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**THREE DIMENSIONAL COSMOLOGICAL GRAVITY:
DYNAMICS OF CONSTANT CURVATURE***

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ABSTRACT

In three space-time dimensions, Einstein's equations with cosmological constant Λ imply that the curvature is constant outside sources. When particles are present, they alter the global properties of the exterior geometry. In the De Sitter case, space is a two-sphere and static many-body solutions are quite different from their $\Lambda = 0$ counterparts; in particular particles lie at antipodal points, with the great-circle wedge between them excised. These configurations are analyzed in terms of the general static solution to the exterior field equations.

I. INTRODUCTION

Three-dimensional Einstein theory with point-particle sources and without cosmological constant is a model where local dynamics is replaced by global effects: locally the curvature vanishes everywhere except at the sources, but the latter affect the global structure of space-time. The resulting spaces were discussed in a previous work [1], where many-body and “Kerr” solutions were analyzed. Here we consider the case of a non-vanishing cosmological constant Λ so that space-time has constant curvature outside sources, being De Sitter for $\Lambda > 0$ or anti-De Sitter for $\Lambda < 0$. We shall again be interested in the global effects of static point sources.

Let us recall the relevant features of the $\Lambda = 0$ analysis. The static solutions are described by properties of space-time’s two-space sections [1]. For one particle, the geometry is given by cutting a sector out of the Euclidean two-plane along two spatial geodesics, *i.e.* straight lines, and identifying the edges to form a cone, whose “angular defect” is proportional to the particle’s mass. For the corresponding De Sitter case, where two-space is a sphere, one would therefore expect cuts from the location of the source along great circles. But on a sphere, these cuts meet again at the antipodal point. The latter then automatically houses a “mirror” source, and there is no true one-particle solution. The global structure of a two-particle solution is that of a sphere minus a wedge, with the edges identified. As we shall show, solutions with more than two particles can also be constructed, but the particle positions and masses are in general constrained in order to allow the appropriate identifications of the wedges’ edges. For anti-De Sitter space, where the sphere is replaced by a hyperboloid, the “antipodes” are infinitely far away and the mirror sources can be ignored, as in the $\Lambda = 0$ case.

The above geometrical picture will be confirmed by analytical calculation. Our problem consists of solving the [static] Einstein equations, which are non-trivial only in their time-time and space-space components. In suitable coordinates, the former is just the Liouville equation, well-known [2] to express possible metric forms of constant curvature two-spaces [here with prescribed point singularities], while the latter produces the conformal Killing equation. Both of these can be solved explicitly and we shall be able to characterize all static metrics in terms of one “master” function. Before turning to the field equations, let us first recall

some properties of De Sitter and anti-De Sitter spaces in the absence of sources which in fact is all we need, since the point sources alter only the ranges of the coordinates, but not the exterior geometry, just as for $\Lambda = 0$.

Because of global peculiarities in De Sitter space, it is important to use an appropriate coordinate system. The full space can be described [in "comoving" coordinates] by the interval [3]

$$ds^2 = \Lambda^{-1} \cosh^2 r \sqrt{\Lambda} (d\chi^2 + \sin^2 \chi d\Phi^2) - dr^2 \quad (1.1)$$

where the spatial sections are two-spheres. If one restricts oneself to "half" the space, the interval may be written as [3]

$$ds^2 = e^{2r'} \sqrt{\Lambda} (d\rho^2 + \rho^2 d\Phi^2) - dr'^2 \quad (1.2)$$

which has flat two-sections. The latter description would lose all the "antipodal" requirements imposed by (1.1), and the mirror sources would not be seen since they lie in the other "half"; with (1.2) one finds solutions with arbitrary source locations as in the $\Lambda = 0$ case. Because it is incomplete, we shall not consider the coordinate system (1.2) any further, but remain with (1.1). In the absence of sources, the range of variables in (1.1) is $-\infty < r < \infty, 0 \leq \chi \leq \pi, 0 \leq \Phi \leq 2\pi$ and the topology is $R^1 \times S^2$. There are no singularities, apart from the trivial ones arising from parametrizing the two-sphere.

