)$;
- 2) $H^{123} = H^{145} = a$ with $|a|^2$ and $\langle V \rangle$ given by Eqs.(18) and (19) and the remaining symmetry is $Sp(4)$;
- 3) $H^{123} = H^{456} = a$ with

$$|a|^2 = \frac{\mu^2}{12\lambda_1 + 2\lambda_2} \quad (20)$$

$$\langle V \rangle = -\frac{3}{2} \frac{\mu^4}{6\lambda_1 + \lambda_2} \quad (21)$$

and the remaining symmetry is $SU(3) \times SU(3)$;

- 4) $H^{123} = H^{345} = H^{246} = a$ with

$$|a|^2 = \frac{\mu^2}{18\lambda_1 + \frac{10}{3}\lambda_2} \quad (22)$$

$$\langle V \rangle = -\frac{27}{4} \frac{\mu^4}{27\lambda_1 + 5\lambda_2} \quad (23)$$

and the remaining symmetry is $SU(3)$.

We notice that the choice of the indices is not unique, but the results of Eqs.(22) and (23) obviously do not depend on this particular choice.

- 5) $H^{123} = H^{345} = H^{246} = H^{156} = a$ with

$$|a|^2 = \frac{\mu^2}{24\lambda_1 + 4\lambda_2} \quad (24)$$

$\langle V \rangle$ given by Eq.(21) and the remaining symmetry is $SU(3) \times SU(3)$.

We see that the last solution leads to the same result as case 3) and cannot therefore be considered as a new solution.

The maximal number of components that can be different from zero in order to satisfy the extremum equations (6) in $SU(6)$ being four, we do not need go further. In order to determine which of the above solutions can be the absolute minimum, we

must again impose the condition that the potential is bounded from below. Taking into account this constraint, we find that for $\lambda_2 > 0$, the absolute minimum is given by case 3) ($H^{123} = H^{456} = a$); and for $-3\lambda_1 < \lambda_2 < 0$ by case 1) ($H^{123} = a$). The solutions 2) and 4) are saddle points, as was checked by calculating the corresponding mass matrices. If one tries to make extrema 2) and 4) lower than 1) and 3), the boundedness conditions are violated. Again we see that only one local minimum is allowed.

- $N \geq 6$: In this case, we could follow the above procedure, determining the maximal number of non-zero components and then find all possible solutions up to that number. This is a rather formidable task we have not fulfilled! We only looked at particular extremal solutions which are in some sense a generalization of the four encountered for $SU(6)$:

- 1) Only components with non-overlapping indices are non-zero :

$$H^{123} = H^{456} = H^{789} = \dots = a.$$

The number j of such components is given by $0 \leq j \leq \lfloor N/3 \rfloor$.

Then we have

$$|a|^2 = -\frac{\mu^2}{6\lambda_1 + \lambda_2} \quad (25)$$

$$\langle V \rangle = -\frac{3}{2} \delta - \frac{\mu^4}{6\lambda_1 + \lambda_2} \quad (26)$$

and the remaining symmetry is $[SU(3)]^j \times SU(N-3j)$. This case has been discussed in Ref. 10).

- 2) Secondly, we can consider the case when all components that take a value different from zero have all one index in common; but the other two are different for each component. For instance : $H^{123}, H^{145}, H^{167}, \dots$. The number n of such components can be $0 \leq n \leq \lfloor (N-1)/2 \rfloor$. Then, at the extremum, we have $H^{123} = H^{145} = \dots = b$ with

$$|b|^2 = -\frac{2}{18n^2 + (n+2n)\lambda_2} \quad (27)$$

$$\langle V \rangle = -\frac{3n}{4} - \frac{\mu^4}{9n\lambda_1 + (2+n)\lambda_2} \quad (28)$$

and the resulting unbroken symmetry is $Sp(n) \times SU[N-(2n+1)]$.

- 3) A third possibility is to take for every group of six indices a, b, c, d, e, f in N , the three components $H^{abc}, H^{cde}, H^{bdf}$ (always one index in common) different from zero.

