NIELSEN'S GENERALIZED POLYLOGARITHMS*

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Abstract. Properties (in particular functional relations and special values) of the functions

$$\begin{aligned} (-1)^{n+p-1} (n-1)! p! S_{n,p}(z) &= \int_0^1 \log^{n-1} t \log^p(1-zt) \frac{dt}{t}, \\ (-1)^{n+p-1} (n-1)! p! L_{n,p}(z) &= \int_0^z \log^{n-1} t \log^p(1-t) \frac{dt}{t}, \\ (-1)^{n-1} (n-1)! p! M_{n,p}(z) &= \int_0^z \log^{n-1} t \log^p(1+t) \frac{dt}{t}, \end{aligned}$$

which play a role in the computation of higher order radiative corrections in quantum electrodynamics, are discussed for complex z and positive integers n and p . The first function is a generalization of the well-known polylogarithms ($p=1$). The discussion is based on results published by Nielsen early this century in a little-known monograph.

Key words. Nielsen's generalized polylogarithms, polylogarithms, Spence functions, logarithmic integrals, Riemann zeta functions, Stirling numbers of the first kind

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1. Introduction. A certain class of logarithmic integrals, the so-called polylogarithms, defined by

$$(1.1) \quad Li_n(x) = \frac{(-1)^{n-1}}{(n-2)!} \int_0^1 \log^{n-2} t \log(1-xt) \frac{dt}{t}, \quad (n \geq 2),$$

has been investigated in the past by many mathematicians, including Euler, Kummer, Abel and Spence. In particular, their properties with respect to transformations of the variable were established for several special cases, for example, for the dilogarithm ($n=2$) and the trilogarithm ($n=3$). A special property of these functions is the relation

$$(1.2) \quad Li_n(1) = \zeta(n)$$

where $\zeta(n)$ is the Riemann zeta function of integer argument.

At the beginning of this century, the Danish mathematician Niels Nielsen collected the known results into a monograph [20], and introduced a new class of functions, namely

$$(1.3) \quad S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)! p!} \int_0^1 \log^{n-1} t \log^p(1-zt) \frac{dt}{t},$$

defined for positive integers n, p , and complex $z = x + iy$. The function $\log \zeta$ is understood to lie on its principal sheet, and we define

$$(1.4) \quad \log \xi = \lim_{\varepsilon \rightarrow 0+} \log(\xi + i\varepsilon) = \log|\xi| + i\pi \quad \text{if } \xi < 0.$$

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Because of the fact that

$$S_{n-1,1}(z) = Li_n(z),$$

these functions are generalizations of the polylogarithms (1.1). It is likely that Nielsen undertook his research on $S_{n,p}(z)$ in the hope of finding expressions for $\zeta(2k+1)$ similar to those known for $\zeta(2k)$, and results for

$$(1.5) \quad \beta(2k) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{2k}} \quad (k=1, 2, \dots),$$

similar to those known for $\beta(2k+1)$. At least he remarked [20, pp. 131, 204] that he was not able to obtain such results from his relations for $S_{n,p}(z)$.

Nielsen's monograph [20] contains many formulae for $S_{n,p}(z)$ and for two related functions $L_{n,p}(z)$ and $M_{n,p}(z)$, in particular involving transformations of the argument z . Although most of his formulae are correct (apart from misprints) for complex $z = x + iy$ with $y \neq 0$, the direct replacement of z by real x may lead to discrepancies when the functions involved are complex-valued for real x . This fact has been overlooked in [12]. As far as the author knows, these formulae have never found their way into any of the relevant handbooks, and it seems that the only reference to this monograph until about fifteen years ago was in the book of Lewin [16]. About this time, however, it became apparent that Nielsen, although he did not succeed in finding results for $\zeta(2k+1)$ and $\beta(2k)$, had introduced a class of functions which are of importance in certain applications, in particular in quantum electrodynamics (see, for example, [2], [3], [6], [15], [18], [19]). With this application in mind, functional relations for $S_{n,p}(z)$ were presented in [12], together with a method for computing $S_{n,p}(z)$ accurately (12–14 digits) for real $z = x$ and $n + p \leq 5$. Also some of Nielsen's ideas, in particular on the special values $s_{n,p} = S_{n,p}(1)$ and on a similar integral, have been investigated further [12], [13], [14]. Jacobs and Lambert [11] have developed a method for computing $S_{1,1}(z)$ and $S_{2,1}(z)$ for complex values of z , and Barlow [4] has considered the computation of $S_{n,p}(z)$ for complex z by continued fraction approximations.