The standard static De Sitter form in Schwarzschild [curvature] coordinates is given by

$$ds^2 = (1 - \Lambda R^2)^{-1} dR^2 + R^2 d\Phi^2 - (1 - \Lambda R^2) dt^2 \quad (1.3)$$

whose singularity at $R = \Lambda^{-\frac{1}{2}}$ is unphysical as is seen through comparison with (1.1). The range $0 \leq R \leq \Lambda^{-\frac{1}{2}}$ corresponds to covering half the spatial sphere ($0 < \chi < \pi/2$), and the second half ($\pi/2 < \chi < \pi$) is covered by passing over the range $\Lambda^{-\frac{1}{2}} \geq R \geq 0$ a second time. Another form - the static version of (1.1) - exhibits the correct covering; it is obtained from (1.3) by the change of variable

$$\sin \Omega = \Lambda^{\frac{1}{2}} R \quad (1.4)$$

with Ω ranging from 0 to $\pi/2$ and then to π as R ranges from 0 to $\Lambda^{-\frac{1}{2}}$ and then back to 0.

$$ds^2 = \Lambda^{-1} [d\Omega^2 + \sin^2 \Omega d\Phi^2] - \cos^2 \Omega dt^2 \quad (1.5)$$

Similar forms apply in the anti-De Sitter ($\Lambda < 0$) case, where space is hyperbolic, and hyperbolic functions replace trigonometric ones in (1.1) and (1.5), while (1.3) remains unchanged: $(1 - \Lambda R^2) \rightarrow (1 + |\Lambda| R^2)$.

Since the anti-De Sitter case is globally less interesting than the $\Lambda > 0$ model, we shall deal primarily with the latter. The former can be studied in a similar fashion, with obvious replacement of spherical by hyperbolic structures.

II. FIELD EQUATIONS

In three-dimensional space-time, Einstein's equations with cosmological constant Λ and sources described by an energy-momentum tensor $T^{\mu\nu}$,

$$\sqrt{-g}[G^{\mu\nu} - \Lambda g^{\mu\nu}] = 8\pi G \sqrt{-g} T^{\mu\nu} \quad (2.1)$$

imply that the geometry has constant curvature outside sources. The reason for this is that the three-dimensional curvature tensor is simply the double dual of the Einstein tensor, so that when $T^{\mu\nu} = 0$

$$\epsilon^{\mu\nu\lambda} \epsilon_{\alpha\beta\sigma} [G_{\lambda}^{\sigma} - \Lambda \delta_{\lambda}^{\sigma}] \equiv R^{\mu\nu}{}_{\alpha\beta} - \Lambda (\delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu} \delta_{\beta}^{\mu}) = 0 \quad (2.2)$$

Thus the exterior solutions are spaces of constant curvature: De Sitter for $\Lambda > 0$, and anti De Sitter for $\Lambda < 0$, locally unaffected by sources just as for $\Lambda = 0$. We shall be interested in obtaining solutions of (2.1) with point particle sources; in this Section, we set up the relevant equations to be solved.

The particle's worldlines must of course be geodesics of the constant curvature space. This geodesic requirement is simply met; it is expressed by the usual equation,

$$\ddot{x}^{\mu} + \Gamma_{\alpha\beta}^{\mu} \dot{x}^{\alpha} \dot{x}^{\beta} = 0 \quad (2.3)$$

which is equivalent to the energy-momentum tensor conservation law as applied to a localized source $T^{\mu\nu}$,

$$D_{\nu}(\sqrt{-g} T^{\mu\nu}) \equiv \partial_{\nu}(\sqrt{-g} T^{\mu\nu}) + \Gamma_{\alpha\beta}^{\mu} \sqrt{-g} T^{\alpha\beta} = 0 \quad (2.4)$$

Here D_{ν} is the covariant derivative. Static solutions are obtainable if [and only if] particles which are at rest initially, $\dot{x}_i = 0$, do not acquire any acceleration, i.e. if Γ_{00}^i vanishes at each particle in an appropriate coordinate frame, for example in (1.3).