Each group of such components gives a resulting $SU(3)$ symmetry. In $SU(N)$ the number t of such groups is at most $\lfloor N/6 \rfloor$. Thus at the extremum, we have $H^{123} = H^{345} = H^{246} = c$, $H^{789} = H^{91011} = H^{81012} = c$; etc., with

$$|c|^2 = \frac{3\mu^2}{54t\lambda_1 + 10\lambda_2} \quad (29)$$

$$\langle V \rangle = -\frac{27}{4}t \frac{\mu^4}{27t\lambda_1 + 5\lambda_2} \quad (30)$$

and the remaining symmetry is $[SU(3)]^t \times SU(N-6t)$.

- 4) We can combine the above three patterns of symmetry breaking in order to get as resulting symmetry

$$[SU(3)]^{(j+t)} \times Sp(n) \times SU(N-6t-3j-2n-1) \quad (31)$$

with $6t + 3j + 2n + 1 \leq N$.

In all these cases, one can calculate quite easily the squared mass matrix of the scalars; it turns out that only the absolute minima given by Eq.(25) with $j = 1$, $[N/3]$ are minima at all.

c) $k = 4$

We consider first possible symmetry breaking patterns of $SU(5)$ and $SU(6)$ and will then use the results for higher N .

- $SU(5)$: For $SU(5)$, there is only one extremal solution, namely $H^{1234} = a$, with

$$|a|^2 = \frac{\mu^2}{24\lambda_1 + 6\lambda_2 + 4\lambda_3} \quad (32)$$

$$\langle V \rangle = -6 \frac{\mu^4}{24\lambda_1 + 6\lambda_2 + 4\lambda_3} \quad (33)$$

and the remaining symmetry is $SU(4)$.

This is the same result (necessarily) as if one breaks $SU(5)$ with a fundamental ($k = 1$) representation.

- $SU(6)$: For $SU(6)$, there are three different solutions, to be compared with the case $k = 2$ 1):

$$1) H^{1234} = a \text{ and the remaining symmetry is } SU(4) \times SU(2) \quad (34)$$

$$2) H^{1234} = H^{3456} = a \text{ and the remaining symmetry is } Sp(4) \times SU(2) \quad (35)$$

$$3) H^{1234} = H^{3456} = H^{1256} = a \text{ and the remaining symmetry is } Sp(6). \quad (36)$$

Again, from appendix C, we know that 2) corresponds to a saddle point.

- $\underline{N} \geq 6$: As in the previous case $k = 3$, we can again find easily many extremal solutions. For $SU(7)$, we use first the known results of $k = 3$ to find solutions for $k = 4$. As in the case $k = 3$, these results can then be used to find new solutions for $k = 4$, $N > 7$. In $SU(7)$, there is, for instance, the solution :

$$H^{1234} = H^{4567} = 0 \quad (37)$$

[corresponding to $H^{567} = H^{123} \neq 0$ for $k = 3$, Eq.(25)] which gives as remaining symmetry $SU(3) \times SU(3)$.

Then we can have

$$H^{4567} = H^{1267} = H^{1357} = 0 \quad (38)$$

[in $k = 3$ this was $H^{123} = H^{345} = H^{246} \neq 0$, Eq.(29)] which breaks $SU(7)$ down to $SU(3)$.

In the same way, we obtain solutions to the extremum conditions with the following breaking patterns :

$$SU(4)^m \times SU(N-4m) \quad N \geq 4m \quad (39.a)$$

$$[SU(2) \times Sp(4)]^m \times SU(N-6m) \quad N \geq 6m \quad (39.b)$$

$$[Sp(6)]^m \times SU(N-6m) \quad N \geq 6m \quad (39.c)$$

$$[SU(3) \times SU(3)]^m \times SU(N-7m) \quad N \geq 7m \quad (39.d)$$

$$[SU(3)]^m \times SU(N-7m) \quad N \geq 7m \quad (39.e)$$

Obviously, these can be combined in a similar way as in Eq.(31).

