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SUPERGRAVITY IN ELEVEN-DIMENSIONAL SPACE-TIME ^{+) "})

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A B S T R A C T

A comprehensive review of supergravity in eleven dimensions is presented with particular emphasis on the seven-sphere compactification. The complete mass spectrum and its $Osp(8,4)$ content are exhibited and discussed. We conclude with some speculative remarks about the effects of quantum gravity.

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1. - FROM SUPERSYMMETRY TO SUPERGRAVITY IN ELEVEN DIMENSIONS

Quantum field theory in Minkowski space-time is marred by ultra-violet divergences which apparently preclude a perturbative analysis of non-renormalizable theories such as gravity. One of the most basic quantities which diverge in quantum field theory is the vacuum energy $\langle 0|H|0\rangle$. Bosonic and fermionic degrees contribute to it with opposite sign, and one may therefore search for theories which possess a symmetry between Bose and Fermi fields such that

$$\langle 0|H|0\rangle = 0 \quad (1.1)$$

This possibility is indeed realized in supersymmetric field theories¹⁾. These are characterized by a fermionic symmetry which relates bosons and fermions and transforms them into each other. In the simplest case, the generators of supersymmetry transformations are given by a Majorana spinor supercharge Q_α which obeys

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma_{\alpha\beta}^\mu P_\mu \quad (\mu = 0, 1, 2, 3) \quad (1.2)$$

If supersymmetry is not spontaneously broken, the supercharge Q annihilates the vacuum, and this leads directly to (1.1). Supersymmetric field theories have further divergence cancellations, and fully convergent supersymmetric models have been constructed. In general, the weakening of ultra-violet divergences implied by (1.2) reduces the radiative corrections of the scalar fields to logarithmic terms, a feature which persists if supersymmetry is softly broken²⁾. In this way one can stabilize large mass scale ratios of the type encountered in grand unified theories. Thus both at the prospective theoretical level and in more direct phenomenological applications, supersymmetry appears promising for further development in the field theoretical description.

It is then natural to investigate invariance under local supersymmetry. By Eq. (1.2), this implies invariance under "local translations" and hence general relativity. In this way a thread leading towards a unified theory of matter and gravitation is brought to light: the graviton helicity state $2(-2)$ must now be accompanied by

a spin- $\frac{3}{2}$ ($-\frac{3}{2}$) degree of freedom, the gravitino, connected to it by the supersymmetric generator Q . This locally supersymmetric graviton-gravitino system constitutes $N = 1$ supergravity³⁾. Full unification is achieved if one requires in addition that the two helicity states of the graviton are connected by supersymmetry transformations. If there are N independent supercharges Q_α^i , the irreducible multiplets cover a range of spin of at least $N/4$. To connect the two helicity states of the graviton, one therefore needs eight helicity- $\frac{1}{2}$ flips; this case corresponds to $N = 8$ supersymmetry and the minimal multiplet obtained in this way is CPT self-conjugate⁴⁾. In the presence of eight supersymmetry charges Q_α^i ($i = 1, \dots, 8$), (1.2) generalizes to

$$\{Q_\alpha^i, \bar{Q}_{\beta j}\} = 2\delta_j^i \gamma_{\alpha\beta}^\mu P_\mu \quad (1.3)$$

The field content of such an " $N = 8$ extended supergravity theory" is determined by the states obtained from the helicity 2 state by the anticommuting Q^i ; hence it comprises 1 spin-2, $\binom{8}{1} = 8$ spin- $\frac{3}{2}$, $\binom{8}{2} = 28$ spin-1, $\binom{8}{3} = 56$ spin- $\frac{1}{2}$ and $\binom{8}{4} = 70$ spin-0 fields. Clearly, no other field system of spin < 2 can be consistently coupled to such an $N = 8$ theory which therefore fully unifies matter and gravitation. Moreover, any supersymmetric theory with $N > 8$ would have to contain particles of spin- $\frac{5}{2}$ or higher and such theories do not appear to be consistent.

We will now rewrite Eq. (1.3) in a more compact and "unified" form. For this purpose, we combine the eight four-component Majorana spinors of (1.3) into one single 32-component Majorana spinor $\tilde{Q}$. Simple counting reveals that the number of space-time dimensions appropriate to accommodate such a spinor is eleven [for a review, see Ref. 5)]. We are thus led to consider an eleven-dimensional space-time of signature $(+ - \dots -)$ instead of the usual four-dimensional space-time. The $\tilde{\Gamma}$ matrices in eleven dimensions can be expressed in terms of four-dimensional γ matrices and eight-dimensional Γ matrices, which generate the Clifford algebra in seven dimensions, according to

$$\tilde{\Gamma}^\mu = \gamma^\mu \otimes \mathbb{1} \quad , \quad \tilde{\Gamma}^m = \gamma^5 \otimes \Gamma^m \quad (1.4)$$

where the indices $\mu, \nu, \dots$ take the values 0, 1, 2, 3 and the indices $m, n, \dots$ are 4, ..., 10. Indices in eleven dimensions will be denoted by $M, N, \dots$. With this, (1.3) becomes (spinor indices α now take the values 1, ..., 32)

$$\begin{aligned} \{\tilde{Q}_\alpha, \bar{\tilde{Q}}_\beta\} &= 2\tilde{\Gamma}_{\alpha\beta}^M P_M \\ \{\tilde{\Gamma}_M, \tilde{\Gamma}_N\} &= 2g_{MN} \end{aligned} \quad (1.5)$$

with $g^{MN} = \text{diag}(1, -1, \dots, -1)$ if, in addition, we stipulate that $P_m = 0$, or

$$P_M = 0 \quad \text{for} \quad M = 4, 5, \dots, 10 \quad (1.6)$$

in order to make (1.5) agree with (1.3). From this equivalence, we conclude that $N = 1$ supergravity in eleven dimensions should lead to an $N = 8$ supergravity theory in four dimensions when the fields do not depend on the extra seven co-ordinates. The action of $N = 1$ supergravity in eleven dimensions has been constructed⁶⁾, and an $N = 8$ supergravity in four dimensions has indeed been obtained in this way⁷⁾. The action in eleven dimensions is unique, and, on the basis of the observation that supergravity in more than eleven dimensions would inevitably contain states of spin higher than two⁸⁾, it has been argued that no supergravity theory exists beyond $d = 11$.

At this point, one may call into question the physical relevance of the constraint (1.6) by which the four-dimensionality of space-time is a priori imposed. There are reasons to expect that, at large scales, the apparent four-dimensionality of space-time is, from quantum theory, the required framework to describe long-range forces. Four is the critical number of dimensions for gauge field theories: in more than four dimensions, these theories are non-renormalizable, so the effective coupling constants should decrease like inverse powers in the low energy limit (or at large distances). It is only in four dimensions that gauge theories are strictly renormalizable and that the effective coupling strengths vary logarithmically. This could explain the existence of long-range forces at present day energies and provide a reason why not all coupling constants are of the same order as the gravitational coupling (whose smallness would be a consequence of the non-renormalizability of gravity). This picture should then naturally emerge from a truly unified theory of all interactions. However, there is no compelling reason why space-time should be four-dimensional at very small scales. Given the uniqueness of the $N = 1$ supergravity in eleven dimensions, this line of argument suggests that this theory is more than just a technical device to construct $N = 8$ supergravity in four dimensions (as it was originally conceived) and should be considered in its own right. We shall therefore drop the constraint $P_m = 0$ and tentatively assume that it is in fact the $d = 11$ theory which describes the physical world.

The assumption that space-time has more than four dimensions is not new and also underlies Kaluza-Klein theories [for reviews and further references, see 9)-12)]. There, the transition from the original higher-dimensional theory at small scales to the required four-dimensional background at large scales can occur naturally already at the classical level through the spontaneous compactification mechanism¹³⁾ which we shall now investigate.

2. - SPONTANEOUS COMPACTIFICATIONS OF ELEVEN-DIMENSIONAL SUPERGRAVITY

2.1 Spontaneous Compactification

The eleven-dimensional $N = 1$ supergravity action is based on the following fields⁶⁾: an elfbein e_M^A (flat indices are labelled by the first letters of the alphabet) which describes ordinary gravity in eleven dimensions, a 32-component Majorana spinor ψ_M which is the analogue of the gravitino in four dimensions and an antisymmetric three-index tensor A_{MNP} defined up to a gauge transformation^{*)}

$$\delta A_{MNP} = D_{[M} \Lambda_{NP]} \quad , \quad D^M \Lambda_{MN} = 0 \quad (2.1)$$

This field is required in order to balance bosonic and fermionic degrees of freedom in eleven dimensions. Defining $F_{MNPQ} \equiv 24A_{[NPQ,M]}$ and $\tilde{\Gamma}^{MN\dots R} \equiv \Gamma^{[M}\tilde{\Gamma}^N\dots\tilde{\Gamma}^R]$, we may write the action as follows (up to four-Fermi terms which can be included through supercovariantized elfbein affinities and field strengths F_{MNPQ})

$$\begin{aligned} S = \int d^{11}x \, e \left\{ -\frac{1}{2} R - \frac{1}{48} F_{MNPQ} F^{MNPQ} - \frac{i}{2} \bar{\psi}_M \tilde{\Gamma}^{MNP} \psi_{P;N} + \right. \\ \left. + \frac{4\sqrt{2}}{(4!)^3} \eta^{M_1\dots M_{11}} F_{M_1\dots M_4} F_{M_5\dots M_8} A_{M_9 M_{10} M_{11}} + \right. \\ \left. + \frac{3\sqrt{2}}{(4!)^2} \left(\bar{\psi}_M \tilde{\Gamma}^{MNPQRS} \psi_N + 12 \bar{\psi}^P \tilde{\Gamma}^{QR} \psi^S \right) F_{PQRS} \right\} \quad (2.2) \end{aligned}$$

Here $\eta^{M_1\dots M_{11}}$ is the eleven-dimensional Levi-Civita tensor. The action (2.2) is invariant under the local supersymmetry transformations

$$\delta e_M^A = -\frac{i}{2} \bar{\epsilon} \tilde{\Gamma}^A \psi_M \quad (2.3)$$

$$\delta A_{MNP} = \frac{\sqrt{2}}{8} \bar{\epsilon} \tilde{\Gamma}_{[MN} \psi_{P]} \quad (2.4)$$

$$\delta \psi_M = \epsilon_{;M} + \frac{2\sqrt{2}i}{(4!)^2} \left(\tilde{\Gamma}^{NPQR}{}_{;M} - 8 \delta_M^N \tilde{\Gamma}^{PQR} \right) F_{NPQR} \epsilon \quad (2.5)$$

Again we have omitted Fermi bilinears which result from supercovariantization of $\delta\psi_M$ in Eq. (2.5).

We have argued that, at large scales, the space-time background should be four-dimensional and it is therefore of interest to inquire whether four-dimensional space-time has a natural classical limit. To this effect, we search for solutions of the classical equations of motion which describe a direct product space

*) Covariant derivatives are denoted by $D_M \phi = \phi_{;M}$.