For the static spaces with which we are concerned, the metric may be parametrized according to

$$-g_{00} = N^2(\vec{r}) \quad (2.5a)$$

$$g_{0i} = 0 \quad (2.5b)$$

$$g_{ij} = \phi(\vec{r})\delta_{ij} \quad (2.5c)$$

We have made the static *Ansatz*, and in (2.5c) we are taking advantage of the fact that two-space is conformally flat, so that its interval can be expressed as

$$d\ell^2 = \phi dx^i dx^i = \phi dz dz^* \quad (2.6)$$

where $z \equiv x + iy = re^{i\theta}$. Various transformations may be applied to (2.5) to pass to the standard static forms (1.3) and (1.5); see below.

For the energy-momentum tensor in (2.1) and (2.4) we take a time-independent expression, non-vanishing only in its time-time component:

$$\sqrt{-g}T^{00} = N\mathcal{E} \quad (2.7)$$

Then only the $\nu = i$ component of (2.4) is non-trivial, giving

$$\Gamma_{00}^i \mathcal{E} = \phi^{-1} N \partial_i N \mathcal{E} = 0 \quad (2.8)$$

which shows that $\mathcal{E}$ must vanish whenever $\phi^{-1} N \partial_i N$ is non-vanishing. Since N is not constant, we conclude that the energy density $\mathcal{E}$ must be local, and we shall take it to be

$$\mathcal{E} = \sum_{n=1}^N m_n \delta(\vec{r} - \vec{r}_n) \quad (2.9)$$

$\mathcal{E}$ describes N particles located at position $\vec{r} = \vec{r}_n$, each with mass m_n . Moreover, the previously stated requirement, *viz.* the vanishing of Γ_{00}^i at the location of each particle, is once again recognized in (2.8) and (2.9).

The time-time component of Eq. (2.1) is

$$N^{-2} \left(\frac{1}{2} \nabla^2 \ln \phi + \Lambda \phi \right) = -8\pi G \mathcal{E} \quad (2.10)$$

while the space-space components give two independent equations, one for the spatial trace

$$\phi^{-1}\nabla^2 N + 2\Lambda N = 0 \quad (2.11a)$$

and the other for the traceless part,

$$\begin{aligned} \partial_i V^j + \partial_j V^i - \delta_j^i \partial_k V^k &= 0 \\ V^i &= \Lambda^{-1} \phi^{-1} \partial_i N \end{aligned} \quad (2.11b)$$

[The cosmological constant is included in the definition of V for later convenience.]

We shall solve these equations in the exterior region [vanishing right-hand side in (2.10)], but we allow singularities in the solution, to reproduce the point particle delta function in (2.10), though of course none in (2.11).

As expected, (2.10) with vanishing right-hand side is the Liouville equation, characterizing the conformal factor of a constant curvature two-space. Although its general solution is known, we shall see that (2.11) imply (2.10) [by virtue of the Bianchi identities] and we need only solve the former set.

The conformal Killing equation (2.11b) has its general solution determined as follows. Define the complex function $V \equiv V_x + iV_y$. Then (2.11b) becomes the Cauchy-Riemann equation for V , which is solved by $V = V(z)$. From the definition of V^i , it follows that

$$2\phi^{-1}\partial_{z^*} N = \Lambda V(z) \quad (2.12)$$

and (2.11a) becomes

$$\partial_z V + V \partial_z \ln \phi + N = 0 \quad (2.13)$$

That (2.13) implies (2.10) is seen by differentiating (2.13) with respect to z^* , and using (2.12) again, noting that $\nabla^2 \equiv 4\partial_z \partial_{z^*}$.