An explanation for the fact that Nielsen's monograph remained undiscovered for so long lies perhaps in the fact that it was published in a journal which (at least for mathematicians and physicists) was little-known and not easy to obtain. Further, it contains an unusually large number of misprints and, as we shall see, genuine errors. It is, however, far from being out-of-date or without interest. Integrals of this type or even, in some respects, more general, such as

$$\int_0^1 \frac{\log^m t \log^n(1-t) \log^p(1+t)}{D(t)} dt$$

where m, n , and p are nonnegative integers, and $D(t) = t, 1-t$, or $1+t$, play a role in the evaluation of Feynman and relativistic phase space integrals, as is indicated by the recent publication of a table by Gastmans and Troost [7]. These authors also discuss certain special values of the polylogarithms, a problem which Nielsen investigated in a more general way for his functions, and give a table of infinite series related to polylogarithmic integrals.

Therefore, for historical and practical reasons, it is perhaps worthwhile to give a systematic and critical presentation of the more important part of Nielsen's work on generalized polylogarithms, which until now has not been generally accessible. For

example, Lewin [17, p. 199], in a new edition of [16], remarked that it was not possible for him to give a discussion of the general results obtained by Nielsen, although he cites Nielsen's monograph many times. Also, the table of Gröbner and Hofreiter [9, pp. 71–73] contains a chapter on “Euler's dilogarithm and its generalisations” and a reference to [20], but only formulae for $p = 1$ are given there.

In the presentation which follows, we shall follow the original monograph fairly closely, but on some occasions we shall present new results or proofs; for example, certain constants appearing in the functional equation relating $S_{n,p}(z)$ to $S_{n,p}(1/z)$ will be expressed in terms of known constants. Special attention will also be given to the correct representation of the functional relations for real arguments.

2. Basic formulae. Nielsen's generalized polylogarithms can be defined by [12], [20]

$$(2.1) \quad S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \log^{n-1} t \log^p(1-zt) \frac{dt}{t}$$

$$(2.2) \quad = \sum_{j=0}^{\infty} \frac{(-1)^j S_{p+j}^{(p)}}{(p+j)!(p+j)^n} z^{p+j} \quad (|z| \leq 1)$$

$$(2.3) \quad = \frac{(-1)^{n-1}}{(n-1)!p!} \left[\frac{\partial^{n+p-1}}{\partial \beta^{n-1} \partial \alpha^p} \frac{1}{\beta} {}_2F_1(\alpha, \beta; \beta+1; z) \right]_{\alpha=\beta=0}$$

where

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt$$

is the Gaussian hypergeometric function, and

$$S_k^{(m)} = \sum_{i=0}^{k-m} \frac{1}{i!} \binom{k-1+i}{k-m+i} \binom{2k-m}{k-m-i} \sum_{j=0}^i (-1)^j \binom{i}{j}^{k-m+i}$$

are the Stirling numbers of the first kind in Schlömilch's representation [5, p. 216], generated by [1], [5, p. 212]

$$(2.4) \quad \log^m(1+x) = m! \sum_{k=m}^{\infty} S_k^{(m)} \frac{x^k}{k!} \quad (|x| < 1).$$

In this paper, we shall use Nielsen's notation

$$S_n(z) = S_{n-1,1}(z)$$

instead of the notation $Li_n(z)$ introduced by Lewin [16], [17].