$\mathcal{M}_7\{y^m\} \times \mathcal{M}_4\{x^\mu\}$ where $\mathcal{M}_7$ is compact and $\mathcal{M}_4$ admits local Lorentz invariance. Any such classical solution provides a "vacuum state" configuration $|\Omega\rangle$ for the Fock-space of normal mode excitations $\phi^{(n)}(x)$. Explicitly, one expands all fields $\phi(x,y)$ of the $d = 11$ theory in terms of a complete set of functions on the internal manifold $\mathcal{M}_7$ according to

$$\phi(x, y) = \sum_n \phi^{(n)}(x) Y^{(n)}(y) \quad (2.6)$$

where we have suppressed indices which might be attached to the various fields. Four-dimensional fields are associated with functions $\phi^{(n)}(x)$ while $Y^{(n)}(y)$ are eigenfunctions of suitable operators on $\mathcal{M}_7$ (explicit examples will be presented later). Inserting the expansion (2.6) into the kinetic term in the higher dimensional space-time, one sees that, quite generally, the part which operates in the internal space becomes a mass term in four dimensions. It is therefore the mass operator which determines the spectrum and the set of normal modes $\{Y^{(n)}(y)\}$ in (2.6). In general, the relevant operators are elliptic since the signature in the internal space is uniform $(-...-)$, and the compactness of the internal manifold $\mathcal{M}_7$ ensures that the spectrum is discrete and that there exists a gap between massless and massive excitations. The mechanism by which a classical solution leads to a spontaneous decomposition of eleven-dimensional space-time into a product of $\mathcal{M}_4$ and $\mathcal{M}_7$, is referred to as spontaneous compactification¹³⁾.

In the absence of fermion condensates, the classical equations which arise from the action (2.2) are

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R &= \\ &= - \frac{1}{48} \left\{ 8 F_{MPQR} F_N{}^{PQR} - g_{MN} F_{PQRS} F^{PQRS} \right\} \end{aligned} \quad (2.7)$$

$$F_{MNPQ}; M = - \frac{\sqrt{2}}{2 \cdot (4!)^2} \gamma^{M_1 \dots M_8 N P Q} F_{M_1 \dots M_4} F_{M_5 \dots M_8} \quad (2.8)$$

and, up to supersymmetry gauge transformations, we must have

$$\psi_M = 0 \quad (2.9)$$

The solutions describing spontaneous compactification on $\mathcal{M}_7 \times \mathcal{M}_4$ are characterized by $g_{\mu m} = F_{\mu m n p} = F_{\mu v n p} = F_{\mu v o p} = 0$. If we impose in addition

$$F_{mnpq} = 0 \quad (2.10)$$

we get from Eqs. (2.8) and (2.9)

$$F_{\mu\nu\sigma} = f \varepsilon_{\mu\nu\sigma} , \quad f = \text{constant} \quad (2.11)$$

$$R_{mn} = -6 m_7^2 g_{mn} \quad (2.12)$$

$$R_{\mu\nu} = 12 m_7^2 g_{\mu\nu} \quad (2.13)$$

$$m_7^2 = f^2/18 \quad (2.14)$$

These solutions were obtained by Freund and Rubin¹⁴⁾ who pointed out that the natural splitting of eleven dimensions in $7 + 4$ which follows from Eq. (2.11) is a consequence of the fact that the field strength F_{MNPQ} has four indices. Note that Eq. (2.8) implies the validity of Eq. (2.11) even when $F_{mnpq} \neq 0$.

2.2 Symmetries of the compactified vacuum

We now examine the possible gauge symmetries and four-dimensional supersymmetries of the vacuum obtained after spontaneous compactification. As in the original Kaluza-Klein theory and its non-Abelian generalizations^{9),10)}, the invariance gauge group on $\mathcal{M}_4$ will contain the isometry group of the internal space $\mathcal{M}_7$ for Freund-Rubin solutions. Indeed, under the transformation

$$\begin{aligned} x'^\mu &= x^\mu \\ y'^m &= y^m + \varepsilon^{(\alpha)}(x) \xi^{(\alpha)m}(y) \end{aligned} \quad (2.15)$$

the seven-dimensional metric is left invariant if the vector field $\xi^{(\alpha)m}(y) \partial_m$ generates an isometry. This is the case if $\xi^{(\alpha)m}(y)$ is a Killing vector, i.e.,

$$2 \xi^{(\alpha)}_{(m;n)} = \xi^{(\alpha)}_{m;n} + \xi^{(\alpha)}_{n;m} = 0 \quad (2.16)$$

The Killing vectors generate a group - the isometry group of $\mathcal{M}_7$ -, and (α) is therefore a group index. Thus, we have

$$[\xi^{(\alpha)m}(y) \partial_m, \xi^{(\beta)n}(y) \partial_n] = f^{(\alpha)(\beta)}_{(\gamma)} \xi^{(\gamma)p}(y) \partial_p \quad (2.17)$$

where $f^{(\alpha)(\beta)}_{(\gamma)}$ are the structure constants of the group. Since the parameter $\xi^{(\alpha)}(x)$ in (2.15) is space-time dependent, the resulting theory in four dimensions will inherit the isometry group of $\mathcal{M}_7$ as a local gauge group. In the presence of matter fields, the actual symmetry may be smaller than the isometry group of $\mathcal{M}_7$.

For $F_{mnpq} \neq 0$, it is reduced to that subgroup of the isometry group which, in addition to (2.16), leaves the tensor F_{mnpq} invariant, viz.,

$$\delta_{\xi} F_{mnpq} = \xi^r D_r F_{mnpq} + 4 D_{[m} \xi^r F_{n]pq]r} = 0 \quad (2.18)$$

Global supersymmetries, on the other hand, may survive spontaneous compactification only if there exists at least one 32-component Majorana spinor $\epsilon(x,y)$ for which¹¹⁾

$$\delta \psi_M = 0 \quad (2.19)$$

This condition is the fermionic analogue of (2.16). We now show that only Freund-Rubin solutions may have supersymmetric ground states if the spinor $\epsilon(x,y)$ factorizes into a product of a four-component and an eight-component spinor according to $\epsilon(x,y) = \epsilon(x)\eta(y)$ ¹⁵⁾ [we will assume that $\epsilon(x,y)$ factorizes in this way although it is far from clear whether such a factorization is really necessary to isolate a local supersymmetry in four dimensions. If space-time is not described by a global product $\mathcal{M}_4 \times \mathcal{M}_8$, but rather by a non-trivial fibre-bundle, this assumption is presumably no longer valid].

From Eq. (2.5) and local Lorentz invariance which implies Eq. (2.11), we may write Eq. (2.16) as

$$\begin{aligned} \epsilon(x) \left\{ \eta(y)_{;m} - \frac{1}{2} \frac{f}{\sqrt{18}} \Gamma_m \eta(y) \right\} + \\ + i \gamma^5 \epsilon(x) \cdot \frac{\sqrt{12}}{(4!)^2} \left\{ \Gamma_m{}^{npqr} - 8 \delta_m{}^n \Gamma^{pqr} \right\} F_{npqr}(y) \eta(y) = 0 \end{aligned} \quad (2.20)$$

$$\left\{ \epsilon(x)_{; \mu} - \frac{f}{\sqrt{18}} \gamma^5 \gamma_{\mu} \epsilon(x) \right\} \eta(y) + \frac{2\sqrt{2}i}{(4!)^2} \gamma_{\mu} \Gamma^{mnpq} F_{mnpq} \eta(y) \epsilon(x) = 0 \quad (2.21)$$

Projecting (2.20) onto chiral components, we see that both the coefficients of ϵ and $i\gamma_5 \epsilon$ in (2.20) must vanish and hence η must be a "Killing spinor" satisfying

$$\eta(y)_{;m} = \frac{1}{2} m_7 \Gamma_m \eta(y) \quad (2.22)$$

where m_7 is related to f by (2.14). The integrability condition of (2.22) reads

$$(R_{mnpq} \Gamma^{pq} + 2 m_7^2 \Gamma_{mn}) \eta(y) = 0 \quad (2.23)$$

which yields Eq. (2.12) upon contraction with Γ^m . However, Eqs. (2.11), (2.12) and (2.14) are consistent with the eleven-dimensional Einstein equation (2.7) if and only if $F^{mnpq} F_{mnpq} = 0$. Hence the only admissible solutions of Eq. (2.17) are given by

$$F^{mnpq} = 0 \quad (2.24)$$

and we recover the Freund-Rubin solutions. One then verifies that the integrability condition of (2.21), namely

$$(R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} - 8 m_7^2 \gamma_{\mu\nu}) \varepsilon(x) = 0, \quad (2.25)$$

is indeed consistent with Eq. (2.13). Hence, only for Freund-Rubin solutions can supersymmetry survive compactification if $\varepsilon(x,y) = \varepsilon(x) \cdot \eta(y)$. That the residual supersymmetry is in fact local follows from (2.20) which implies $\delta\psi_m = 0$ for arbitrary $\varepsilon(x)$ while such transformations induce only a four-dimensional supersymmetry gauge term in $\delta\psi_\mu$.

The largest number of supersymmetries available in four dimensions is equal to the number of linearly independent solutions of (2.22), namely eight. This can only happen if the integrability conditions (2.23) and (2.25) are identically satisfied. Hence

$$R_{mnpq} = - m_7^2 (g_{mp} g_{nq} - g_{mq} g_{np}) \quad (2.26)$$

$$R_{\mu\nu\rho\sigma} = 4 m_7^2 (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \quad (2.27)$$

and there are only two fully supersymmetric compactifications¹⁶⁾

$$(A) : \quad m_7^2 = 0 \quad \mathcal{M}_4 \times \mathcal{M}_7 = \text{Minkowski} \times T^7 \quad (2.28)$$

$$(B) : \quad m_7^2 \neq 0 \quad \mathcal{M}_4 \times \mathcal{M}_7 = (\text{AdS})_4 \times S^7 \quad (2.29)$$

Case (A) corresponds to the original $N = 8$ theory of Cremmer and Julia⁷⁾ when only the zero modes are retained. Case (B) corresponds to the compactification on the seven sphere S^7 of (arbitrary) radius $|m_7|^{-1}$ to an anti-de Sitter $(\text{AdS})_4$ space-time. This solution was first exhibited by Duff and Pope who were led to it by requiring the existence of eight unbroken supersymmetries^{12),18)}. Since the seven sphere S^7 is the coset space $SO(8)/SO(7)$ which admits an $SO(8)$ isometry group, this compactification describes a theory with local $SO(8)$ invariance. If only the zero-mass supermultiplet of states is retained, this theory is presumably equivalent to gauged $N = 8$ supergravity¹⁷⁾. This expectation has been partially confirmed by explicit calculations at the linearized level^{16),18)} where one may identify fields and states.