To solve (2.11a) multiply it by V and use (2.12) to get, upon multiplication by ϕ

$$2V\partial_z \partial_{z^*} N + 2N\partial_{z^*} N = 2V\partial_z \partial_{z^*} N + \partial_{z^*} N^2 = 0$$

After integration with respect to z^* , the above yields a Riccati equation,

$$2V\partial_z N + N^2 = c(z) \quad (2.14)$$

where $\epsilon(z)$ is an integration "constant". Next introduce the variable

$$\zeta = \frac{1}{2} \left[\int^z \frac{dw}{V(w)} + \int^{z^*} \frac{dw^*}{V^*(w^*)} \right] \quad (2.15)$$

so that (2.14) reads

$$\partial_\zeta N + N^2 = \epsilon \quad (2.16)$$

The only real solutions are those for which ϵ is a real constant, *i.e.* z -independent; see below. Then the general solution of (2.16) is

$$N = \sqrt{\epsilon} \tanh \sqrt{\epsilon}(\zeta - \zeta_0) \text{ or } \sqrt{\epsilon} \coth \sqrt{\epsilon}(\zeta - \zeta_0) \quad (2.17)$$

The conformal factor is obtained from (2.12)

$$\phi = \frac{\epsilon}{\Lambda V(z) V^*(z^*) \cosh^2 \sqrt{\epsilon}(\zeta - \zeta_0)} \text{ or } \frac{-\epsilon}{\Lambda V(z) V^*(z^*) \sinh^2 \sqrt{\epsilon}(\zeta - \zeta_0)} \quad (2.18)$$

Eq. (2.18) is in fact the general solution of the Liouville equation; this independently validates the result. Indeed the field equations (2.10,11) or equivalently (2.10, 12, 13) may be solved in reverse order in the following way: for ϕ we take the known general real solution of the Liouville equation, which is parametrized by one arbitrary function $F(z)$ [and its complex conjugate]; we find N from (2.13) in terms of the two arbitrary functions $V(z)$, $F(z)$, and verify that (2.12) is satisfied. However, this procedure produces a complex N ; demanding that it be real gives one relation between V and F , thereby regaining (2.17) and (2.18).

We now proceed to specialize the general exterior metric (2.17, 18) to obtain solutions with desired singularities corresponding to point particles. That is, we seek spaces of constant curvature except on prescribed worldline singularities.

III. FORMS OF THE METRIC

We require ϕ to be positive, to insure a positive spatial interval (2.6). Hence for $\Lambda > 0$ [De Sitter space] only the first solution of (2.17, 18) is acceptable and $\epsilon > 0$,

$$\phi = \frac{\epsilon}{\Lambda V(z) V^*(z^*) \cosh^2 \sqrt{\epsilon}(\zeta - \zeta_0)} \quad (3.1a)$$

$$N = \sqrt{\epsilon} \tanh \sqrt{\epsilon}(\zeta - \zeta_0) \quad (3.1b)$$

$$\epsilon > 0$$

For $\Lambda < 0$ [anti-De Sitter space], the second solution of (2.17, 18) is acceptable, and ϵ is unrestricted.

Here we find instead of (3.1)

$$\phi = \frac{\epsilon}{|A|V(z)V^*(z^*)\sinh^2 \sqrt{\epsilon}(\zeta - \zeta_0)} \quad (3.2a)$$

$$N = \sqrt{\epsilon} \coth \sqrt{\epsilon}(\zeta - \zeta_0) \quad (3.2b)$$

Returning to the $\Lambda > 0$ case of interest, we note that the transformation

$$\sin \Omega = \frac{1}{\cosh \sqrt{\epsilon}(\zeta - \zeta_0)} \quad (3.3a)$$

$$\Phi = \frac{\sqrt{\epsilon}}{2i} \left[\int^z \frac{dw}{V(w)} - \int^{z^*} \frac{dw^*}{V^*(w^*)} \right] \quad (3.3b)$$

brings us back to the standard spherical form of the De Sitter metric (1.5), but generically with unconventional ranges for the variables, producing changes in the global geometry. The further change of coordinates (1.4) casts the metric into the De Sitter-Schwarzschild form (1.3), again with unconventional ranges for the variables. Similar transformations apply in the anti-De Sitter case, with hyperbolic functions replacing circular ones.

