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MONTE CARLO STUDY OF THE BREAKDOWN OF
TRANSLATIONAL INVARIANCE IN A LIOUVILLE MODEL

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ABSTRACT

We detect the existence of highly metastable non-translationally invariant states in a regularized Liouville model. The relation with ideas of Fubini and D'Hoker and Jackiw is discussed.

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The four-dimensional Poincaré group has a favoured place among the symmetries describing fundamental interactions at experimentally accessible distance scales. However, once away from its experimental sanctuary, the position of the Poincaré group is less secure. In particular, in various gravitational theories in higher dimensions, ranging between five and eleven, it is assumed that a partial breakdown of the Poincaré group can occur. It is thus of interest to gain an understanding of the dynamics of such a breakdown. As this phenomenon is supposed to occur in gravitational theories which are rather complex, one may wish to search for it also in simpler theories.

Classically, a breakdown of the translation group can be forced upon a model by a twist of the boundary conditions in such a way that the lowest state acquires a non-trivial spatial structure. Such a mechanism is possible in theories containing topological solitons ¹⁾. However, when one attempts to quantize around a soliton, one finds translational zero modes, which, depending on their normalizability properties, may restore translational invariance. Another general mechanism with a more dynamical origin was proposed by Fubini ²⁾ some time ago. Any system which possesses the conformal group as the complete space-time symmetry groups will contain the Poincaré group as a subgroup. Thus if a spontaneous breakdown of the full conformal symmetry takes place, it may induce a partial breakdown of the Poincaré group, and the new vacuum may, for example, not be translationally invariant.

This phenomenon was shown to occur in conformally invariant quantum mechanics ²⁾. In this case the potential is repulsive and of the form $1/x^2$. The state of lowest energy ($E=0$) is classically found at $x=\pm\infty$. Quantum mechanically, the fluctuations are, however, not normalizable around this state. The system has a continuous spectrum of states with energy $E > 0$, but no state with energy $E=0$. It was suggested ²⁾ that instead of diagonalizing the Hamiltonian, one should diagonalize a compact operator which is a linear superposition of the Hamiltonian and the conformal operator. This superposition introduces a scale and makes the new ground state well defined but not translationally invariant. It was claimed that this phenomenon would occur for scale invariant field theories in higher dimensions, the vacuum of the system being obtained by expanding around a non-trivial classical solution to the equations of motion. Such theories are characterized by interactions of the form $\phi^{2D/D-2}$ where D is the dimension of space-time. One problem with these theories is that the renormalization prescription breaks the classical conformal invariance, such that they are not, strictly speaking, conformally invariant at the quantum level.

Recently, however, scale invariant field theories have reappeared. Some Yang-Mills supersymmetric theories have been shown to be superconformally invariant. This was demonstrated in the light cone gauge for the $N=4$ case ³⁾ and for some $N=2$ models as well ⁴⁾. It has not yet been demonstrated that such theories have an ill-defined naïve vacuum. Theories containing gravity may also turn out to be scale invariant in some dimension. It could be that a theory which is finite in $D=11$ spontaneously breaks down to $D=4$.

In two dimensions there exist infinitely many theories that are conformally invariant at the classical level. At least one of them, the Liouville model, is believed to be conformally invariant also on the quantum level. The Lagrangian describing this model is given by

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 - \frac{m^2}{\alpha^2} e^{\alpha \phi} \quad (1)$$

where the time derivative is denoted by a dot and the space derivative by a prime. Neither the space-time scale parameter m nor the field scale parameter α have physical significance at the classical level since they can be removed by a linear transformation of the field. This model has been extensively studied ⁵⁾.

Recently, D'Hoker and Jackiw ⁶⁾ have suggested that a spontaneous breakdown of spatial translational invariance will occur in this model. As in conformal quantum mechanics, there is a steep potential and a remote classical vacuum at $\phi = -\infty$. Since perturbation theory is ill defined around $\phi = -\infty$, it may well be that this state does not belong to the Hilbert space of the system. In the language of statistical mechanics the relaxation time for the system to reach equilibrium at $\phi = -\infty$ is infinite. D'Hoker and Jackiw suggest that the system instead will pick a vacuum concentrated around one of the static solutions of the classical model. They indicate that perturbation theory around one of these solutions appears to be well defined; the Goldstone modes that communicate between equivalent solutions are excluded by their lack of normalizability from the spectrum of quantum fluctuations around them.