Solutions with $N < 8$ supersymmetries have also been obtained with $N = 1, 2$ and 4, and it has recently been shown that these are the only permitted values¹⁹⁾. One example of a solution with $N = 1$ supersymmetry is the "squashed sphere" of

Ref. 20) that corresponds to a compactification on the coset space $Sp(2) \times Sp(4) / Sp(2)' \times Sp(2)''$ ²¹⁾ [the primes indicate that $Sp(2)' \neq Sp(2) \neq Sp(2)''$] with an isometry group $Sp(2) \times Sp(4)$. An example with $N = 2$ supersymmetry is obtained with $\mathcal{M}_7 = (SU(3) \times SU(2) \times U(1)) / (SU(2) \times U(1) \times U(1)'')$ for particular embeddings of the $U(1)$ subgroups into $SU(3) \times SU(2) \times U(1)$ ²²⁾. Both spaces have $m_7 \neq 0$. Finally, an example with $N = 4$ supersymmetry and $m_7 = 0$ is also known ²³⁾.

2.3 Supersymmetry breaking

We shall now show that for each solution having $N \geq 1$ supersymmetry one can associate at least one other solution where supersymmetry is fully broken by a mechanism having a geometrical significance in the unseen dimensions ¹⁵⁾. The new solution which has $F_{mnpq} \neq 0$, can be interpreted as a spontaneously broken version of the old one.

Consider Eq. (2.22) and eventually the equation obtained from it by reversing the sign in front of m_7 . We write

$$\eta^\pm_{;m} \mp \frac{1}{2} m_7 \Gamma_m \eta^\pm = 0 \quad (2.30)$$

The η^+ is a covariant constant spinor characterizing a residual supersymmetry. The η^- satisfies a different differential equation in the same co-ordinate system ^{*}) and its existence is not guaranteed by the necessary condition (2.23) and in some cases it indeed does not exist. We look for a new solution of (2.8) with $F_{mnpq} \neq 0$ in the old Einstein space $\mathcal{M}_7$ but we allow for a constant rescaling of the $\mathcal{M}_4$ space to cope with an eventual shift of the cosmological constant. We have

$$F_{\mu\nu\sigma} = f' \eta_{\mu\nu\sigma}, \quad f' = \text{constant} \quad (2.31)$$

and F^{mnpq} must be a solution of

$$F^{mnpq}_{;m} = \frac{\sqrt{2}}{4!} f' \varepsilon^{npqrst} F_{rstu} \quad (2.32)$$

It is readily checked that

$$\begin{aligned} A^\pm_{mnp} &= \xi \bar{\eta}^\pm \Gamma_m \Gamma_n \Gamma_p \eta^\pm = \\ &= \xi \bar{\eta}^\pm \Gamma_{mnp} \eta^\pm \quad (\bar{\eta} \equiv \eta^+) \end{aligned} \quad (2.33)$$

^{*}) In order to avoid confusion we distinguish the two equations explicitly despite the fact that the sign of m_7 is not fixed.

solves (2.32) provided that

$$\pm m_7 = - f'/\sqrt{8} \quad (2.34)$$

and is dual or antidual to its field strength F_{pqrs} . The arbitrary constant ξ is determined, up to a sign, by Eq. (2.7) and one finds $\xi^2 = 2/(4!)^2$. The new solution(s) corresponding to η^+ (and η^- if the latter exists) has (have) $F_{mnpq} \neq 0$ and hence all supersymmetries are broken. The metric is unchanged except for a shift of the four-dimensional cosmological constant which amounts to a constant re-scaling $g_{\mu\nu} \rightarrow \frac{6}{5}g_{\mu\nu}^0$ on $\mathcal{M}_4$. The field F_{mnpq} has a geometrical interpretation. Consider the quantities $S_{mnp}^\pm$, given by

$$S_{mnp}^\pm = \pm m_7 \bar{\eta}^\pm \Gamma_{mnp} \eta^\pm \quad (2.35)$$

Using

$$1 = \eta \bar{\eta} + \Gamma_P \eta \bar{\eta} \Gamma^P \quad (2.36)$$

one gets

$$\begin{aligned} S_{mnp;q} &= S_{[mnp,q]} = \\ &= S_{mn} S_{pq} + m_7^2 (g_{mp} g_{nq} - g_{mq} g_{np}) \end{aligned} \quad (2.37)$$

and the curvature tensor associated with the connection $\{\overset{m}{n}_p\} + S_{np}^m$, where $\{\overset{m}{n}_p\}$ is the Levi-Civita connection, takes the form

$$\begin{aligned} R_{mnpq} (\{ \overset{r}{st} \} + S^r_{st}) &= \\ &= R_{mnpq} (\{ \overset{r}{st} \}) + m_7^2 (g_{mp} g_{nq} - g_{mq} g_{np}) = \\ &= C_{mnpq} (\{ \overset{r}{st} \}) \end{aligned} \quad (2.38)$$

where C_{mnpq} is the Weyl tensor of the Einstein space $\mathcal{M}_7$. Hence

$$R_{mn} (\{ \overset{r}{st} \} + S^r_{st}) = 0 \quad (2.39)$$

Thus up to a constant, the dual of F^{mnpq} is interpretable as a totally antisymmetric torsion which renders $\mathcal{M}_7$ Ricci-flat. The total antisymmetry of S_{mnp} guarantees that metric properties are preserved, namely $g^{mn}_{;p} = 0$ both for the Levi-Civita and for the total connexion.

The new solution(s) differs from the old one by a change in the four-dimensional cosmological constant due to $F^{mnpq} \neq 0$. In this way all the supersymmetries of the old solution are broken by a spontaneously induced torsion which Ricci-"flattens" the compact manifold. These solutions with supersymmetry breaking have been explicitly constructed for S^7 ¹⁶⁾, for the squashed seven sphere ^{15),21),*)} and for the coset space $(SU(3) \times SU(2) \times U(1)) / (SU(2) \times U(1) \times U(1))$ ²²⁾.

*) We find ourselves in disagreement with the authors of Ref. 34) who distinguish between left and right squashing and claim that a solution with squashing and torsion can only be obtained for the right squashed sphere. The difference between left and right squashing corresponds to $m_7 < 0$ and $m_7 > 0$, respectively (P. Spindel, private communication). However, the existence of a covariantly constant spinor is determined by the relative sign between the parameters m_7 and f' in (2.34). If this sign is a priori fixed as in Ref. 34), it is indeed true that a solution exists only for one handedness; if, however, this sign is reversed, a solution with opposite handedness is obtained. The statement that the existence of a solution with squashing and torsion is linked to a definite handedness of the squashed sphere is therefore incorrect.

3. - THE SEVEN SPHERE

3.1 The geometry of S^7

We have seen that $S^7 \times (\text{AdS})_4$ is the only non-"trivial" compactification which preserves full $N = 8$ local supersymmetry in four dimensions. It turns out that S^7 has remarkable geometric properties which fit in a surprising way into the structure of the eleven-dimensional theory. We shall therefore first review these exceptional properties which are easily revealed by the octonion parametrization of the sphere.

The real octonion algebra is an eight-dimensional vector space over the real number field endowed with a non-associative multiplication law²⁴⁾. The latter can be defined by choosing a basis formed by the identity and seven "imaginary" units e_a whose multiplication table is

$$e_a e_b = -\delta_{ab} + \sum_{c=1}^7 a_{abc} e_c \quad (3.1)$$

Here a_{abc} is totally antisymmetric and equal to ± 1 , when the indices a, b, c are equal to 126, 315, 234, 465, 714, 736 and 725; otherwise, $a_{abc} = 0$. To each octonion $X = x_0 + \sum_a x_a e_a$ one associates a complex conjugate octonion $\bar{X} = x_0 - \sum_a x_a e_a$; the scalar product of X and Y is then defined by

$$\langle XY \rangle = \text{Tr } X \bar{Y} = \text{Tr } \bar{X} Y ; \quad \begin{aligned} \text{Tr } 1 &= 1 \\ \text{Tr } e_a &= 0 \end{aligned} \quad (3.2)$$

The non-degenerate norm $N(x)$ defined by

$$N(x)^2 = \langle XX \rangle = x_0^2 + \sum_{a=1}^7 x_a^2 \quad (3.3)$$

satisfies the composition law

$$N(X) N(Y) = N(XY) \quad (3.4)$$

This property, Eq. (3.4), is shared with the algebra of real numbers, complex numbers and octonions. In contradistinction to those, however, it is not associative and one verifies from Eq. (3.1) that the associator defined by

$$[X, Y, Z] = (XY)Z - X(YZ) \quad (3.5)$$

is totally antisymmetric in X, Y and Z .

We restrict ourselves to octonions of unit norm which describe Euclidean vectors of unit length in E_8 ; they may thus be used to parametrize points on the unit seven sphere, $|m_7| = 1$. At each point X of this S^7 we can construct an orthonormal tangent 7-bein $e_a X$ as is easily verified from Eqs. (3.2) and (3.5). Such a 7-bein field which is globally defined because Eq. (3.4) defines an absolute parallelism on S^7 : tangent vectors are called parallel if they have the same components in the two frames defined at their respective origin. In fact, one easily constructs two seven-parameter families of parallelism from the 7-bein fields²⁵⁾

$$u_a^+(X) = (e_a A)(\bar{A} X), \quad u_a^-(X) = (X \bar{A})(A e_a) \quad (3.6)$$

where A is an arbitrary constant unit octonion.

The above construction of absolute parallelisms can be viewed as a generalization of the parallelism induced on S^1 and S^3 in an identical manner using complex numbers and quaternions, respectively. However, as S^1 and S^3 are group spaces, their parallelizability also follows from group theory while the S^7 parallelism constitutes a unique feature rooted in the existence of the octonion composition algebra²⁶⁾. Clearly, the occurrence of distinct sets of (+) and (-) parallelisms defined by (3.6) is a consequence of the non-commutative character of (+) while the dependence of each set on seven arbitrary parameters follows from its non-associative character. All the (+) parallelisms [and all the (-) parallelisms] are conjugate under an $SO(8)$ rotation on the seven sphere so that to study the parallelized geometries it suffices to put $A = 1$ in (3.6).