Our solutions (3.1) and (3.2), and therefore the metric, are determined by a "master" function $V(z)$, whose form is fixed by source singularities, *i.e.* by $\mathcal{E}$, in the following way. In order that T^{00} describe point particles, ϕ must have zeros and/or poles, so that $N^{-2}\nabla^2 \ln \phi$ give rise to delta functions as a consequence of the Green's function equation $\nabla^2 \ln r = 2\pi\delta^2(\vec{r})$. The strengths and locations of these singular points determine the masses and locations of the particles. But ϕ is determined by V , and the zeros/poles of the former are produced by the latter, hence we study different forms for V . Note also that the consistency condition that Γ_{00}^i vanish at the source is equivalent to NV vanishing at the source.

The simplest case is

$$V = c^{-1}z \quad (3.5a)$$

corresponding to a zero with strength $1/c$, and to

$$2\zeta = c \ln z + c^* \ln z^* \quad (3.5b)$$

Single valuedness of ζ and of ϕ requires $c = c^*$, so that

$$\zeta = c \ln r \quad (3.5c)$$

Defining $\zeta_0 = c \ln r_0$, we obtain [in the r -coordinate system (2.5)]

$$\begin{aligned} \phi &= \frac{\epsilon c^2}{\Lambda r^2 \cosh^2 \sqrt{\epsilon c} \ln r/r_0} \\ N &= \sqrt{\epsilon} \tanh \sqrt{\epsilon c} \ln r/r_0 \end{aligned} \quad (3.6)$$

At the singularity $r = 0$, $\frac{1}{2} \ln \phi \sim (|\sqrt{\epsilon c}| - 1) \ln r$, $N^2 \sim \epsilon$. We may take $c > 0$ without loss of generality and rescale time so that $N^2 = 1$ at the singularity. This means that (3.6) corresponds to a source

$$4G\mathcal{E} = (1 - \sqrt{\epsilon c})\delta^2(\vec{r}) \quad (3.7)$$

i.e. a particle at the origin with mass $4Gm = 1 - \sqrt{\epsilon c}$. Defining the mass defect

$$\alpha \equiv 1 - 4Gm = \sqrt{\epsilon c} \quad (3.8)$$

we may write (3.8) as

$$\begin{aligned} \phi &= \frac{4\alpha^2}{\Lambda r^2 [(\frac{r}{r_0})^\alpha + (\frac{r}{r_0})^{-\alpha}]^2} \\ N &= \frac{(\frac{r}{r_0})^\alpha - (\frac{r}{r_0})^{-\alpha}}{(\frac{r}{r_0})^\alpha + (\frac{r}{r_0})^{-\alpha}} \end{aligned} \quad (3.9)$$

The consistency condition is satisfied, since $V(z)$ vanishes at $z = 0$.

Transforming according to (3.3b) to the covering circular coordinates (1.5), we see that $\Phi = \alpha\theta$; hence Φ ranges over $(0 \rightarrow 2\pi\alpha)$ as θ ranges over $(0 \rightarrow 2\pi)$ and Ω passes from 0 to $\pi/2$ and then to π as r goes from 0 to r_0 and then to ∞ . We have a sphere with an excised wedge running from the north pole ($\Omega = 0$, equivalently $r = 0$) to the south pole ($\Omega = \pi$, equivalently $r = \infty$). Sufficiently near each point, where the effects of curvature are negligible, this corresponds to the beginnings of a cone, as in the $\Lambda = 0$ case [1].