The static classical solutions are characterized by the integration constant

$$\mathcal{E} = \phi'^2 - 2 \frac{m^2}{\alpha^2} e^{\alpha \phi} \quad (2)$$

and by c , the centre of the solution. The generic form is

$$\phi(x) = -\frac{2}{\alpha} \log \left[\frac{m}{\alpha \sqrt{\epsilon}} \sinh \left(\frac{\alpha \sqrt{\epsilon}}{2} (x-c) \right) \right] \quad (3)$$

For $\epsilon \geq 0$ space is divided into two separate parts, $x < c$ and $x > c$, which do not communicate with each other. (The normal modes and their first derivatives all vanish at $x=c$.) For $\epsilon < 0$ space is completely compactified into non-communicating sectors of length $2\pi/\alpha\sqrt{\epsilon}$. Although these states carry infinite energy, this cannot be held against them, once the "natural" vacuum at $\phi = -\infty$ has been banished from the spectrum.

As is clear from Fubini's work ²⁾, this remedy for the theory is not unique, but both Refs. 2) and 6) share the view that part of the Poincaré symmetry should be broken by the correct theory. Stimulated by these works, we have tried to construct a lattice version of the Liouville model on which these ideas could be tested.

Some of the obstacles which one encounters in attempting to regularize the theory on a lattice are intrinsic to the lattice formulation whereas others are of a more general nature. A conformally invariant regularization seems to be impossible in a simple lattice with a fixed distribution of lattice sizes, but may be possible in a random lattice. We have not investigated this possibility but have chosen a rectangular lattice with periodic boundary conditions. If, however, the theory is naïvely implemented on the lattice by replacing derivatives by finite differences, it is easy to see that there is only one static solution, $\phi = -\infty$. None of the other classical solutions survive due to the fact that they have singularities, which cannot appear as solutions to a finite difference equation. We have observed that this model never reaches equilibrium but from any starting configuration displays a slow drift towards $\phi = -\infty$.

In order to overcome the problem, we are forced to replace the Liouville model (1) by a model in which the field ϕ never reaches infinite values. As soon as ϕ is given a finite range, this may be transformed into the interval $-1 \leq \phi \leq 1$ by a linear transformation. Instead of making a brutal cut-off, it is more convenient to do it dynamically by choosing the Lagrangian

$$\mathcal{L} = \beta \left\{ \frac{\frac{1}{2}(\dot{\phi}^2 - \phi'^2)}{1 - \phi^2} - \frac{m^2}{\lambda^2} e^{\lambda \phi} \right\} \quad (4)$$

where λ is a new parameter and $\beta \equiv \lambda^2/\alpha^2$. This corresponds to fixing ϕ within the finite range $-\lambda/\alpha \leq \phi \leq \lambda/\alpha$ in the original model (1). At least formally, the cut-off model will approach the Liouville model in the limit $\lambda \rightarrow \infty$. In a region of space-time where $|\phi| \ll 1$ this is identical to the Liouville Lagrangian. Regions where ϕ previously reached positive infinite values have been replaced by regions where $\phi \approx 1$. The regions where ϕ previously reached negative infinite values have been replaced by a true minimum at $\phi \approx -1$.