The integral curves of Eq. (3.6) (with $A = 1$) are the unit circles parametrized by

$$\exp(\lambda e_a) X = \cos \lambda X + \sin \lambda e_a X \quad (3.7)$$

where λ varies from 0 to 2π . Hence, metric geodesics are preserved by the parallelisms and hence the corresponding linear connections are characterized by totally antisymmetric torsion tensors $S_{rst}^\pm$. Thus, the two parallelized geometries satisfy

$$R^m{}_{npq} (\{ \overset{r}{s} t \} + S^r{}_{st}) = 0 \quad (3.8)$$

Contracting (3.8) to R_{nq} and comparing with (2.39), we see that these geometries are precisely those constructed from the η^+ and η^- spinors defined on the seven sphere. Hence in addition to the Riemann seven sphere S_7 , eleven-dimensional supergravity admits compactification on the two exceptional "flat" seven sphere geometries $S_7^\pm$ ²⁷⁾. These may be viewed as spontaneously broken phases of the S_7

compactifications. All the supersymmetries are broken^{16),28)} and the $SO(8)$ gauge group is reduced to the isometry invariance group which preserves a parallelism, this is readily shown to be the covering group of $SO(7)$, namely $Spin(7)$ ^{29)-31),*)}. Moreover it has been shown that, when supersymmetry is defined by η^+ Killing spinors, the breaking induced by the $(-)$ parallelized solution occurs (at the linearized level) within the Fock space of zero-mass excitations defined from the S_7 compactification^{16),33)}. This is a peculiar feature which, in general, is not a property of spontaneous compactification. For example, the squashed solution of Ref. 20) cannot be interpreted in this way because the covariantly constant spinor is not a linear combination of the eight covariantly constant spinors of the compactification on the round S^7 ^{21),34)}. In the case of the parallelized solution of Ref. 27), the identification is made possible by the existence of covariantly constant spinors for both signs in Eq. (2.30). The fact that this solution can be interpreted as a spontaneously broken version of gauged $N = 8$ supergravity is also strongly suggested by the existence of stationary points of the $N = 8$ potential with $Spin(7)$ symmetry^{35),29)}.

3.2 The mass spectrum

The bare mass spectrum of the four-dimensional theory can be obtained by varying the fields g_{MN} , A_{MNP} , ψ_M around their background values. The bosonic field equations (2.7) and (2.8) are linear in δg_{MN} , δA_{MNP} and the corresponding linearized equation for the fermionic field is, from Eq. (2.2),

$$- i \tilde{F}^{MNP} \psi_{P;N} + \frac{\sqrt{2}}{288} m_7 \left\{ \tilde{F}^{MNPQRS} + 12 g^{MP} g^{NQ} \tilde{F}^{RS} \right\} \overset{o}{F}_{PQRS} \psi_N = 0 \quad (3.9)$$

where the superscript o labels a background value. One expands the components of ψ_M , δg_{MN} and δA_{MNP} in complete sets of eigenfunctions of suitable differential operators on S_7 according to Eq. (2.6). The masses are then obtained in terms of the corresponding eigenvalues. However, the actual calculation reveals a mixing between the various fields of the theory and one has in fact to diagonalize a mass matrix.

The diagonalization can be performed by field redefinitions, and such a method was used to obtain fields describing the zero-mass supermultiplet, namely one graviton, eight gravitinos, 28 $SO(8)$ gauge fields, 56 spin- $\frac{1}{2}$ fields, 35 scalars and 35 pseudo-

*) The $Spin(7)$ invariance may also be deduced from the results of Ref. 32).

scalars^{16),18)}. By retaining only these fields in the full non-linear theory, one would expect to recover the gauged $N = 8$ supergravity theory of Ref. 17). But such an expectation can only be borne out if this truncation is consistent, namely if no massive mode can be varied into a massless mode by a supersymmetry transformation in four dimensions. As such a mixing does indeed occur, a consistent truncation cannot be defined from the ansätze as given in Refs. 16) and 18), and can therefore be obtained only if non-linear modifications are taken into account²⁹⁾ (in particular, the supersymmetry transformation parameter $\epsilon(x,y)$ must be redefined in such a way that it is no longer a Killing spinor for non-vanishing fluctuations, i.e., outside the background).

There is another, and perhaps related problem, which motivates an analysis of the complete four-dimensional field content, including the massive modes, which arise from the seven-sphere compactification. It was conjectured that the supersymmetry breaking induced by different compactifications, which are topologically equivalent to the seven-sphere, may lead to a "level crossing" in the sense that massive modes may become massless in the new compactified geometry¹⁶⁾; recently, it was shown that level crossing indeed occurs in the squashed sphere compactification^{34),21)} which reduces the supersymmetries from $N = 8$ to $N = 1$ ²⁰⁾. Finally, we wish to point out that the emphasis on the zero-mass sector based on its possible physical relevance for a low energy phenomenology does not appear to be justified a priori. Massive states have masses of the order of the inverse seven-sphere radius $|m_7|$ which is also a measure of the cosmological constant. Thus, compactifications topologically equivalent to the seven sphere can acquire physical significance only if quantum effects could cancel the cosmological term and this will be touched upon in the last section. All these arguments indicate that a selection of the zero-mass sector on physical grounds in the seven-sphere compactification appears premature at least.

We now turn to the evaluation of the full bare mass spectrum, which, as we shall see, contains in fact additional zero mass states not included in the basic $N = 8$ supermultiplet³⁶⁾. We first note that the mass matrix contains spurious modes describing the gauge degrees of freedom of the theory. These can be eliminated at the outset by fixing a gauge. Such a procedure has the advantage that a convenient gauge choice can be made to undo mixings between remaining physical fields. In this way, field redefinitions become superfluous; moreover, while these redefinitions were useful at the level of the zero-mass supermultiplet, they would have to be extended in a mass dependent, and hence a non-local way, in order to cope with the full spectrum.

The gauge invariances of the linearized equations for field fluctuations are

$$\delta_G h_{MN} = 2 \xi_{(M,N)} \quad (3.10)$$

$$\delta_G a_{MNP} = \Lambda_{[MN;P]} , \quad \Lambda^{MN}{}_{;N} = 0 \quad (3.11)$$

$$\delta_G \psi_M = \overset{\circ}{D}_M \chi \quad (3.12)$$

Here $h_{MN} \equiv -\delta g_{MN}$ ($h^{MN} \equiv \delta g^{MN}$), $a_{MNP} \equiv \delta A_{MNP}$, and the symbol δ_G labels the eleven co-ordinate, the 45 Maxwell, and the 32 supersymmetry gauge transformations. From hereon, covariant derivatives are taken with respect to the background and indices are raised and lowered by the background metric tensor. Correspondingly, $\overset{\circ}{D}_M$ is a background supercovariant derivative and, from (2.5), we have

$$\overset{\circ}{D}_\mu \chi = \chi_{;\mu} - m_7 \gamma^5 \gamma_\mu \chi \quad (3.13)$$

$$\overset{\circ}{D}_m \chi = \chi_{;m} - \frac{1}{2} m_7 \Gamma_m \chi \quad (3.14)$$

Convenient gauge choices are^{36),37)}

$$h^{\mu\mu}{}_{;m} = h^{\nu\nu}{}_{;v} = 0 \quad (10 \text{ conditions}) \quad (3.15)$$

$$h^M{}_M = 0 \quad (1 \text{ condition}) \quad (3.16)$$

$$a^{\mu\nu\rho}{}_{;p} = a^{\mu\kappa\rho}{}_{;p} = a^{\kappa\nu\rho}{}_{;p} = 0 \quad (15 + 24 + 6 = 45 \text{ conditions}) \quad (3.17)$$

$$\tilde{\Gamma}^M \psi_M = 0 \quad (32 \text{ conditions}) \quad (3.18)$$

Now we consider the fermionic sector³⁷⁾. By substituting the Freund-Rubin expectation value for $\overset{\circ}{F}_{MNPQ}$ in (3.9) and by writing out four- and seven-dimensional indices explicitly, we obtain

$$\begin{aligned} & \gamma^{\mu\nu\rho} \psi_{\rho;v} + \gamma^5 \gamma^{\mu\nu} \Gamma^m \psi_{m;v} - \gamma^5 \gamma^{\mu\nu} \Gamma^m \psi_{v;m} + \\ & + \gamma^\mu \Gamma^{mn} \psi_{n;m} + \frac{3}{2} m_7 \gamma^5 \gamma^{\mu\nu} \psi_\nu = 0 \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} & \gamma^5 \gamma^{\mu\nu} \Gamma^m \psi_{v;\mu} + \gamma^\mu \Gamma^{mn} \psi_{\mu;n} - \gamma^\mu \Gamma^{mn} \psi_{n;\mu} + \\ & + \gamma^5 \Gamma^{\mu\nu\rho} \psi_{\rho;n} - \frac{3}{2} m_7 \gamma^5 \Gamma^{mn} \psi_n = 0 \end{aligned} \quad (3.20)$$

Clearly, (3.19) and (3.20) mix the four-dimensional spin- $\frac{3}{2}$ and spin- $\frac{1}{2}$ fields. In Refs. 16) and 18), Eqs. (3.19) and (3.20) were diagonalized by a linear field redefinition

$$\begin{aligned}\psi_\mu &= \psi'_\mu + \frac{1}{2} \gamma^5 \gamma_\mu \Gamma^m \psi'_m \\ \psi_m &= \psi'_m\end{aligned}\quad (3.21)$$

However, it is not difficult to see that this procedure only works for the zero modes and fails to decouple the massive modes. Instead of using (3.21), we therefore simply substitute the gauge condition (3.18) which reads

$$\tilde{\Gamma}^M \psi_M = \gamma^\mu \psi_\mu + \gamma^5 \Gamma^m \psi_m = 0 \quad (3.22)$$

into (3.19) and (3.20). In this way, we get

$$\begin{aligned}-\gamma^\nu \psi^\mu{}_{;\nu} - m_7 \gamma^5 \gamma^\mu \gamma^\nu \psi_\nu &= \\ &= -\frac{3}{2} m_7 \gamma^5 \psi^\mu + \gamma^5 \Gamma^m \psi^\mu{}_{;m}\end{aligned}\quad (3.23)$$

and

$$\begin{aligned}-\gamma^5 \gamma^\mu \psi^m{}_{;\mu} &= \Gamma^\mu \psi^m{}_{;\mu} - \\ &- m_7 \left(\Gamma^m \Gamma^\mu \psi_\mu - \frac{3}{2} \psi^m \right)\end{aligned}\quad (3.24)$$

Consequently, the equations for the four-dimensional spin- $\frac{1}{2}$ and gravitino fields have been completely decoupled by the gauge choice (3.18). Inspection now shows that, up to an additive constant - the canonical mass term for a massless gravitino in $(\text{AdS})_4$ space - the spin- $\frac{3}{2}$ mass matrix is given by the eigenvalues of the Dirac operator on S^7 . Similarly, the spin- $\frac{1}{2}$ mass matrix is given by the eigenvalues of the operator on the right-hand side of (3.24). The eigenmodes of the Dirac operator on S^7 can be easily constructed as follows. The spherical harmonics $Y(y)$ on S^7 obey the equations³⁸⁾

$$\Delta Y(y) = k(k+6) Y(y), \quad k = 0, 1, 2, \dots \quad (3.25)$$

where Δ is the Laplace-Beltrami operator on S^7 . To solve the equation $\nabla\phi = \lambda\phi$ and to find the eigenvalues λ , we make the ansatz

$$\varphi(y) = \alpha Y(y) \eta(y) + \beta Y_{,m}(y) \Gamma^m \eta(y) \quad (3.26)$$

where $\eta(y)$ is the covariantly constant spinor on S^7 . After a little calculation, one finds that, for each $k \geq 1$, there are two eigenvalues whereas there is only one for $k = 0$, namely

$$\begin{aligned}\lambda &= \frac{7}{2} \quad \text{for } k = 0 \\ \lambda &= k + \frac{7}{2} \quad \text{and} \quad -k - \frac{5}{2} \quad \text{for } k \geq 1\end{aligned}\tag{3.27}$$

with

$$\frac{\alpha}{\beta} = \lambda + \frac{5}{2} \quad \left(\lambda \neq -\frac{5}{2} \right)\tag{3.28}$$