It follows from the series definition (2.2) that, for the polylogarithms,

$$(2.5) \quad S_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n} = \frac{z}{1^n} + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots \quad (|z| \leq 1) \quad (n > 1).$$

For $n = 1$ and $n = 0$, we have as degenerate cases

$$(2.6) \quad S_1(z) = -\log(1-z), \quad S_0(z) = \frac{z}{1-z}, \quad (|z| < 1).$$

For $p > 1$, the series (2.2) becomes considerably more complicated. By writing [5, p. 217]

$$(2.7) \quad S_{j+2}^{(2)} = (-1)^j (j+1)! \sum_{k=1}^{j+1} \frac{1}{k},$$

$$(2.8) \quad S_{j+3}^{(3)} = (-1)^j (j+2)! \left[\left(\sum_{k=1}^{j+2} \frac{1}{k} \right)^2 - \sum_{k=1}^{j+2} \frac{1}{k^2} \right],$$

we find, for example

$$(2.9) \quad S_{n,2}(z) = \frac{z^2}{2^{n+1}} + \left(1 + \frac{1}{2}\right) \frac{z^3}{3^{n+1}} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{z^4}{4^{n+1}} + \dots,$$

$$(2.10) \quad S_{n,3}(z) = \frac{1}{2} \left[\left(1 + \frac{1}{2}\right)^2 - \left(1 + \frac{1}{4}\right) \right] \frac{z^3}{3^{n+1}} \\ + \frac{1}{2} \left[\left(1 + \frac{1}{2} + \frac{1}{3}\right)^2 - \left(1 + \frac{1}{4} + \frac{1}{9}\right) \right] \frac{z^4}{4^{n+1}} + \dots.$$

From (2.1) and (2.2) one may easily obtain

$$(2.11) \quad \frac{d}{dz} S_{n,p}(\alpha z) = \begin{cases} \frac{(-1)^p}{p!} \frac{1}{z} \log^p(1 - \alpha z), & (n=1), \\ \frac{1}{z} S_{n-1,p}(\alpha z), & (n \geq 2), \end{cases}$$

$$(2.12) \quad S_{n,p}(\alpha z) = \int_0^z \frac{1}{\xi} S_{n-1,p}(\alpha \xi) d\xi, \quad (n \geq 1), \\ S_{n,p}(z) = \frac{1}{p! p^n} z^p + O(z^{p+1}), \quad (z \rightarrow 0),$$

and from (5.12) below,

$$S_{n,p}(z) = \frac{(-1)^p}{(n+p)!} \log^{n+p}(-z) + O(\log^{n-1}(-z)), \quad (z \rightarrow \infty).$$

In order to derive his relations, Nielsen introduces in addition the functions

$$(2.13) \quad L_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)! p!} \int_0^z \log^{n-1} t \log^p(1-t) \frac{dt}{t}, \quad (n \geq 1, p \geq 1),$$

$$(2.14) \quad M_{n,p}(z) = \frac{(-1)^{n-1}}{(n-1)! p!} \int_0^z \log^{n-1} t \log^p(1+t) \frac{dt}{t}, \quad (n \geq 1, p \geq 1).$$

Using the binomial theorem, it is easy to see that

$$(2.15) \quad S_{n,p}(z) = \sum_{j=0}^{n-1} \frac{\log^j z}{j!} L_{n-j,p}(z),$$

$$(2.16) \quad L_{n,p}(z) = \sum_{j=0}^{n-1} \frac{(-1)^j \log^j z}{j!} S_{n-j,p}(z),$$

and

$$(2.17) \quad (-1)^p S_{n,p}(-z) = \sum_{j=0}^{n-1} \frac{\log^j z}{j!} M_{n-j,p}(z),$$

$$(2.18) \quad M_{n,p}(z) = (-1)^p \sum_{j=0}^{n-1} \frac{(-1)^j \log^j z}{j!} S_{n-j,p}(-z).$$