Instead of describing the system by ϕ , it is much more convenient to use the unconstrained variable ψ defined by

$$\phi = \cos \psi \quad (5)$$

such that

$$\mathcal{L} = \beta \left\{ \frac{1}{2} \dot{\psi}^2 - \frac{1}{2} \psi'^2 - \frac{m^2}{\lambda^2} e^{\lambda \cos \psi} \right\} \quad (6)$$

The potential seen by ψ is periodic with sharp peaks at $\psi = 2\pi p$ where p is integer (see Fig.1). The classical static solutions to this model are again characterized by the integration constant

$$\mathcal{E} = \frac{\phi'^2}{1-\phi^2} - 2 \frac{m^2}{\lambda^2} e^{\lambda \phi} = \psi'^2 - 2 \frac{m^2}{\lambda^2} e^{\lambda \cos \psi} \quad (7)$$

and the centre of the solution, c . There are three types of solutions (see Fig.2) corresponding to the three types of solutions of the Liouville model. For $\mathcal{E} > -2m^2/\lambda^2 e^{-\lambda}$ the range of the ψ is infinite. This makes ϕ periodic and the solutions must be considered to be multisoliton states. For $\mathcal{E} = -2m^2/\lambda^2 e^{-\lambda}$ the solution is aperiodic in ψ ; it is a true soliton. For $-2m^2/\lambda^2 e^{\lambda} < \mathcal{E} < -2m^2/\lambda^2 e^{-\lambda}$ the range of ψ is finite, $-\psi_0 \leq \psi \leq \psi_0$ and the solutions are interpreted as soliton-antisoliton states. The energy of a single soliton is in this case ($\lambda \rightarrow \infty$).

$$E = 2 \sqrt{2\pi\lambda} \frac{e^{\lambda/2}}{\lambda^2} m \quad (8)$$

The regulated model has a well-defined translationally invariant vacuum state at $\phi = -1$ and only metastable configurations around the classical solutions. Thus if the continuum theory has indeed a vacuum which is not translationally invariant, we expect that for $\lambda \rightarrow \infty$ the metastable states will become increasingly more stable, and that the system when started out in a disordered state will have essentially infinite relaxation time for

reaching the true ground state, and instead get trapped into one of the numerous metastable states. These features have clearly been borne out by the Monte Carlo study which we have carried out.

The Euclidean action which we have chosen is

$$S = \beta \sum_{x,t} (1 - \cos \Delta_t \psi + 1 - \cos \Delta_x \psi + \mu^2 e^{-\lambda(1 - \cos \psi)}) \quad (9)$$

Comparison with Eq. (6) shows that, in the naïve continuum limit,

$$\mu = ma \frac{e^{2/2}}{\lambda} \quad (10)$$

where a is the lattice spacing. We have chosen the compact form of the kinetic energy to allow the multisoliton states to exist on a finite lattice^{*)}. We have measured the Monte Carlo (Metropolis) relaxation time T for the system to reach the perturbative vacuum, defined in practice as $\psi = \cos \psi$ being negative everywhere. We chose an asymmetric lattice (typically 50×20) in order to encourage the formation of soliton states independent of the coordinate ("time") in the shorter dimension. Since the renormalization group structure of the model is not known, we were unable to search for continuum behaviour. The best we could do was to make a survey of an interesting region of parameter space. In order to give soliton-like states a maximum probability to form, it is necessary that the kinetic and potential terms in Eq. (9) be roughly the same order of magnitude. This requires μ to be of order one. Further, the temperature should be rather low (i.e., β should be large) in order to avoid randomizing the system; this is also suggested by the relation $\beta = \lambda^2 / \alpha^2$.

In Fig. 3 we display the λ dependence of T for a 50×20 lattice with a one-soliton-like start for $\mu = 1.0$, $\beta = 20$. The system was first seen to adjust the configuration as closely as possible to a classical solution, which, however, after a time $\sim T$ broke down to the vacuum. In Fig. 4, we display the λ dependence of T for a disordered start with $\mu = 1.0$, $\beta = 20$.

In both cases we observe an extremely sharp boundary in λ . For sufficiently large λ the system is effectively trapped in a highly metastable state. The relaxation time becomes too long to measure. In the case of a

*) The continuous $U(1)$ symmetry was in fact replaced by $Z(100)$ to increase the computational speed; we always worked well within the Coulomb region where the mass gap was irrelevant.

random start, translational invariance is broken in time as well as in space. Metastable islands are formed in the lattice (see Fig.5). The exact nature of the metastable state depends on the sequence of random numbers delivered by the generator. Different seeds lead to different final configurations.