The spin- $\frac{1}{2}$ mass matrix is obtained in an analogous fashion, although the calculation is considerably more tedious [details will be given in Ref. 37)]. To determine the mass spectrum, it follows from (3.24) that one has to solve the eigenvalue equation

$$\Gamma^n \psi^m_{;n} - \Gamma^m \Gamma^n \psi_n = \lambda \psi^m\tag{3.29}$$

It can be shown that Eq. (3.29) gives rise to six towers of massive states. Two of these are spurious gauge modes of the form $\psi_m = \mathcal{D}_m \phi$; their eigenvalues are given by

$$\lambda = k + \frac{3}{2} \quad \text{and} \quad -k - \frac{3}{2} \quad , \quad k \geq 1\tag{3.30}$$

The four other towers are physical and characterized by $\mathcal{D}_m \psi^m = 0$; their mass eigenvalues are

$$\lambda = k + \frac{7}{2} \quad (k \geq 1) \quad \text{and} \quad -k - \frac{5}{2} \quad (k \geq 2)\tag{3.31}$$

$$\lambda = k - \frac{5}{2} \quad \text{and} \quad -k - \frac{17}{2} \quad (k \geq 1)\tag{3.32}$$

It should be understood that the eigenmodes corresponding to (3.27), (3.30), (3.31) and (3.32) belong to irreducible representations of $SO(8)$; we have suppressed their representation labels for simplicity. The symmetry assignments of all the modes will be the subject of Section 3.3.

The existence of gauge mode solutions to Eq. (3.29) follows from the gauge condition Eq. (3.18) which mixes seven- and four-dimensional subspaces. These modes determine four-dimensional constraints which must be imposed on the massive spin- $\frac{3}{2}$ fields to eliminate spurious degrees of freedom. Indeed, combining Eq. (3.18) with the equation of motion (3.9), we get

$$\begin{aligned}\psi^m_{;m} &= \psi^\mu_{;\mu} + \psi^m_{;m} = \\ &= \frac{1}{2} \gamma^5 \gamma^\mu \psi_\mu = -\frac{1}{2} \Gamma^m \psi_m\end{aligned}\tag{3.33}$$

Thus, using (3.22) and (3.33), we may rewrite the gauge condition as

$$\hat{\mathcal{D}}^M \psi_M = \hat{\mathcal{D}}^\mu \psi_\mu + \hat{\mathcal{D}}^m \psi_m = 0 \quad (3.34)$$

For spurious modes $\hat{\mathcal{D}}^m \psi_m \neq 0$ and hence Eq. (3.34) determines the required constraint on $\hat{\mathcal{D}}^\mu \psi_\mu$ for a spin- $\frac{3}{2}$ field having the same $SO(8)$ content as the gauge mode. To understand why the solution of Eq. (3.29) naturally splits into a space of gauge modes $\psi_m = \hat{\mathcal{D}}_m \chi$ and the orthogonal space of physical solutions $\hat{\mathcal{D}}_m \psi^m = 0$, we apply the operator $\hat{\mathcal{D}}_m$ to the eigenvalue equation (3.29); we obtain

$$\Gamma^m (\hat{\mathcal{D}}^n \psi_n)_{;m} = (\lambda + 1) \hat{\mathcal{D}}^n \psi_n \quad (3.35)$$

If one now decomposes ψ_m into pieces which are transversal and longitudinal with respect to the operator $\hat{\mathcal{D}}_m$, viz.

$$\psi_m = \chi_m + \hat{\mathcal{D}}_m \chi, \quad \hat{\mathcal{D}}^m \chi_m = 0 \quad (3.36)$$

one infers from (3.34) that this decomposition is maintained by the spin- $\frac{1}{2}$ mass operator. This means that the eigenspace of this operator decomposes into the space of spurious states with $\hat{\mathcal{D}}^m \psi_m \neq 0$ and the space of physical states associated with the eigenvalues (3.31) which obey

$$\hat{\mathcal{D}}^m \psi_m = 0 \quad (3.37)$$

Equation (3.37) is a genuine seven-dimensional gauge condition. Note that the decomposition

$$\psi_m = \chi'_m + \Gamma_m \chi', \quad \Gamma^m \chi'_m = 0 \quad (3.38)$$

does not leave these two subspaces invariant. In fact the two towers (3.31) are traceless while the two others, Eq. (3.32), are not. According to the gauge condition (3.18), the modes given by Eq. (3.32) can thus be classified according to the same $SO(8)$ representation as the spin- $\frac{1}{2}$ fields; this correspondence, however, occurs between different supermultiplets, as will become evident in Section 3.3.

The masses of the spin- $\frac{1}{2}$ and spin- $\frac{3}{2}$ particles are defined by the eigenvalues $m_{\frac{1}{2}}$ and $m_{\frac{3}{2}}$ of the four-dimensional differential operators appearing in the left-hand side of Eqs. (3.23) and (3.24). They are given, in units of m_7 , by Eqs. (3.27), (3.31) and (3.32), up to an additive constant. We thus have

$$m_{3/2}^{(1)} = k + 2, \quad k \geq 0 \quad (3.39)$$

$$m_{3/2}^{(2)} = -k - 4, \quad k \geq 1 \quad (3.40)$$

$$m_{1/2}^{(1)} = k - 1, \quad k \geq 1 \quad (3.41)$$

$$m_{1/2}^{(2)} = -k - 7, \quad k \geq 1 \quad (3.42)$$

$$m_{1/2}^{(3)} = k + 5, \quad k \geq 1 \quad (3.43)$$

$$m_{1/2}^{(4)} = -k - 1, \quad k \geq 2 \quad (3.44)$$

The superscripts label the towers. The members of the "massless" $N = 8$ supermultiplet are at the bottom of the towers $m_{\frac{3}{2}}^{(1)}$ and $m_{\frac{1}{2}}^{(1)}$. Note that for convenience the "massless" gravitino which is anyway not conformal³⁹⁾ has been given the value +2. We do not list the $SO(8)$ content of the modes here as this will be discussed in the next section.

The fermionic sector illustrates all the essential features of the procedure. Thus, for the bosonic sector we shall simply list the results³⁶⁾. The $(\text{mass})^2$ operators are defined by the following differential operators

$$\text{spin-0:} \quad \eta_{;\mu}{}^{;\mu} - 8 m_7^2 \eta = -m_0^2 \eta \quad (3.45)$$

$$\text{spin-1:} \quad (\eta_{\mu;\nu} - \eta_{\nu;\mu})^{;\nu} = -m_1^2 \eta_\mu \quad (3.46)$$

$$\text{spin-2:} \quad \eta_{(\mu\nu);;\rho}{}^{;\rho} - 16 m_7^2 \eta_{(\mu\nu)} = -m_2^2 \eta_{(\mu\nu)} \quad (3.47)$$

$(\eta^\mu{}_\mu = \eta^{\mu\nu}{}_{;\nu} = 0)$

In this way, the spin-0 and spin-1 members of the "massless" supermultiplet have indeed a zero mass while the (non-conformal) graviton has been assigned a mass +8 in units of m_7^2 . In these units, one obtains

$$m_2^2 = (k+3)^2 - 1, \quad k \geq 0 \quad (3.48)$$

$$m_{1+}^2 = (k+3)^2 - 1, \quad k \geq 1 \quad (3.49)$$

$$(m_{1-}^{(1)})^2 = k^2 - 1, \quad k \geq 1 \quad (3.50)$$

$$(m_{1-}^{(2)})^2 = (k+6)^2 - 1, \quad k \geq 1 \quad (3.51)$$

$$(m_{0-}^{(1)})^2 = k^2 - 1, \quad k \geq 1 \quad (3.52)$$

$$(m_{0-}^{(2)})^2 = (k+6)^2 - 1, \quad k \geq 1 \quad (3.53)$$

$$(m_{0+}^{(1)})^2 = (k-3)^2 - 1, \quad k \geq 2 \quad (3.54)$$

$$(m_{0+}^{(2)})^2 = (k+9)^2 - 1, \quad k \geq 0 \quad (3.55)$$

$$(m_{0+}^{(3)})^2 = (k+3)^2 - 1, \quad k \geq 2 \quad (3.56)$$

The massless supermultiplet is given by the lowest value of k in the towers m_2 , $m_{1-}^{(1)}$, $m_{0-}^{(1)}$ and $m_{0+}^{(1)}$. Note the appearance of an additional zero-mass supermultiplet for $k = 4$ and a multiplet with $m^2 = -1$ for $k = 3$ in the scalar tower $m_{0+}^{(1)}$.

To end this section, we wish to stress that the expression of the modes in terms of fields depends on the gauge choice while the spectrum is, of course, gauge invariant. This is best illustrated in the case of the massless spin- $\frac{1}{2}$ mode. In Refs. 16) and 18), it was proved that the ansatz

$$\begin{aligned} \psi_m(x, y) = \chi^{IJK}(x) \{ & \gamma^I(y) \bar{\gamma}^J(y) \Gamma_m \gamma^K(y) - \\ & - \frac{1}{9} \Gamma_m \Gamma^n \gamma^I(y) \bar{\gamma}^J(y) \Gamma_n \gamma^K(y) \} \end{aligned} \quad (3.57)$$

describes the 56 massless spin- $\frac{1}{2}$ fields of $N = 8$ supergravity [I, J, K are $SO(8)$ indices]; in the proof, use was made of the field redefinition (3.21). In the gauge (3.18), however, the result is different, and one easily checks that the right-hand side of (3.24) vanishes for

$$\psi_m(x, y) = \chi^{IJK}(x) \gamma^I(y) \bar{\gamma}^J(y) \Gamma_m \gamma^K(y) \quad (3.58)$$

i.e., the trace term has disappeared. One immediately verifies that (3.58) also satisfies the gauge condition (3.37) whereas (3.57) does not. The notion of ansatz is therefore a gauge-dependent concept in general.