d) $k > 4$

It should be now clear how to generalize the previous analysis to k greater than four. Again, we have not found all the solutions, but many of them can be easily given. As a generalization of relations (39.a)-(39.e), we can write down immediately :

$$[SU(k)]^m \times SU(N - mk) \quad *) \quad N \geq mk \quad (40.a)$$

$$[SU(k+2-2j) \times Sp(2j)]^m \times SU(N - m(k+2)) \quad N \geq m(k+2) \quad (40.b)$$

$$0 \leq j \leq \left[\frac{k+2}{2} \right]$$

$$[SU(3) \times SU(3) \times SU(k-3)]^m \times SU(N - m(k+3)) \quad N \geq m(k+3) \quad (40.c)$$

$$[SU(3) \times SU(k-3)]^m \times SU(N - m(k+3)) \quad N \geq m(k+3) \quad (40.d)$$

$$[Sp(4) \times SU(k-2)]^m \times SU(N - m(k+3)) \quad N \geq m(k+3) \quad (40.e)$$

One gets Eq.(40.b) by comparison with $k = 2$; Eqs.(40.c)-(40.e) by looking at $k = 3$.

3. SYMMETRIC REPRESENTATION OF DEGREE k OF $SU(N)$

We consider briefly the symmetry breaking due to one irreducible k symmetric tensor, the completely symmetric tensor product of k fundamental representations (a Young tableau with k boxes arranged in one line). However, we will discuss it in less detail as we did for the antisymmetric case. We denote the elements of the tensor by

$$H_{a_1, \dots, a_k} \quad (41)$$

which satisfy

$$H_{P(a_1, \dots, a_k)} = H_{a_1, \dots, a_k} \quad (42)$$

where $P(a_1, \dots, a_k)$ is a permutation of $(a_1, \dots, a_k)$.

The complex conjugated fields are

$$(H_{a_1, \dots, a_k})^* = H^{a_1, \dots, a_k} \quad (43)$$

*) See appendix C for the saddle point solutions.

The expressions for the potential and the extremum equations are again given by Eqs.(4.a), (6) and (7) with the condition (42) instead of (2). Note that only if $N = 3$ (4) are there cubic (quartic) terms as in $\tilde{V}$ (4.b). Again, the equation for the unbroken generators is given by Eq.(10) with condition (42).

a) $k = 2$

For completeness, we mention also this case which has already been considered in Ref. 1). The number j of components that can be different from zero take the value $0 \leq j \leq N$. From the extremum equations, one obtains that all the components that are different from zero take the same value a :

$$H^{11} = H^{22} = H^{33} = \dots = a$$

with
$$|a|^2 = \frac{\mu^2}{j\lambda_1 + \lambda_2} \quad (44)$$

The value of the potential is then

$$\langle V \rangle = -\frac{1}{4} j - \frac{\mu^4}{j\lambda_1 + \lambda_2} \quad (45)$$

and the resulting symmetry is $SU(N-j) \times O(j)$.

The absolute minimum for $\lambda_2 > 0$ is reached for $j = N$, and for $-\lambda_1 \leq \lambda_2 \leq 0$ for $j = 1$. This is obtained if one takes into account the fact that the potential must be bounded from below. The other extrema are again saddle points, as in the antisymmetric case.

b) $k > 2$

In this case we did not find a convenient normal form. However, it is obvious that there are solutions to the extremum equations, where the components that are different from zero have all equal indices ($H_{a_1 \dots a_k}$ with $a_1 = a_2 = \dots = a_k$). The number j of components different from zero is then $0 \leq j \leq N$. We have

$$H^{11\dots 1} = H^{22\dots 2} = \dots = H^{jj\dots j} = a$$

with
$$|a|^2 = \frac{\mu^2}{j\lambda_1 + \sum_{i=2}^{k+1} \lambda_i} \quad (46)$$

$$\langle V \rangle = -\frac{1}{4} \delta \frac{r^4}{\delta \lambda_1 + \sum_{i=2}^{[k/2]} \lambda_i} \quad (47)$$

($[k/2]$ means integer part of $k/2$) and the remaining symmetry is $SU(N-j)$.