For large λ , highly metastable states were formed whenever the initial configuration had high enough energy. For example, starts with $\psi = 0$ everywhere or ψ randomly distributed between 0.1 and 0.3 always froze into metastable states. However, starts of low energy, such as ψ randomly distributed between $\pi/2$ and $3\pi/2$, fell rapidly to the vacuum.

In the Metropolis method, a system is generally brought to equilibrium by adjusting the step size so as to obtain an acceptance ratio of about one-half. However, in our case, that value would be dictated by the large regions in configuration space which are near the vacuum. Therefore we usually demanded that the step be smaller to reflect the smaller fluctuations occurring around the top of the potential. The critical value of λ did not seem to be sensitive to the prescription. For those values of λ for which the system reached the translationally invariant vacuum, the energy corresponded within a few per cent to the value $1/28 + \mu e^{-2\lambda}$ of perturbation theory.

Despite the clear structure apparent in the above results, there are several important difficulties involved in their interpretation. One difficulty is related to the fact that rapid cooling of a system which contains non-trivial spatial structures may result in trapping of the complicated configuration, although the system has a well-defined translationally invariant vacuum. An extreme example is the $D=2$ Ising model with periodic boundary conditions. There rapid cooling may result in a stable domain structure. We have observed that even in this extreme case this seems to occur in roughly 50% of the random starts in lattices up to size 50×20 , while in the Liouville case, once above the "critical" λ value, the non-translationally invariant states occur for all disordered starting configurations. This phenomenon seems to persist for other lattice sizes as well.

Another difficulty is related to the fact that in a Metropolis relaxation, tunnelling out of a metastable state is rather unlikely because it involves a large energy increase. We believe, however, that this is in fact what we are looking for, since we wish to detect the existence of precisely these metastable states. Furthermore, the metastable states found here are stabilized not by their topological structure, but by their local "singularities", which for large λ result in a large effective mass in the regions

of $\phi \approx 1$. Thus the survival of the metastable states is completely independent of boundary conditions placed on the lattices, which is, of course, not the case for the usual solitons. In a suitably defined continuum limit, we envisage that these states become stable.

As long as the renormalization group flow of the model is not known, our analysis remains an exploratory tour in the coupling constant manifold in search of a new realization of the theory.

We conclude, as we have started, with some rather speculative thoughts. Preliminary results of measuring the correlation functions suggest that the correlations across "singularities" is much smaller than the intra-mural correlations. This is what would be expected in a nearly compactified ensemble of islands. It is as yet unclear what the particle structure in this theory is. Even if the various islands are separated only by a finite wall, it may be that the band structure of the particles propagating in a random medium will contain localized states ⁷⁾.

Assuming that translational invariance is in fact broken in the Liouville model, it is still not clear whether this is due mainly to the problems with the naïve vacuum, i.e., to remoteness in functional space and the lack of a consistent perturbation theory around it, or whether a similar phenomenon may also be possible in other theories with steep potentials and "good" classical vacua. If one wants a theory with a true ground state and the "natural" vacuum is ill-defined, one is forced to search for other candidates. However, the ϕ^4 model or the sinh-Gordon model (with a $\cosh \phi$ potential) are presumably consistent when constructed around the classical vacuum at $\phi=0$. Yet, these theories may also be consistent when constructed around any of the singular soliton solutions, analogous to those of the Liouville model, which break translational invariance. These solitons are prevented from decaying to the classical vacuum by their singularities and could be stable even if they have infinite energy above the classical vacuum. What physical sector the theory chooses depends entirely on the initial conditions. We believe that a Monte Carlo simulation of these systems would yield results of the same nature as those that we have found for the Liouville model. On the other hand, there are now quite a few systems under study whose classical vacua seem to be at infinity ⁸⁾. For each such system, we claim that alternative vacua should be considered.

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NOTE ADDED

After this work was completed, we received a preprint by E. D'Hoker, D. Freedman and R. Jackiw, "SO(2,1) invariant quantization of the Liouville theory", MIT CPT 1072. They make a detailed study of the perturbation theory around a singular solution and argue that there are no infra-red singularities in any order. In addition, they point out that a similar treatment of the ϕ^4 model also yields a stable theory - in agreement with the observations made here. However, the theory which emerges does not have an obvious particle interpretation.