3.3 Group theory: $Osp(8,4)$ classification

Up to this point, the symmetry assignments of the various modes have not been discussed in any detail. The mass spectrum of $N = 8$ supergravity on S^7 in the bosonic³⁶⁾ and fermionic case³⁷⁾ has been determined by solving the appropriate eigenvalue equations, and no explicit reference to the $SO(8)$ and supersymmetry content of the modes was necessary. The $SO(8)$ assignments can be deduced from those of the spherical (scalar, vector and tensor) harmonics on S^7 which are known³⁸⁾, but this is not sufficient to group the various states into supermultiplets. For a complete classification, one has to make use of the full invariance of the S^7 ground state. This group contains not only the 28 rotations of $SO(8)$ corresponding to the 28 Killing vectors on S^7 but also eight spinorial translations which correspond to the eight Killing spinors on S^7 . Together, the generators associated with these bosonic and fermionic transformations constitute the graded Lie algebra $Osp(8,4)$, and a rigorous proof of the $Osp(8,4)$ invariance of the S^7 ground state has been given in Ref. 40). The excitations corresponding to the fluctuations about the ground state should therefore form irreducible representations of $Osp(8,4)$. From the general Kaluza-Klein theory^{9),10)} and the absence of higher spin fields in eleven-dimensional supergravity, it follows that the relevant representations are those with maximum spin 2. The latter have been classified in Ref. 41); and we will restrict our attention to these representations here. For a comprehensive review of the general construction and properties of unitary irreducible representations of $Osp(N,4)$, we refer the reader to Ref. 42).

We have already mentioned that the masses of the excited states are proportional to the inverse radius $|m_7|$ of the seven sphere. Thus, in the limit $m_7 \rightarrow 0$ where the space becomes flat, all masses tend to zero. In this limit, the relevant superalgebra is the Poincaré superalgebra, and we conclude that in this contraction limit, the massive representations of $Osp(8,4)$ become massless representations of $N = 8$ Poincaré supersymmetry^{*)}. This has the very important consequence that all massive representations of $Osp(8,4)$ with maximum spin 2 must be obtainable from massless representations of $N = 8$ supersymmetry with the same spin limit. There is only one such multiplet with maximum spin 2, namely the massless $N = 8$ multiplet already mentioned in the introduction. It contains one graviton [$\underline{1}$ of $SO(8)$], eight gravitinos^{**)} ($= 8_s$), 28 spin-1 fields ($= 28$), 56 spin- $\frac{1}{2}$ fields ($= 56_s$), 35 scalars

^{*)} In $N = 8$ Poincaré supersymmetry, massive (Poincaré) multiplets go up to spin 4, if $SO(8)$ is preserved, and can be restricted to stop at spin 2 only if central charges are introduced which break $SO(8)$; Our argument shows, however, that no inconsistency with this result arises⁴¹⁾.

^{**)} The assignment of the spin- $\frac{3}{2}$ fields to the 8_s representation (instead of 8_v or 8_c) is a matter of conventions.

($= 35_v$) and 35 pseudoscalars ($= 35_c$) [for the group theoretic conventions, see Ref. 43)]. Hence, one should be able to derive all massive $Osp(8,4)$ multiplets from products of the form

$$R \otimes \{1, 8_s, 28, 56_s, 35_v, 35_c\} \quad (3.59)$$

where R is an as yet unspecified representation of $SO(8)$.

To facilitate the discussion, we next introduce Dynkin labels to classify the representations of $SO(8)$ ⁴³⁾. Each irreducible representation of $SO(8)$ can be uniquely labelled by a set $(a_1 a_2 a_3 a_4)$ of four non-negative integers a_1, a_2, a_3, a_4 . Since the massless graviton which belongs to the massless $N = 8$ multiplet is an $SO(8)$ singlet, the charged massive gravitons will carry the same label as the relevant irreducible representation. One now realizes that the representation R which occurs in (3.59) is no longer arbitrary, since we know from the explicit calculations ^{9),10)} that the massive gravitons are in one-to-one correspondence with the eigenfunctions of the Laplacian on S^7 , (3.25), i.e., the spherical harmonics on S^7 . These are characterized by the Dynkin labels $(n000)$, $n \in \mathbb{N}$, which correspond to the symmetric and traceless $SO(8)$ tensors with n indices. To obtain the full $Osp(8,4)$ multiplet, we replace R in (5.2) by $(n000)$, perform the multiplication and identify the irreducible components in this product. The Dynkin labels of the massless representation are given by

$$\begin{aligned} 8_s &= (0001) \\ 28 &= (0100) \\ 56_s &= (1010) \\ 35_v &= (2000) \\ 35_c &= (0020) \end{aligned} \quad (3.60)$$

The result of this multiplication, which is given in Ref. 41), is, however, not yet the final answer. One still has to add lower helicity states to the spin-2, spin- $\frac{3}{2}$ and spin 1 fields to make them massive. The lower helicity states which are absorbed must belong to the same representation as the gauge field into which they are absorbed. The final result which is obtained after absorbing these states reads ^{*)}:

$$\begin{aligned} \text{spin-2:} & \quad (n000) \\ \text{spin-}\frac{3}{2}: & \quad (n001) \oplus (n-1010) \end{aligned}$$

^{*)} For the special case $n = 1$, this result was first obtained in Ref. 34).

$$\begin{aligned}
 \text{spin} - 1^+ &: (n-1 \ 0 \ 1 \ 1) \\
 \text{spin} - 1^- &: (n \ 1 \ 0 \ 0) \oplus (n-2 \ 1 \ 0 \ 0) \\
 \text{spin} - \frac{1}{2} &: (n+1 \ 0 \ 1 \ 0) \oplus (n-1 \ 1 \ 1 \ 0) \oplus \\
 &\quad \oplus (n-2 \ 1 \ 0 \ 1) \oplus (n-2 \ 0 \ 0 \ 1) \\
 \text{spin} - 0^+ &: (n+2 \ 0 \ 0 \ 0) \oplus (n-2 \ 2 \ 0 \ 0) \oplus \\
 &\quad \oplus (n-2 \ 0 \ 0 \ 0) \\
 \text{spin} - 0^- &: (n \ 0 \ 2 \ 0) \oplus (n-2 \ 0 \ 0 \ 2) \tag{3.61}
 \end{aligned}$$

where, whenever an integer is negative, the associated representation does not exist; for example, the second $\text{spin} - \frac{3}{2}$ tower starts only at $n = 1$. For each n , (3.61) is an irreducible representation of $\text{Osp}(8,4)$, and the integer n therefore labels the "floors" of the massive tower^{*)}.

The $\text{SO}(8)$ content of the spectrum as calculated directly by solving the relevant eigenvalue equations^{36),37)} is in complete agreement with (3.61) which was obtained in a completely different manner. For example, the two eigenmodes of the Dirac operator found before [see (3.27)] exactly correspond to the two representations in (3.61), and the absence of the "ground-floor" for the second tower was also obtained here. Similarly, the four $\text{spin} - \frac{1}{2}$ towers of (3.31) and (3.32) have their counterparts (3.61), if one properly adjusts the relation between n and k . Analogous considerations apply to the bosonic sector.

We can express the mass formulas of Section 3.2 in terms of the supersymmetry index n and hence the spectrum is completely characterized by Eqs. (3.39)-(3.44) and Eqs. (3.48)-(3.56). The results are summarized in the Table, where we have marked the Dynkin labels of the towers containing massless multiplets by an asterisk, and relabelled the various towers.

Besides spin and $\text{SO}(8)$ content, the complete characterization of the $\text{Osp}(8,4)$ states requires the knowledge of the lowest eigenvalues E_0 of the "energy operator" M_{04} of the $\text{SO}(2,3)$ subalgebra of $\text{Osp}(8,4)$. The energy labels are most easily found by using the $\text{Osp}(1,4) \times \text{SO}(7)$ subalgebra of $\text{Osp}(8,4)$ and the known energy labels of $\text{Osp}(1,4)$ representations⁴⁴⁾. The results are given in Ref. 37) and can be summarized by the following universal relations between mass and energy:

*) It is perhaps instructive to note that the massive modes on the hypertorus T^7 [whose massless sector corresponds to the $N = 8$ theory of Ref. 7)] have a much simpler structure. The relevant group is $[U(1)]^7$ in this case, and after recombining helicities, one easily sees that the massive multiplets all have the same multiplicities (1,8,27,48,42).

$$E_0 = \frac{3}{2} \pm \frac{1}{2} \sqrt{m^2 + 1} \quad \text{for bosons} \quad (3.62)$$

$$E_0 = \frac{3}{2} + \frac{1}{2} |m| \quad \text{for fermions} \quad (3.63)$$

where all the bosonic states obey relation (3.62) with the $+$ sign except the 35_v massless scalars for which the $-$ sign is required.

The universality of Eqs. (3.62) and (3.63) implies that E_0 has a dynamical significance. In fact, we know from Ref. 45) that the relation (3.62) for spin-0 fields characterizes modes which die fast enough at infinity to ensure energy conservation in AdS. The reality of E_0 is guaranteed by the fact that $m^2 \geq -1$ ⁴⁵⁾, the lower limit being reached in the $O^{+(1)}$ tower for $n = 2$ ($k = 3$) with a multiplet of 112 scalars. For $n = 3$ ($k = 4$) the same tower contains again conformal massless modes (294 scalars) with vanishing energy flow at spatial infinity. However, they must satisfy different boundary conditions, characterized by the $+$ sign in Eq. (3.62) ⁴⁵⁾, than the 35_v in order not to break supersymmetry. In this way they fit indeed as massless members in the "massive" supermultiplet $n = 2$. Note from the Table that, quite generally, in contrast to Poincaré supersymmetry, states belonging to the same supermultiplet characterized by n may have different masses because of the non-commutativity of the energy operator (M_{04}) with supersymmetry generators. Thus we see that for scalar modes, the supersymmetric spectrum is consistent with a Hilbert space of functions with boundary condition preventing energy flow in and out of AdS and hence admitting well-defined Cauchy data in this otherwise unviable space. We infer that such a property holds for all the modes because of the universality of Eqs. (3.62) and (3.63), a conjecture that could be checked explicitly following the method of Ref. 45). It follows from the positivity of E_0 that the seven sphere is stable against small fluctuations belonging to this Hilbert space, a fact which also follows from supersymmetry ⁴⁶⁾.

The emergence of such a Hilbert space in the context of the seven-sphere compactification in an AdS background may have an important consequence. At least we cannot dismiss a priori that inclusion of quantum effects and strong localized disturbances could lead to a smooth Minkowskian limit when $|m_7| \rightarrow 0$. This possibility would not only cure the physical problem posed by the AdS cosmological constant but would put into new terms the long-standing chirality issue. Indeed, the arguments preventing complex fermions to appear in this Kaluza-Klein approach ^{11), 47), 48)} have to be confronted with the infinite dimensional space of modes which would become relevant in this limit. This will be discussed in Section 4, whose speculative content, needless to say, cannot as yet be based on any firm mathematical basis.

4. - TOWARDS QUANTUM SUPERGRAVITY

Spontaneous compactifications provide classical solutions which play the rôle of a "mean field" average over quantum metric fluctuations. Hence at least a qualitative understanding of the nature of these fluctuations seems necessary in order to appreciate the relevance of such "mean field" configurations. The difficulty which then appears at the non-perturbative level clearly required here is the same as in ordinary gravity. Indeed, even if perturbative ultra-violet behaviour would turn out to be controllable, we cannot avoid the indefinite sign of R (in the Euclidean sector) due to non-perturbative scale fluctuations, which make the gravitational action unbounded from below⁴⁹⁾. How could then the path integral over metrics be defined at all? Moreover, in compactified supergravity, a new question arises which cannot be eliminated even by a fine tuning procedure: what are the fluctuations which can produce the vacuum energy required to make the cosmological constant vanish? This leads naturally to a third question: Is it conceivable that such fluctuations could provide a mechanism to stabilize the path integral? We suggest that the answer to these questions lies in the structure of space-time revealed by the "cosmological puzzle"⁵⁰⁾, and we speculate about the possibly related chirality issue raised at the end of the last section.