The absolute minima are again reached for the case $j = 1$ and $j = N$, owing to the condition of boundedness of the potential. The other extrema are again saddle points, as seen by calculating the scalars' squared mass matrix.

4. SUMMARY AND DISCUSSION

In this paper, we have considered the symmetry breaking of $SU(N)$ by one completely antisymmetric or symmetric tensor of rank k .

For the antisymmetric tensor, we give a "normal" form which makes it easier to find all non-equivalent solutions to the extremum conditions. It implies that all components of the tensor that have a non-zero vacuum expectation value have at least two different indices between each other ^{*)}. [For $k = 2$, this corresponds to bringing the tensor into "diagonal" form ¹⁾.] The number of components which satisfy this rule is still rather large, and its upper bound (saturated up to 10) for $k = 3$ is $[N(N-1)]/6$ (N odd), $[N(N-2)]/6$ (N even). Furthermore, for each number of non-zero components, there might be different solutions to the extremum conditions. We have determined all such solutions up to $k = 3$, $N = 6$; for larger N, k , we give many solutions which can be obtained from those of $k = 3$, $N = 6$; they are given by Eqs.(40.a)-(40.e).

Despite that large number of solutions to the extremum equations, we can give, for the cases considered, the symmetry breaking pattern. For $k = 2$, there are the solutions $SU(N) \rightarrow Sp(2j) \times SU(N-2j)$ ^{1), 10)}, $j = 1 \dots [N/2]$ ($[]$ means the integer part). But it turns out that only $j = 1$ or $[N/2]$ correspond to a minimum of the potential; the others are saddle points.

*) The potential of the scalars consists of a "normal" part [Eq.(4.a)] and a piece [Eq.(4.b)] which only arises for special values of k, N . We have in general omitted it; its absence implies an extra global $U(1)$ invariance. If this piece is present, our rule does not hold.

For $k = 3$, $N = 6$, where we have determined all extremal solutions, the solution (with no overlapping indices) which gives the lowest value for the potential is the unique minimum ^{*)}, the others are saddle points. For larger k , N , all solution which we give lead to the same result.

We have so far not been able to use the general properties of the potential to show that all other solutions correspond also to saddle points (or, at most to equivalent situations as the minima with non-overlapping indices [see previous footnote]); all (scattered) examples we have considered are, however, saddle points. It seems therefore quite plausible to conjecture that only the above absolute minima remain; they lead to breaking $SU(N) \rightarrow SU(k) \times SU(N-k)$ or $SU(k)^{\lfloor N/k \rfloor}$ ($k > 2$), according to the sign of the self-interacting couplings.

For the symmetric tensor we have not found a convenient normal form [except, of course, for $k = 2$ ¹⁾]. We give here only the obvious solution, where all indices of the non-zero components are equal. Their number being j ($j = 1 \dots N$), their associated breaking is $SU(N) \rightarrow SU(N-j)$ [for $k = 2$, $SU(N-j) \times O(j)$ ^{1),9)}]. Also in this case, only $j = 1$, N give minima, other value of j give saddle points. If it turns out to be the case for all extrema, as we expect, the breaking pattern of the symmetric tensor is also uniquely determined.

ACKNOWLEDGEMENTS

We would like to thank H. Ruegg, G. Felder and A. Bovier for many useful comments and suggestions. One of us (P.J.) also benefited from discussions with R. Anishetty and Ch. Schmid. We thank J. Prentki for a careful reading of the manuscript.

*) It should be noted that among the other three solutions, there is one, with the maximum number of non-zero components, which is equivalent to one of the above.

APPENDIX A

In this appendix, we give a general proof of the rule given in the text, in section 2.