In particular,

$$(2.19) \quad L_{1,p}(z) = S_{1,p}(z), \quad M_{1,p}(z) = (-1)^p S_{1,p}(-z).$$

The values of $S_{n,p}(z)$ for $z=1$, -1 and $\frac{1}{2}$ are of special interest. These values are real, and we define

$$(2.20) \quad \begin{aligned} s_{n,p} &= S_{n,p}(1) = L_{n,p}(1), \\ \sigma_{n,p} &= (-1)^p S_{n,p}(-1) = M_{n,p}(1), \\ a_{n,p} &= S_{n,p}\left(\frac{1}{2}\right). \end{aligned}$$

We also write

$$s_n = s_{n-1,1}, \quad \sigma_n = \sigma_{n-1,1}, \quad a_n = a_{n-1,1}.$$

Properties of $s_{n,p}$, $\sigma_{n,p}$ and $a_{n,p}$ are discussed in §9. For the moment, we note only that $s_{n,p}$ can be expressed in $\zeta(k)$, that

$$(2.21) \quad s_{n,p} = s_{p,n},$$

which follows from the definition integral (2.1) by partial integration, and that

$$(2.22) \quad s_n = \zeta(n), \quad \sigma_n = (1 - 2^{1-n})\zeta(n),$$

which follows from (2.5), using well-known properties of the Riemann zeta function.

3. Some analytical properties of $S_{n,p}(z)$, $L_{n,p}(z)$, and $M_{n,p}(z)$. With the general definition of the logarithm function, $S_{n,p}(z)$, $L_{n,p}(z)$, and $M_{n,p}(z)$ are multi-valued functions of z . By restricting $\log \zeta$ to its principal sheet, using (1.4), and cutting the z -plane appropriately, we obtain single-valued functions which are holomorphic at all finite points z with the exception of the branch-cuts. In the remainder of this section, we define these cuts along sections of the real axis $z=x$ and describe the behavior of these functions on this axis, in particular on the cuts. These properties will be required later.

3.1. $S_{n,p}(z)$. In this case, we have as branch points $z=1$ and $z=\infty$, and we cut the z -plane along the real axis from $x=1$ to ∞ . For $x \leq 1$, $S_{n,p}(x)$ is real. For the behavior on the cut, we have from (1.3) and (1.4)

$$(3.1) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0+} S_{n,p}(x - i\varepsilon) &= S_{n,p}(x), \\ \lim_{\varepsilon \rightarrow 0+} S_{n,p}(x + i\varepsilon) &= S_{n,p}^*(x), \end{aligned} \quad (x > 1),$$

where the asterisk denotes the complex conjugate. Therefore, $S_{n,p}(z)$ is, on the cut, continuous from below.

3.2. $L_{n,p}(z)$ ($n > 1$). Because of (2.19), we can make the restriction $n > 1$. As branch points, we find $z=0$, $z=1$ and $z=\infty$, and we cut the z -plane along the real axis from $x=1$ to ∞ , and from $x=0$ to $-\infty$. For $0 \leq x \leq 1$, $L_{n,p}(x)$ is real. On the cuts, we have from (2.16),

$$(3.2) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0+} L_{n,p}(x+i\varepsilon) &= L_{n,p}(x), \\ \lim_{\varepsilon \rightarrow 0+} L_{n,p}(x-i\varepsilon) &= L_{n,p}^*(x), \end{aligned} \quad (x < 0),$$

and

$$(3.3) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0+} L_{n,p}(x-i\varepsilon) &= L_{n,p}(x), \\ \lim_{\varepsilon \rightarrow 0+} L_{n,p}(x+i\varepsilon) &= L_{n,p}^*(x), \end{aligned} \quad (x > 1),$$

so that $L_{n,p}(z)$ is continuous from above for $x < 0$, and continuous from below for $x > 1$.

3.3. $M_{n,p}(z)$. From (2.19) and (3.1), we find for $n=1$ that the branch points are $z=-1$ and $z=\infty$, and we cut the z -plane from $x=-1$ to $-\infty$. On the cut, we have

$$(3.4) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0+} M_{1,p}(x+i\varepsilon) &= M_{1,p}(x), \\ \lim_{\varepsilon \rightarrow 0+} M_{1,p}(x-i\varepsilon) &= M_{1,p}^*(x), \end{aligned} \quad (x < -1),$$

so that $M_{1,p}(z)$ is continuous from above for $x < -1$. For $x \geq -1$, $M_{1,p}(x)$ is real.