It is well known that the conventional hot big bang theory is hardly acceptable because of the tuning of initial conditions required. In this theory, the evolution of the early Universe is determined mainly by the set of asymptotically free massless fields in thermal equilibrium with the photon and the expansion is adiabatic. The cosmological state is then defined by the temperature T , the number of degrees of freedom $\nu(T) [\equiv \nu_{\text{bose}}(T) + \frac{7}{8} \nu_{\text{fermi}}(T)]$ and the scale factor a . These parameters are related by Einstein's equations and by the black-body formulae [see, e.g., Weinberg's book⁵¹⁾] ($\hbar = c = G = k = 1$)

$$aT = \frac{3}{2\pi} \left[\frac{5\mathcal{S}}{\nu(T)} \right]^{1/3} \quad (4.1)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8}{30} \pi^3 \nu(T) T^4 \quad (4.2)$$

where $\mathcal{S}$ is the total conserved entropy within a volume $4\pi a^3/3$ and $k = \pm 1$ (the limiting case $k/a^2 = 0$ need not be considered separately) according to the closed or open character of the Universe. The presently observed background radiation in the volume $(4\pi/3)H^{-3}$ [$H \equiv \dot{a}/a$] yields $\mathcal{S} \geq 10^{87}$ [we used $a \geq H^{-1}$ because the Universe is either open ($a > H^{-1}$) or anyway nearly flat]. As $\mathcal{S}$ is conserved, we have $a \geq 10^{29}/\nu^{1/3}(T)$ from Eq. (4.1) when T approaches 1 ($\sim 10^{19}$ GeV). Thus the classical "mean field" values $(\dot{a}/a)^2$ and $\frac{8}{30} \pi^3 \nu(T) T^4$ in Eq. (4.2) have to be tuned to

an accuracy of the order of k/a^2 , ($a \gtrsim 10^{29}/\sqrt[3]{1}$), when quantum fluctuations are of the order of 1. If $v(1)$ remains bounded, such a tuning appears absurd in the absence of a mechanism producing an initial correlated classical state.

A solution to the tuning problems posed by the "flatness" of the Universe at proper times $t \approx 0(1)$ requires that prior to the adiabatic expansion, the Universe experienced a different régime in which the scale factor could grow rapidly in a natural way from $0(1)$ to, say, $0(10^{30})$. Such a primordial "inflation" was initially proposed in the context of the creation of the Universe itself⁵²⁾. Indeed, in some cases one finds a semi-classical solution of the coupled gravity-matter system such that $a(t)$ grows exponentially:

$$a(t) = e^{t/\tau} \quad \left[H = \frac{1}{\tau} \right] \quad t \gtrsim 1 \quad (4.3)$$

The initial value $a(1)$ was then attributed to a quantum effect. Such a solution can in fact occur because of the "wrong" sign of R for the scale mode constituted by the cosmological expansion; in more intuitive terms, matter can be created at the expense of a negative expansion energy where the energy is measured in the conformal Minkowskian background. However, matter quanta must be higher than the Planck mass⁵³⁾ in order to sustain the semi-classical solution and a reinterpretation of these primordial quanta in terms of black holes⁵⁰⁾ was proposed. Other inflationary scenarios leading to Eq. (4.3), based on the cosmological constant induced by a phase transition at some grand unified scale, have been proposed^{54),55)}. We shall not restrict ourselves here to a specific model leading to Eq. (4.3), but we shall simply assume that this equation correctly describes the primordial phase of the Universe.

Equation (4.3), however, does not fully solve the initial tuning problem because there seems to be no way to stabilize an initial spatial curvature ($k/a^2 \approx \pm 1$). The latter is indeed inconsistent with Eq. (4.3) and could only arise therefore from quantum fluctuations at $t \lesssim 1$. As, however, any curvature appears flat for short distances, we may conclude that Eq. (4.3) solves the tuning problem if the initial classical state $a \approx 0(1)$ extends only over a finite proper length $L \approx a \approx 0(1)$. An observer within the Universe would hardly feel the finiteness of it as a consequence of Birkhoff's theorem⁵¹⁾ and thus the presently seen Universe can emerge from an initial classical state characterized by

$$L = 0(1) \quad \text{when} \quad t = 0(1) \quad (4.4)$$

Here comes the puzzle; the naturalness of the classical initial condition, Eq. (4.4), means that this initial state may well arise from a quantum metric fluctuation. But a fluctuation on which background? A natural answer appears to be on a

"mean field" which should be Minkowskian at scales large compared to the Planck size. This is indeed suggested both by the experimental fact that space is indeed flat for such scales, and by the theoretical fact mentioned earlier that in such a background the expansion energy is negative and can ensure energy conservation for the quantum transition giving birth to the Universe. If this is the case, however, a "Universe" can be born anywhere and at any time within our own "Universe" and quantum theory tells us that "we" must include such Universe-like configurations in the path integral over metrics.

A possible solution to the puzzle is that such configurations do exist, namely that within any "Universe" one can construct other "Universes" of comparable proper space-time extensions but which may be viewed as Planckian fluctuations with respect to the first one. To see that this does not lead to contradictions, we rewrite the (de Sitter) metric describing the exponential expansion of such a fluctuation

$$ds^2 = dt^2 - e^{2t/\tau} d\vec{x}^2, \quad |\vec{x}| \lesssim L \quad (4.5)$$

in the conformally Minkowskian form

$$ds^2 = (1 - \bar{t}/\tau)^{-2} \{ d\bar{t}^2 - d\vec{x}^2 \}, \quad |\vec{x}| \lesssim L \quad (4.6)$$

We see that around $\bar{t} = 0$, the co-ordinate system $(\bar{t}, \vec{x})$ may be used to parametrize the fluctuation in our Minkowskian background. The whole exponential expansion reduces to a quasi instantaneous event ($\bar{t} < \tau$) for the external Minkowskian observer and in fact the subsequent adiabatic period will also last for a time $\bar{t}$ of order τ . Indeed, the interior of the Planckian Universe is only protected from the outside in the space-time region bounded by the light cones originating from the "edge" $|\vec{x}| \approx L$. From the conformally flat metric, Eq. (4.6), this time $\bar{t}_f = O(L) = O(1)$. After the time $\bar{t}_f$ the Planckian Universe will then be destroyed by the vacuum fluctuations from which it originated. More generally, one can show that no paradox arises from TCP reversed configuration⁵⁰⁾. Thus, a "Universe" may be viewed as a Planckian fluctuation, hence as an unstable excitation of quantum gravity, defined on the Minkowskian background of some other Universe. Stability for "us" is then simply a consequence of the enormous contraction of the proper time due to the exponential expansion.

The number of distinct metric configurations describing ... Universes within Universes within ... has the power of the continuum and may well dominate the path integral of quantum gravity in such a way as to stabilize it in the vicinity of this "foam of Universes". One may thus adopt the point of view that the lower unboundedness of the Euclidean gravitational action is not just a technical nuisance which may be got rid of by rotating the contour of integration for the scale fluctuations into

the complex plane⁴⁹⁾ but that it has a physical significance. The unboundedness of the action should therefore be compensated for by a decrease of the functional measure of the associated configurations, and we conjecture that it is precisely the foamy structure of space-time that provides the needed mechanism (the relation of this scenario with Hawking's space-time foam⁵⁶⁾ is not clear). This line of thought also suggests that the unperturbed four-dimensional space-time is transmuted to a fractal space-time. The Hausdorff dimension of the fractal⁵⁷⁾ set available for quantum mechanical propagations may well be less than four: propagation within any Universe would be damped by a factor involving the ratio of the Planck size to the proper size of the Universe and would therefore be practically unavailable. To test this picture of quantum gravity, one should thus try to construct a fractal space endowed with an intrinsic metric.

Interestingly enough, the above picture of ... Universes within Universes within ... fits quite naturally into the Kaluza-Klein picture as is most easily illustrated by a two-dimensional analogy. Consider the space $R_1 \times R_1$ compactified on $R_1 \times S_1$; this is just a cylinder where R_1 is the analogue of $\mathcal{M}_4$ and S_1 is the analogue of the internal space. Clearly, one can construct new two-dimensional manifolds by locally opening the original cylinder, inserting a new one, and ... repeating the procedure. This construction is illustrated in the Figure. Clearly, as viewed from one cylinder (one "Universe"), other cylinders are either small (Planckian) disturbances or define a background from which it itself originates. In this way, the description of ... Universes within Universes within ... is recovered in the special gauge in which all Universes belong to the same (four-dimensional) space-time but in the general gauge illustrated in the Figure, one gets instead a "space-time proliferation" in which all Universes are manifestly of the same type. Note that matrix elements between different Universes must exist and would already be induced in the tube picture of the Figure by tube loops connecting different tubes.

To conclude this section, we wish to point out that it is conceivable that space-time proliferation in eleven-dimensional supergravity is in fact the mechanism restoring a zero-cosmological constant^{*)}. Indeed, when, in a given AdS background, the mass of massive modes exceeds the Planck mass, their vacuum fluctuations may give rise through gravitational interactions to new Universes hence to space-time proliferation. These will thus affect the vacuum energy in a way which is dependent on the radius of the compact manifold and hence on the cosmological constant itself. Hopefully, stability will be achieved for cosmological constants which are small compared to the scale of one Universe, a conjecture strengthened by the fact that the full eleven-dimensional theory does not admit a cosmological constant^{8),58)}.

*) An entirely different mechanism to cancel the cosmological constant through fermion condensates was proposed in Ref. 59).

If the (approximately) zero cosmological constant is stabilized by space-time proliferation, then the whole spectrum of excitations may become relevant. Following this line of thought, we conclude that there is no reason to give a preferred status to the original zero-mass supermultiplet and we suggest instead a different approach to make contact with low energy phenomena. It has been shown in the last paper of Ref. 10) that the massive spectrum of ordinary five-dimensional Kaluza-Klein theory exhibits a non-compact $O(2,1)$ symmetry. In a similar vein, one may conjecture that the full spectrum on the seven-sphere possesses a non-compact symmetry group of which $Osp(8,4)$ is only a subgroup. The full spectrum would then transform as one single irreducible representation of this larger (super)group but would be reducible under $Osp(8,4)$. In fact, the supersymmetry in eleven dimensions provides a hint that such a larger group must exist. For the fermionic generators of $Osp(8,4)$ just correspond to the lowest eigenmode of the Dirac operator on S^7 in the expansion of the eleven-dimensional supersymmetry transformation parameter $\epsilon(x,y)$. Higher modes in this expansion correspond to more complicated fermionic charges which will transform states at different levels of the massive tower into each other. This infinite-dimensional superalgebra, which may possibly be extended to include dynamical symmetries, may contain non-compact Lie algebras which could be used to classify the physical states. At this point, one may invoke the curious fact that real unitary representations of non-compact groups may contain representations which are complex under a compact subgroup, a property which has no counterpart in the theory of compact groups^{*)}. If such subgroups are relevant, chirality may be spontaneously generated by this mechanism, and one may hope that the relevant representations are dynamically isolated. It is remarkable that the concept of "spontaneous breaking of reality" in a vector-like theory seems to require the existence of an infinite number of states and provides one more reason why one should expect the number of space-time dimensions to be greater than four.