Thus one suspects [correctly!] that the above solution is in fact a two-particle one. The second particle is at $r = \infty$, i.e. at $\Omega = \pi$, as can be seen from the singularity structure of ϕ in (3.9). The particle at infinity has the same $|\alpha|$ as the one at the origin, [just as the particle at infinity for $\Lambda = 0$ corresponds [1] to $\alpha \rightarrow -\alpha$]. That the two sources are antipodal can also be checked by direct computation of the proper

distance between them, which comes out $\pi/\sqrt{\Lambda}$ – half the circumference of a great circle. As $\Lambda \rightarrow 0$, this distance becomes infinite, and one regains the results of the previous analysis [1]: in the metric (3.9) a rescaling of r_0 by $\Lambda^{\frac{1}{2}}$ leads in the limit to $\phi \propto r^{-2+2\alpha}$, $N \rightarrow 1$. The case $\alpha = 1$ represents the source-free theory, while for $\alpha > 1$, the sphere is multiply covered, but incompletely if α is not an integer. Since $\alpha > 1$ corresponds to negative Gm, we shall regard these solutions as unphysical.

In Schwarzschild coordinates (1.3), R ranges from 0 to $\Lambda^{-\frac{1}{2}}$ as Ω goes from 0 to $\pi/2$ (equivalently $0 \leq r \leq r_0$) and then back to 0 as Ω passes from $\pi/2$ to π (equivalently $r_0 \leq r \leq \infty$), but the range of Φ is modified as above. The limit $\Lambda \rightarrow 0$ is readily taken in these coordinates.

Another choice for $V(z)$ is

$$V(z) = nc^{-1}z^{1-n} \quad n \neq 0 \quad (3.10a)$$

$$2\zeta = cz^n + c^*z^{*n} = 2cr^n \cos(n\theta + \arg c) \quad (3.10b)$$

Without loss of generality we may take $\arg c$ to be zero, which selects the origin for θ , while single valuedness requires that n be an integer. In (3.1a), we define ϵ to be r_0^{-n}/c , rename $\sqrt{\epsilon}\xi_0$ as η and find

$$\begin{aligned} \phi &= \frac{(r/r_0)^{2n}}{\Lambda n^2 r^2 \cosh^2[(\frac{r}{r_0})^n \cos n\theta + \eta]} \\ N &= \frac{\tanh[(\frac{r}{r_0})^n \cosh \theta + \eta]}{\tanh \eta} \end{aligned} \quad (3.11)$$

[Time has been rescaled as well.] The limit $\Lambda \rightarrow 0$ may be taken in (3.11) after rescaling r_0^n by $\Lambda^{\frac{1}{2}}$; this yields $\phi \propto r^{-2+2n}$, $N \rightarrow 1$.

For $n > 0$, ϕ vanishes at $r = 0$, where $\frac{1}{2}N^{-2}\ell n\phi$ behaves as $(n-1)\ell nr$, corresponding to a source located at the origin

$$4G\mathcal{E} = (1-n)\delta^2(\vec{r}) \quad (3.12a)$$

Equivalently the mass defect is

$$\alpha = 1 - 4Gm = n \quad (3.12b)$$

This case includes no source ($n = 1$), as well as unphysical solutions with negative Gm ($n \geq 2$).

We have taken non-zero η ; if η vanishes, N cannot be normalized to unity at $r = 0$, rather it vanishes as r^n and $\frac{1}{2}N^{-2}\nabla^2 \ln r$ corresponding to a singularity stronger than a delta function. Also when $n < 0$, the

singularity in $\frac{1}{2}N^{-2}\nabla^2 \ln \phi$ at $r = 0$ cannot correspond to a delta function source, and we shall not consider this possibility.