For $n > 1$, we find from (2.18)

$$\lim_{\varepsilon \rightarrow 0+} M_{n,p}(x \pm i\varepsilon) = (-1)^p \sum_{j=0}^{n-1} \frac{(-1)^j \log^j(x \pm i\varepsilon)}{j!} S_{n-j,p}(-x \mp i\varepsilon).$$

Because of (1.4) and (3.1), it follows that $M_{n,p}(z)$ has branch points at $z=0$ and $z=\infty$, and we cut the z -plane from $x=0$ to $-\infty$. Then

$$(3.5) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0+} M_{n,p}(x+i\varepsilon) &= M_{n,p}(x), \\ \lim_{\varepsilon \rightarrow 0+} M_{n,p}(x-i\varepsilon) &= M_{n,p}^*(x), \end{aligned} \quad (x < 0).$$

For $x \geq 0$, $M_{n,p}(x)$ is real. Thus, $M_{n,p}(z)$ is continuous from above for $x < 0$ ($n \geq 1, p \geq 1$).

4. A lemma. In order to establish some of the functional relations for the generalized polylogarithms, Nielsen needs the following

LEMMA. Let the complex quantities x_j , y_j ($j=1, \dots, n$), and α , for $n=1, 2, 3, \dots$, be related by

$$(4.1) \quad x_n = \sum_{j=0}^{n-1} \frac{\alpha^j}{j!} y_{n-j}.$$

Then

$$(4.1a) \quad y_n = \sum_{j=0}^{n-1} (-1)^j \frac{\alpha^j}{j!} x_{n-j},$$

and

$$(4.1b) \quad \sum_{j=0}^{n-1} \frac{\alpha^j}{j!} \binom{n+r-j-1}{r} y_{n+r-j} = \sum_{m=0}^r (-1)^m \frac{\alpha^m}{m!} \binom{n+r-m-1}{r-m} x_{n+r-m}.$$

Proof. The first result (4.1a) is well known. Nielsen obtains (4.1a) by using relations between integrals (see Lewin [17, p. 266]), but mentions that it can be proved directly. An elementary proof follows from the well-known binomial inversion theorem [21, p. 45], namely

$$a_n = \sum_{j=0}^n \binom{n}{j} b_{n-j}, \quad b_n = \sum_{j=0}^n (-1)^j \binom{n}{j} a_{n-j},$$

by setting $a_0 = b_0 = 0$ and

$$a_k = k! \alpha^{-k} x_k, \quad b_k = k! \alpha^{-k} y_k.$$

Note that, for fixed p , $L_{n,p}(x)$, $S_{n,p}(x)$, and $M_{n,p}(x)$, $(-1)^p S_{n,p}(-x)$ are, according to (2.15)–(2.18) pairs of functions related by this lemma.

In order to prove (4.1b), Nielsen uses a form of Vandermonde's convolution theorem (see, for example, Riordan [21, pp. 8–10]), namely

$$(4.2) \quad \sum_{m=0}^r (-1)^m \binom{j}{m} \binom{n+r-m-1}{r-m} = \binom{n-j+r-1}{r},$$

in particular

$$(4.3) \quad \sum_{m=0}^r (-1)^m \binom{n+k}{m} \binom{n+r-m-1}{r-m} = 0, \quad (0 \leq k \leq r-1).$$

Denoting the left-hand side of (4.1b) by $A_{n,r}$, we obtain, using (4.2),

$$(4.4) \quad \begin{aligned} A_{n,r} &= \sum_{j=0}^{n-1} \frac{\alpha^j}{j!} \sum_{m=0}^r (-1)^m \binom{j}{m} \binom{n+r-m-1}{r-m} y_{n+r-j} \\ &= \sum_{m=0}^r (-1)^m \frac{\alpha^m}{m!} \binom{n+r-m-1}{r-m} \sum_{j=0}^{n-m-1} \frac{\alpha^j}{j!} y_{n+r-m-j}. \end{aligned}$$