Assuming $\tilde{V} = 0$, the extremal conditions on the scalar potential V given in (4.a) become

$$\frac{\partial V}{\partial H_{a_1 \dots a_k}} = -\frac{k}{2} H_{a_1 \dots a_k} + \sum_{i=1}^{[k/2]+1} \frac{\lambda_i}{2} H_{a_1 \dots a_{k-i+1}, b_{k-i+2} \dots b_k} X_{a_{k-i+2} \dots a_k}^{b_{k-i+2} \dots b_k} = 0 \quad (A.1)$$

with

$$X_{a_{k-i+2} \dots a_k}^{b_{k-i+2} \dots b_k} \equiv H_{b_1 \dots b_{k-i+1}, b_{k-i+2} \dots b_k} H_{b_1 \dots b_{k-i+1}, a_{k-i+2} \dots a_k}$$

the irreducible (traceless) tensor of dimension

$$\frac{(N-2i+3)}{N+1} \left[C \binom{N+1}{i-1} \right]^2$$

However, we know that a $SU(N)$ gauge transformation can always rotate the adjoint tensor X_a^b ($i = 2$) into a diagonal form:

$$X_a^b \longrightarrow x_b \delta_a^b \quad (A.2)$$

such that the off-diagonal fields disappear from the extremal equations (A.1). One can easily check that if the rule in the text (all non-equivalent extremal solutions are such that non-vanishing components have at least two indices different from each other) is satisfied, then Eq. (A.2) holds automatically. On the other hand, one could, in agreement with (A.2), violate this rule by a suitable choice of relative phases for the vacuum expectation values. For instance, in the case of $k = 3$ with non-zero H^{123} , H^{124} , H^{563} and H^{564} , one could in principle require $X_4^3 \equiv H^{123} H_{124} + H^{563} H_{564} = 0$.

However, due to the $U(1)$ global symmetry of the potential ($\tilde{V} = 0$), the phases are irrelevant. Therefore, this equation must hold for a phase convention where all the vacuum expectation values are taken positive, and thus the above conclusion remains valid.

APPENDIX B

In this appendix, we give the maximal number of components of the antisymmetric tensor of rank three such that, at least, two indices are different between each component.

The problem is easily solved if formulated in a geometrical way. Take N points, connected to each other by a line. Then we look how many triangles we can draw so that they have at most a point in common (no common side).

The total number of lines is $N(N-1)/2$; therefore, one can have at most $(1/3)(N(N-1)/2)$ triangles satisfying the above condition. For even N we can even give a sharper bound: at each point, $N-1$ lines come together. Therefore, every point can only be common to $(N-2)/2$ triangles, and thus at any point, there is one line not used up by triangles. Therefore, there are at least $N/2$ lines which can be omitted; only $(N(N-1)/2) - N/2 = (N(N-2)/2)$ are "effective". Thus we have for the maximum number of components ℓ : ($[\]$ means the integer part):

$$N \text{ odd} \qquad \ell \leq \left[\frac{N(N-1)}{6} \right] \qquad (B.1)$$

$$N \text{ even} \qquad \ell \leq \left[\frac{N(N-2)}{6} \right] \qquad (B.2)$$

By explicit evaluation for $N = 3$ to 10 , we found the values of ℓ to be equal to the bounds (B.1) and (B.2); the only exception being $N = 5$ where $\ell = 2$ although (B.2) yields three. It is presumably possible to give similar bounds for higher k , but the easy geometrical interpretation does not work there.

APPENDIX C

In this appendix, we compute the squared mass matrix for the real component of the complex scalar fields, when indices of the non-vanishing v.e.v.'s do not overlap at all. In this case indeed, the value of the potential at the minimum will take the following form:

$$\langle V \rangle = -\frac{\mu^2}{2} \sum_{j=1}^{[N/k]} v_j^2 + \frac{\lambda}{4} \left(\sum_j v_j^2 \right)^2 + \frac{\lambda'}{4} \sum_j v_j^4 \quad (C.1)$$

with $v_j \equiv \langle H_{(j-1)k+1} \dots jk \rangle$

The extremal conditions are

$$\frac{\partial V}{\partial v_j} = 0 = v_j \left\{ -\mu^2 + \lambda \sum_l v_l^2 + \lambda' v_j^2 \right\} \quad (C.2)$$

compatible with the following set of solutions:

$$\begin{aligned} \text{solution (j)} \quad v_1^2 = \dots = v_j^2 \equiv v^2 \neq 0 \quad 1 \leq j \leq \left[\frac{N}{k} \right] \\ v^2 = \frac{\mu^2}{j(\lambda + \lambda')} \end{aligned} \quad (C.3)$$