While it appears that the solution (3.11), with n a positive integer and η non-zero, describes the same physical situation as (3.9) with $\alpha = n$, there is an important difference: in the present case the consistency condition is not satisfied, since V and hence Γ_{00}^i are non-vanishing at $\vec{r} = 0$. Hence we do not have a self-consistent solution to the gravitational field equations and to the static geodesic equation; however, we shall meet this geometry again when we discuss multiparticle configurations.

To summarize: a simple zero or pole in V gives rise to a delta function source. In the latter case $4Gm$ is negative and $1 - 4Gm$ is an integer, equal to the order of the pole. The consistency condition is satisfied in the former case, but not in the latter. Other forms for $V(z)$, *e.g.* $z^a \ln^b z$ *etc.*, do not correspond to delta function sources.

Since delta function sources arise from simple zeros or poles in V , it is clear that the next simplest form for V is the product of two zeros.

$$V = \tilde{c}^{-1}(z - z_1)(z - z_2) \quad (3.13a)$$

Demanding single-valuedness, we find

$$\zeta - \zeta_0 = c[\ln \frac{\rho_1}{\rho_1^0} - \ln \frac{\rho_2}{\rho_2^0}] \quad (3.13b)$$

where $z_{21} \equiv z_2 - z_1$, $\rho_i \equiv |z - z_i| = |\vec{r} - \vec{r}_i|$, ρ_i^0 is an arbitrary scale, and $c \equiv \tilde{c}/z_{21}$ is real. The metric functions become

$$\begin{aligned} \phi &= \frac{\alpha^2 \rho_{12}^2}{A \rho_1^2 \rho_2^2 \cosh^2 \alpha (\ln \frac{\rho_1}{\rho_1^0} - \ln \frac{\rho_2}{\rho_2^0})} \\ N &= \tanh \alpha (\ln \frac{\rho_1}{\rho_1^0} - \ln \frac{\rho_2}{\rho_2^0}) \\ \vec{\rho}_{12} &= \vec{r}_1 - \vec{r}_2 \end{aligned} \quad (3.14)$$

This clearly describes two particles located at $\vec{r} = \vec{r}_i$, with equal [positive] mass defects. There is no particle at infinity, since $\zeta - \zeta_0$ approaches a constant there, and the condition $\Gamma_{00}^i = 0$ is fulfilled at each particle.

As before, the two particles are antipodal, since $\sin \Omega = 0$ at r_1 and r_2 . This is borne out by determining the geodesic distance between them, which can be calculated explicitly to be $\pi/\sqrt{A}$. To effect the calculation, the following argument is used: Place the two particles at $\vec{r}_1 = 0, \vec{r}_2 = (x_2, 0)$. Then the geodesic distance

is the minimal value of

$$l = \int_1^2 dx \sqrt{\phi[1 + (dy/dx)^2]}^{\frac{1}{2}} \quad (3.15)$$

While it is difficult to solve the minimization problem, one expects that the minimal distance is achieved when $\frac{dy}{dx} = 0$. A path lying entirely along the x-axis would either join $\bar{r}_1$ to $\bar{r}_2$ directly, or go out to $-\infty$ from $\bar{r}_1$, and back from $+\infty$ to $\bar{r}_2$. The integral along these paths turns out to be independent of the value of x_2 , and of the $\frac{\rho_2}{\rho_1}$ ratio by scaling. The quadrature along these paths can be performed explicitly, and is given for either path by $\pi/\sqrt{A}$, which shows the x-axis to be a great circle. Moreover, a small deformation analysis verifies that our path is minimal against the examined variations,* making it eminently plausible that a global minimum has also been achieved.

Thus this solution is really equivalent to the first one! The wedge effect is recognized by the fact that the angular coordinate Φ is given by

$$\Phi = -\alpha(\theta_1 - \theta_2) \quad (3.16)$$

where $z - z_n = \rho_n e^{i\theta_n}$. Hence Φ ranges over $[\alpha \text{ times}]$ the angles subtended by the two masses at the observation point, i.e. $0 \leq \Phi \leq 2\pi\alpha$.