From (4.1), we have

$$(4.5) \quad \sum_{j=0}^{n-m-1} \frac{\alpha^j}{j!} y_{n+r-m-j} = x_{n+r-m} - \alpha^{n-m} \sum_{k=0}^{r-1} \frac{\alpha^k}{(n-m+k)!} y_{r-k}.$$

Substituting (4.5) into (4.4), we find as coefficient for y_{r-k} the expression

$$- \frac{\alpha^{n+k}}{(n+k)!} \sum_{m=0}^r (-1)^m \binom{n+k}{m} \binom{n+r-m-1}{r-m}, \quad (0 \leq k \leq r-1)$$

which vanishes by (4.3). The result (4.1b) follows immediately.

5. The transformations $z \rightarrow 1-z$ and $z \rightarrow 1/z$. Nielsen was able to find many formulae relating the functions $S_{n,p}(z)$, $L_{n,p}(z)$, and $M_{n,p}(z)$ for different arguments. He proved these relations mainly by considering indefinite integrals for $L_{n,p}(z)$ and $M_{n,p}(z)$, for example by writing

$$L_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int \log^{n-1} z \log^p(1-z) \frac{dz}{z} + C,$$

and fixing the constant in the resulting expression by considering special values.

In the following, we shall prove these formulae of Nielsen by using the definitions (2.13) and (2.14) of $L_{n,p}(z)$ and $M_{n,p}(z)$. We start with

5.1. The reflection $z \rightarrow 1-z$. We split the range of integration in (2.13), and obtain by partial integration, for $\text{Im } z \neq 0$, using (2.20),

$$\begin{aligned} L_{n,p}(z) &= s_{n,p} + \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_1^z \log^{n-1} t \log^p(1-t) \frac{dt}{t} \\ &= s_{n,p} + \frac{(-1)^{n+p-1}}{n!p!} \log^n z \log^p(1-z) \\ &\quad - \frac{(-1)^{n+p-1}}{(p-1)!n!} \int_0^{1-z} \log^{p-1} t \log^n(1-t) \frac{dt}{t}, \end{aligned}$$

which gives

$$(5.1) \quad L_{n,p}(z) + L_{p,n}(1-z) = s_{n,p} - \frac{(-1)^{n+p}}{n!p!} \log^n z \log^p(1-z)$$

as the reflection formula for the function $L_{n,p}(z)$. Using (2.15) yields

$$(5.2) \quad S_{n,p}(z) = \sum_{j=0}^{n-1} \frac{\log^j z}{j!} \left\{ s_{n-j,p} - L_{p,n-j}(1-z) - \frac{(-1)^{n+p-j}}{(n-j)!p!} \log^{n-j} z \log^p(1-z) \right\}$$

whence, using (2.16) and a well-known property of the binomial coefficients,

$$\begin{aligned} (5.3) \quad S_{n,p}(z) &= \sum_{j=0}^{n-1} \frac{\log^j z}{j!} \left\{ s_{n-j,p} - \sum_{k=0}^{p-1} \frac{(-1)^k \log^k(1-z)}{k!} S_{p-k,n-j}(1-z) \right\} \\ &\quad + \frac{(-1)^p}{n!p!} \log^n z \log^p(1-z). \end{aligned}$$

Equation (5.3) is the reflection formula for the generalized polylogarithms. In the case of polylogarithms ($p=1$), this equation reduces to

$$(5.4) \quad S_n(z) = \sum_{j=0}^{n-2} \frac{\log^j z}{j!} \{ s_{n-j} - S_{1,n-j-1}(1-z) \} - \frac{1}{(n-1)!} \log^{n-1} z \log(1-z).$$