We notice that all these v.e.v.'s can be made real owing to the global U(1) symmetry of the potential. This implies that real and imaginary parts of the complex scalar fields do not mix together. Moreover, the value of the potential at any extremal point being at least quadratic in the non-vanishing v.e.v.'s, as can immediately write the squared mass matrix for the real part:

$$M_{ij}^2 \equiv \frac{\partial^2 V}{\partial v_i \partial v_j} = -\lambda' v^2 \delta_{ij} + 3\lambda' v_i^2 \delta_{ij} + 2\lambda v_i v_j \quad (C.4)$$

namely

$$M^2 = 2 \left(\begin{array}{c|c} \overbrace{\begin{pmatrix} \lambda + \lambda' & \lambda & \dots & \lambda \\ \lambda & \lambda + \lambda' & \dots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \dots & \dots & \lambda + \lambda' \end{pmatrix}}^{\delta} & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{pmatrix} -\frac{\lambda'}{2} & 0 & \dots & 0 \\ 0 & -\frac{\lambda'}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{\lambda'}{2} \end{pmatrix} \end{array} \right) v^2$$

for the solution (j).

From this squared mass matrix we conclude that for $1 < j < [N/k]$, the eigenvalues are

$$2\lambda'\nu^2 \quad \text{or} \quad -\lambda'\nu^2$$

implying obviously a saddle point in each case. The true minima are therefore given by the solutions

$$\begin{aligned} j=1 & \quad \text{with} \quad \begin{aligned} \lambda + \lambda' &> 0 \\ \lambda' &< 0 \end{aligned} \\ j = [N/k] & \quad \text{with} \quad \begin{aligned} [N/k]\lambda + \lambda' &> 0 \\ \lambda' &> 0 \end{aligned} \end{aligned} \tag{C.5}$$

REFERENCES

- 1) L.F. Li, Phys. Rev. D9 (1974) 1723.
- 2) M. Magg and Q. Shafi, Z. Phys. C4 (1980) 63.
- 3) F. Buccella, H. Ruegg and C. Savoy, Nucl. Phys. B169 (1980) 68.
- 4) H. Ruegg, Phys. Rev. D22 (1980) 2040.
- 5) J.E. Kim, Univ. of Pennsylvania preprint, UPR-0141T (1980);
M. Abud and G. Sartori, Rio University preprint (1982);
S. Frautschi and J. Kim, Nucl. Phys. B196 (1982) 301;
J. Kim, Nucl. Phys. B196 (1982) 285, and B197 (1982) 174;
C. Chang-qi, DESY preprint 82-042 (1982).
- 6) P. Mitra, Tata Report 80-0288 (1980).
- 7) F. Buccella, H. Ruegg and C. Savoy, Phys. Lett. 94B (1980) 491.
- 8) F. Buccella and H. Ruegg, Nuov. Cim. 67A (1982) 61.
- 9) D.D. Wu, Nucl. Phys. B199 (1982) 523.
- 10) J.E. Kim, Phys. Rev. D26 (1982) 2009.
- 11) P. Frampton, S. Nandi and J.J.G. Scanio, Phys. Lett. B85 (1979) 225.
- 12) P. Kalyniak and J.N. Ng, Phys. Rev. D26 (1982) 890.
- 13) P. Eckert, J.-M. Gérard, H. Ruegg and T. Schucker, Phys. Lett. B125 (1983) 385.
- 14) L. Michel and L. Radicati, Ann. Phys. (N.Y.) 66 (1971) 758.
- 15) L. Michel, CERN preprint TH.2716 (1979), unpublished.
- 16) See, e.g., H. Georgi, Nucl. Phys. B156 (1979) 126.