Is there a more general two-body solution? Consider

$$\frac{1}{V} = \frac{c_1}{z - z_1} + \frac{c_2}{z - z_2} \quad (3.17a)$$

Then, as before,

$$\zeta - \zeta_0 = c_1 \ln \frac{\rho_1}{\rho_1^0} + c_2 \ln \frac{\rho_2}{\rho_2^0} \quad (3.17b)$$

where single-valuedness fixes the c_i to be real, but otherwise arbitrary, in contrast to (3.13b) where $c_2 = -c_1$. Furthermore, in addition to the particles at $(\bar{r}_1, \bar{r}_2)$, there is a particle at infinity, since there $V^{-1} \rightarrow (c_1 + c_2)z^{-1}$, $\zeta - \zeta_0 \rightarrow (c_1 + c_2)\ell n r$. Also at the center-of-mass point $z \equiv (c_1 + c_2)^{-1}(c_2 z_1 + c_1 z_2)$, V has a pole, giving rise to an "unphysical" delta function source, as in (3.10-12) with $n = 2$. [These novel features disappear when $c_1 + c_2 = 0$, which reproduces the previous case.] For all three points $(\bar{r}_1, \bar{r}_2, \infty)$, $\sin \Omega$ vanishes, so at least two of them coincide [the source at infinity has strength $(c_1 + c_2)$]. There are two separate values

*The use of elliptic coordinates simplifies the calculation.

of z for which $\Omega = \pi/2$; taking $c_1 = c_2$ for simplicity, the point at which $\zeta - \zeta_0$ vanishes is $\rho_1 \rho_2 / \rho_1^0 \rho_2^0 = 1$; this equation has two roots in general and according to (3.3a) both correspond to $\Omega = \pi/2$. One may suppose that what is involved here is a double cover of the sphere. However, we emphasize that in spite of its richness, this is not a consistent solution to the static problem, since V and hence Γ_{00}^i do not vanish at the center-of-mass point where there is a source.

It is now clear how to proceed to more general configurations. For a consistently static N -point metric we take

$$V(z) = c^{-1} \prod_{n=1}^N (z - z_n) \quad (3.18a)$$

$$\begin{aligned} \zeta &= \sum_{n=1}^N \left(\frac{c_n}{2} \ln(z - z_n) + \frac{c_n^*}{2} \ln(z^* - z_n) \right) \\ c_n &= \prod_{\substack{n'=1 \\ n' \neq n}}^N \frac{c}{z_n - z_{n'}}, \quad \sum_{n=1}^N c_n = 0 \end{aligned} \quad (3.18b)$$

For single valuedness, we must have real c_n ; this gives $N - 1$ conditions which constrain the positions of the N particles. Note that there is no particle at infinity, since $\zeta - \zeta_0$ approaches a constant there. Also the mass defects are constrained: they are given by the absolute value of quantities which sum to zero. Note further that the mapping (3.3a) puts all the particles at $\sin \Omega = 0$, i.e. at $\Omega = 0$ or π . Thus we are dealing with a multiple covering of a sphere, which provides a consistent solution to the static N -body problem, since V vanishes at all the sources.

We do not pursue here the general analysis of the geometry of this N -body solution, beyond remarking that for three bodies the reality constraint gives two conditions which force the particles to lie on a line, one being antipodal to the other two.

One may also consider forms for V of the type

$$\frac{1}{V(z)} = \sum_{n=1}^N \frac{c_n}{z - z_n} \quad (3.19a)$$

with real c_n for single valuedness, and

$$\zeta - \zeta_0 = \sum_{n=1}^N c_n \ln \frac{\rho_n}{\rho_n^0} \quad (3.19b)$$