Note that the reflection $z \rightarrow 1-z$ for $S_{n,p}(z)$ as expressed by (5.3) remains within the set of generalized polylogarithms (apart from known constants and logarithms). Equation (5.4) shows, however, that it is not possible to represent the reflection for the polylogarithms within the set of these functions if $n \geq 3$. Thus, for the trilogarithm, one

has

$$(5.5) \quad S_3(z) = s_3 - S_{1,2}(1-z) + \log z [s_2 - S_2(1-z)] - \frac{1}{2} \log^2 z \log(1-z).$$

This relation contains the function $S_{1,2}(1-z)$ which is not a polylogarithm. It is interesting to note, however, that the reflection formula provides an expression for the particular generalized polylogarithms $S_{1,p}(z)$ in terms of ordinary polylogarithms, since we have

$$(5.6) \quad S_{1,p}(z) = \frac{(-1)^p}{p!} \int_0^1 \log^p(1-zt) \frac{dt}{t} = s_{p+1} + \frac{(-1)^p}{p!} \log z \log^p(1-z) \\ - \sum_{k=0}^{p-1} (-1)^k \log^k \frac{(1-z)}{k!} S_{p-k+1}(1-z).$$

We now consider the case $z=x$ real. By setting $z=x+i\epsilon$ it follows from (3.2), (3.3) and (1.4) that, for $x < 0$, (5.1) remains unchanged, while for $x > 1$, both sides must be replaced by their complex conjugate. From this and the fact that $L_{n,p}(x)$ is real for $0 \leq x \leq 1$, we see that (5.1) remains valid, as it stands, for arbitrary real $z=x$. By a similar argument, this is also true for (5.3) relating $S_{n,p}(z)$ to $S_{n,p}(1-z)$, and for the special case (5.4). Therefore, the restriction $\text{Im } z \neq 0$ made above is no longer necessary.

5.2. The inversion $z \rightarrow 1/z$. This case is more complicated. We follow Nielsen in first establishing an inversion formula for $M_{n,p}(z)$ but instead of considering indefinite integrals for $M_{n,p}(z)$ and $M_{n,p}(1/z)$ as he does, we start with the definition integral (2.14), which gives for $\text{Im } z \neq 0$, using (2.20),

$$M_{n,p}(z) = \sigma_{n,p} + \frac{(-1)^{n-1}}{(n-1)!p!} \int_1^z \log^{n-1} t \log^p(1+t) \frac{dt}{t} \\ = \sigma_{n,p} - \frac{1}{(n-1)!p!} \int_1^{1/z} \log^{n-1} t [\log(1+t) - \log t]^p \frac{dt}{t} \\ = \sigma_{n,p} - \frac{1}{(n-1)!p!} \sum_{k=0}^{p-1} (-1)^k \binom{p}{k} \int_1^{1/z} \log^{n+k-1} t \log^{p-k}(1+t) \frac{dt}{t} \\ - \frac{(-1)^p}{(n-1)!p!} \int_1^{1/z} \log^{n+p-1} t \frac{dt}{t} \\ = \sigma_{n,p} - (-1)^n \sum_{k=0}^{p-1} \binom{n+k-1}{k} \sigma_{n+k,p-k} \\ + (-1)^n \sum_{k=0}^{p-1} \binom{n+k-1}{k} M_{n+k,p-k} \left(\frac{1}{z} \right) - \frac{(-1)^n}{(n+p)(n-1)!p!} \log^{n+p} z.$$

Denoting the constant part by $C_{n,p}$, we have

$$(5.7) \quad M_{n,p}(z) = C_{n,p} + (-1)^n \sum_{k=0}^{p-1} \binom{n+k-1}{k} M_{n+k,p-k} \left(\frac{1}{z} \right) - \frac{(-1)^n}{(n+p)(n-1)!p!} \log^{n+p} z \\ (\text{Im } z \neq 0, \text{ or } \text{Im } z = 0 \text{ and } \text{Re } z \geq 0).$$