R E F E R E N C E S

- 1) See, for instance
S.J. Avis and C.J. Isham, Recent developments in gravitation, Cargèse 1978,
p. 360, edited by M. Lévy and S. Deser.
- 2) S. Fubini, Nuovo Cimento 34A (1976) 521;
V. De Alfaro, S. Fubini and G. Furlan, Nuovo Cimento 34A (1976) 569.
- 3) S. Mandelstam, Berkeley preprint UCB-PTH-82115 (1982); CERN preprint
TH.3385 (1982);
L. Brink, O. Lingren and B. Nilsson, University of Texas preprint UTTG-1-82
(1982).
- 4) K. Stelle, P. Howe and P. West, Santa Barbara preprint NSF-ITP-83-09 (1983);
P.S. Howe, K. Stelle and P.K. Townsend, to be published;
K. Stelle, Imperial College preprint ICTP/82/83-13 (1983).
- 5) J. Liouville, J.Math.Pures Appl. 18 (1853) 71;
A.M. Polyakov, Phys.Letters 103B (1981) 207;
T. Curtright and C. Thorn, Phys.Rev.Letters 48 (1982) 1309; 48 (1982) 1786;
J.L. Gervais and A. Neveu, Ecole Normale Supérieure preprint LPTENS 81/26,
82/7 (1982);
P. Mansfield, Nuclear Phys. B208 (1982) 277;
O. Alvarez, Cornell preprint CLNS 82/539 (1982);
For more references, see, e.g.,
R. Jackiw, MIT preprint CTP 1049 (1982).
- 6) E. D'Hoker and R. Jackiw, Phys.Rev.Letters 50 (1983) 1719.
- 7) For a review, see, for example
E. Fradkin, CEN Saclay preprint DPh-T/82-72 (1982).
- 8) For SUSY QCD see :
M. Srednicki, private communication;
G. Veneziano, unpublished;
I. Affleck, H. Dine and N. Seiberg, to be published.
For the non-compact $O(N,1)/O(N)$ model see :
Y. Cohen and E. Rabinovici, Physics Letters 124B (1983) 371;
D. Amit and A.C. Davis, Institute of Advanced Studies Princeton preprint
(1983).

FIGURE CAPTIONS

- Figure 1 : The regulated potential for $\lambda = 2$. In the regions $\psi = \frac{\pi}{2} + p\pi$, $p = 0, \pm 1, \pm 2, \dots$ this potential resembles the Liouville potential.
- Figure 2 : Static configurations in the potential of Fig. 1 (with $\lambda = 2$) obtained by Runge-Kutta integration of the equations of stability derived by variation of Eq. (6). In order to ease the comparison with the lattice case, we have chosen the unit of length to be $a = 1$ and define μ by Eq. (10).
- a) A single period of a multisoliton solution corresponding to $\lambda = 2$, $\mu = 1$ and $\epsilon = 0$.
 - b) The aperiodic solution with $\lambda = 2$, $\mu = 1$, $\epsilon = -2\mu e^{-2\lambda} = -0.036$.
 - c) A period of the soliton-antisoliton solution with $\lambda = 2$, $\mu = 1$, $\epsilon = -0.1$.
- Figure 3 : Relaxation time (in sweeps) for single soliton starts. Points with arrows above then indicate that the relaxation time was longer than the run. The crosses and circles correspond to different random generator seeds. The points are reasonably fit by $\ln \tau \approx 1.2 \exp(\lambda/2)$.
- Figure 4 : Same as Fig. 3, but with random starts. The points are reasonably fit by $\tau = 80 \exp(\lambda/4)$.
- Figure 5 : Typical lattice configuration obtained for $\lambda = 19$ with a random start. The cross-hatched areas indicate regions close to the vacuum ($\phi = \cos \psi < 0$), whereas the white areas indicate regions far from the vacuum. This configuration persists for a large number of sweeps ($> 10'000$).

FIG. 1

FIG. 2a

FIG. 2b

FIG. 2c

FIG. 5