^{*)} We are grateful to M. Günaydin for bringing this result to our attention.

Spin	SO(8) content	(Mass) ² in units of m ₇ ²
2 ⁺	(n 0 0 0)* n ≥ 0	(n+3) ² - 1
$\frac{3}{2}^{(1)}$	(n 0 0 1)* n ≥ 0	(n+2) ²
$\frac{3}{2}^{(2)}$	(n-1 0 1 0) n ≥ 1	(n+4) ²
1 ⁻⁽¹⁾	(n 1 0 0)* n ≥ 0	(n+1) ² - 1
1 ⁺	(n-1 0 1 1) n ≥ 1	(n+3) ² - 1
1 ⁻⁽²⁾	(n-2 1 0 0) n ≥ 2	(n+5) ² - 1
$\frac{1}{2}^{(1)}$	(n+1 0 1 0)* n ≥ 0	n ²
$\frac{1}{2}^{(2)}$	(n-1 1 1 0) n ≥ 1	(n+2) ²
$\frac{1}{2}^{(3)}$	(n-2 1 0 1) n ≥ 2	(n+4) ²
$\frac{1}{2}^{(4)}$	(n-2 0 0 1) n ≥ 2	(n+6) ²
0 ⁺⁽¹⁾	(n+2 0 0 0)* n ≥ 0	(n-1) ² - 1
0 ⁻⁽¹⁾	(n 0 2 0)* n ≥ 0	(n+1) ² - 1
0 ⁺⁽²⁾	(n-2 2 0 0) n ≥ 2	(n+3) ² - 1
0 ⁻⁽²⁾	(n-2 0 0 2) n ≥ 2	(n+5) ² - 1
0 ⁺⁽³⁾	(n-2 0 0 0) n ≥ 2	(n+7) ² - 1

TABLE: The spectrum of supergravity on the seven-sphere.
The states marked by an asterisk contain the
zero-mass supermultiplet.

REFERENCES

- 1) D. Volkov and V.P. Akulov - Phys.Letters 46B (1973) 109;
J. Wess and B. Zumino - Nuclear Phys. B70 (1974) 39.
- 2) J. Wess and B. Zumino - Phys.Letters 49B (1974) 52;
J. Iliopoulos and B. Zumino - Nuclear Phys. B76 (1974) 310.
- 3) S. Ferrara, D.Z. Freedman and P. van Nieuwenhuizen - Phys.Rev. D13 (1976) 3214;
S. Deser and B. Zumino - Phys.Letters 62B (1976) 335.
- 4) A. Salam and J. Strathdee - Nuclear Phys. B80 (1974) 499;
D.Z. Freedman - in "Recent Developments in Gravitation", Cargèse 1978, eds.
M. Levy and S. Deser (Plenum Press, 1979).
- 5) J. Scherk - same as last reference in 4).
- 6) E. Cremmer, B. Julia and J. Scherk - Phys.Letters 76B (1978) 409.
- 7) E. Cremmer and B. Julia - Phys.Letters 80B (1978) 48; Nuclear Phys. B159 (1979)
141.
- 8) W. Nahm - Nuclear Phys. B135 (1978) 149.
- 9) Th. Kaluza - Sitzungsber.Preuss.Akad.Wiss. K1 (1921) 966;
O. Klein - Z.Phys. 37 (1926) 895.
- 10) B. De Witt - Dynamical Theory of Groups and Fields (Gordon and Breach, New York,
1965);
R. Kerner - Ann.Inst. H. Poincaré 9 (1968) 143;
A. Trautmann - Rep.Math.Phys. 1 (1970) 29;
Y.M. Cho and P.G.O. Freund - Phys.Rev. D12 (1975) 1711;
C. Orzalesi - Fortschr.Phys. 29 (1981) 413).
A. Salam and J. Strathdee - Ann.Phys. 141 (1982) 316.
- 11) E. Witten - Nuclear Phys. B186 (1981) 412.
- 12) M.J. Duff - in "Supergravity '81", eds. S. Ferrara and J.G. Taylor (Cambridge
University Press, 1982);
M.J. Duff and D.J. Toms - in "Unification of the Fundamental Interactions II",
eds. S. Ferrara and J. Ellis (Plenum Press, 1982).
- 13) E. Cremmer and J. Scherk - Nuclear Phys. B103 (1976) 399.
- 14) P.G.O. Freund and M.A. Rubin - Phys.Letters 97B (1980) 233.
- 15) F. Englert, M. Rومان and P. Spindel - Phys.Letters 127B (1983) 47.
- 16) B. Biran, B. de Wit, F. Englert and H. Nicolai - Phys.Letters 124B (1983) 45.
- 17) B. de Wit and H. Nicolai - Phys.Letters 108B (1981) 285; Nuclear Phys. B208
(1982) 323.
- 18) M.J. Duff and C.N. Pope - in "Supersymmetry and Supergravity '82", eds.
S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen (World Scient.Pub.Comp.,
1983).
- 19) P.G.O. Freund, CERN Preprint TH.3655 (1983).
- 20) M.A. Awada, M.J. Duff and C.N. Pope - Phys.Rev.Letters 50 (1983) 294.
- 21) F.A. Bais, H. Nicolai and P. van Nieuwenhuizen - Nuclear Phys. B228 (1983) 333.
- 22) L. Castellani, R. D'Auria and P. Fré - Torino Preprint IFTT427 (1983).
- 23) M.J. Duff, B.E. Nilsson and C.N. Pope - "Compactification of d=11 Supergravity
on $K_3 \times T^3$ ", Texas University Preprint (1983).
- 24) M. Günaydin and F. Gürsey - J.Math.Phys. 14 (1973) 1651.
- 25) M. Rومان - "Eleven-Dimensional Supergravity, and Octonions", Preprint Université
Libre de Bruxelles (1983).
- 26) E. Cartan and J. Schouten - Proc.Kon.Akad.Wet. Amsterdam 29 (1926) 933.
- 27) F. Englert - Phys.Letters 119B (1982) 339.
- 28) R. D'Auria, P. Fré and P. van Nieuwenhuizen - Phys.Letters 122B (1983) 225.
- 29) B. de Wit and H. Nicolai - Preprint NIKHEF-H/83-8 (1983).
- 30) F. Englert, M. Rومان and P. Spindel - "Symmetries in Eleven-Dimensional Super-
gravity Compactified on a Parallelized Seven Sphere", Preprint Université Libre
de Bruxelles (1983), to appear in Phys.Letters B.
- 31) L. Castellani and N.P. Warner - Preprint CALT-68-1033 (1983).
- 32) J. Lukierski and P. Minaert - Preprint Université de Bordeaux I (1983).
- 33) M.J. Duff - in the Proceedings of the Marcel Grossmann Meeting, Shanghai
(1982), to appear.
- 34) M.J. Duff, B.E.W. Nilsson and C.N. Pope - Phys.Rev.Letters 50 (1983) 2043.
- 35) N.P. Warner - Preprint CALT-68-1008 (1983).

- 36) B. Biran, A. Casher, F. Englert, M. Rومان and P. Spindel - "The Fluctuating Seven Sphere in Eleven Dimensional Supergravity", Preprint Université Libre de Bruxelles (1983).
- 37) A. Casher, F. Englert, H. Nicolai and M. Rومان - in preparation.
- 38) S. Gallot and D. Meyer - J.Math. Pures et Appliquées 54 (1975) 259;
G.W. Gibbons and M.J. Perry - Nuclear Phys. B146 (1978) 90.
- 39) S. Deser and R. Nepomechie - "Gauge Invariance Versus Masslessness in de Sitter Space", Preprint Brandeis University (1983).
- 40) R. D'Auria and P. Fré - Phys.Letters 121B (1983) 141.
- 41) D.Z. Freedman and H. Nicolai - "Multiplet Shortening in $Osp(N,4)$ ", MIT Preprint (1983).
- 42) M. Günaydin - in Proceedings of the XI International Colloquium on Group Theoretical Methods in Physics, Istanbul (1982). Lecture Notes in Physics 180 (Springer Verlag);
I. Bars and M. Günaydin - CERN Preprint TH. 3350 (1983), to appear in Commun. Math.Phys.
- 43) R. Slansky - Physics Reports C79 (1981) 1.
- 44) W. Heidenreich - Phys.Letters 110B (1982) 461.
- 45) P. Breitenlohner and D.Z. Freedman - Phys.Letters 115B (1982) 197; Ann.Phys. 144 (1982) 249.
- 46) G.W. Gibbons, C.M. Hull and N.P. Warner - Nuclear Phys. B218 (1983) 173.
- 47) E. Witten - in Proceedings of the Shelter Island II Conference, to appear.
- 48) C. Wetterich - Nuclear Phys. B223 (1983) 109.
- 49) S.W. Hawking - in General Relativity, eds. S.W. Hawking and W. Israel (Cambridge University Press, 1979) and references therein.
- 50) A. Casher and F. Englert - Phys.Letters 104B (1981) 117.
- 51) S. Weinberg - "Gravitation and Cosmology", J. Wiley (1972).
- 52) R. Brout, F. Englert and E. Günzig - Gen.Rel.Grav. 10 (1979) 1.
- 53) R. Brout, F. Englert and P. Spindel - Phys.Rev.Letters 43 (1979) 417.
- 54) A. Guth - Phys.Rev. D23 (1981) 347.
- 55) A. Linde - Phys.Letters 108B (1982) 389.
- 56) S.W. Hawking - Nuclear Phys. B144 (1978) 349.
- 57) P. Mandelbrot - "Fractals, Form, Chance and Dimension", Freeman (San Francisco, 1977).
- 58) H. Nicolai, P. Townsend and P. van Nieuwenhuizen - Lettere al Nuovo Cimento 30 (1981) 315.
- 59) C. Destri, C.A. Orzalesi and P. Rossi - Ann.Phys. 147 (1983) 147;
M.J. Duff and C.A. Orzalesi - Phys.Letters 122B (1983) 37;
C.A. Orzalesi - CERN Preprint TH. 3647 (1983);
C.A. Orzalesi and G. Venturi - CERN Preprint TH. 3648 (1983).

- Figure -

... Universes within Universes within ...
in a two-dimensional Kaluza-Klein analogy.