Formula (5.7) is also valid for real $z = x \geq 0$, in which case it is real. For $x < 0$, we set $z = x + i\varepsilon$ and obtain with the help of (1.4) and (3.5) the relation

$$(5.8) \quad M_{n,p}(x) = C_{n,p} + (-1)^n \sum_{k=0}^{p-1} \binom{n+k-1}{k} M_{n+k,p-k}^* \left(\frac{1}{x} \right) \\ - \frac{(-1)^n}{(n+p)(n-1)!p!} \log^{n+p} x, \quad (x \text{ real}).$$

This equation enables us to express the constants (where the sum is zero for $p = 1$)

$$(5.9) \quad C_{n,p} = (1 - (-1)^n) \sigma_{n,p} - (-1)^n \sum_{k=1}^{p-1} \binom{n+k-1}{k} \sigma_{n+k,p-k}$$

in terms of known constants. This will be discussed further in §7.

In order to establish the inversion formula for the generalized polylogarithms $S_{n,p}(z)$, we start from (5.7) and write, using (2.17)

$$(5.10) \quad (-1)^p S_{n,p}(-z) = \sum_{j=0}^{n-1} \frac{\log^j z}{j!} (-1)^{n-j} \sum_{k=0}^{p-1} \binom{n-j+k-1}{k} M_{n-j+k,p-k} \left(\frac{1}{z} \right) \\ - \frac{1}{p!} \log^{n+p} z \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{(n-j+p)(n-j-1)!j!} + \sum_{j=0}^{n-1} \frac{\log^j z}{j!} C_{n-j,p}.$$

The coefficient of $\log^{n+p} z$ can be written as [8, No. 3.191 3]

$$(5.11) \quad -\frac{1}{p!} \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{(n-j+p)(n-j-1)!j!} = \frac{1}{p!(n-1)!} \int_0^1 \sum_{j=0}^{n-1} (-1)^{n-j} \binom{n-1}{j} u^{n-j+p-1} du \\ = \frac{1}{p!(n-1)!} \int_0^1 u^p (1-u)^{n-1} du = \frac{1}{(n+p)!}.$$

We now apply Lemma (4.1b) and write, remembering that $M_{n,p}(z)$ and $(-1)^p S_{n,p}(-z)$ are related through (4.1),

$$\sum_{j=0}^{n-1} (-1)^j \frac{\log^j z}{j!} \binom{n-j+k-1}{k} M_{n-j+k,p-k} \left(\frac{1}{z} \right) \\ = (-1)^{p-k} \sum_{m=0}^k \frac{\log^m z}{m!} \binom{n+k-m-1}{k-m} S_{n+k-m,p-k} \left(-\frac{1}{z} \right)$$

whence, replacing z by $-z$ in (5.10)

$$(5.12) \quad S_{n,p}(z) = (-1)^n \sum_{k=0}^{p-1} (-1)^k \sum_{m=0}^k \frac{\log^m(-z)}{m!} \binom{n+k-m-1}{k-m} S_{n+k-m,p-k} \left(\frac{1}{z} \right) \\ + (-1)^p \left\{ \sum_{j=0}^{n-1} \frac{\log^j(-z)}{j!} C_{n-j,p} + \frac{1}{(n+p)!} \log^{n+p}(-z) \right\}.$$

It is easy to see that (5.12) is also valid for real $z=x$ if $x \leq 0$. In this case it contains only real functions. For $x > 0$, we proceed analogously to the case of $M_{n,p}(z)$, set $z = x + i\varepsilon$ and obtain, using (1.4) and (3.1),

(5.13)

$$S_{n,p}(x) = (-1)^n \sum_{k=0}^{p-1} (-1)^k \sum_{m=0}^k \frac{\log^m(-x)}{m!} \binom{n+k-m-1}{k-m} S_{n+k-m,p-k}^* \left(\frac{1}{x} \right) \\ + (-1)^p \left\{ \sum_{j=0}^{n-1} \frac{\log^j(-x)}{j!} C_{n-j,p} + \frac{1}{(n+p)!} \log^{n+p}(-x) \right\} \quad (x \text{ real}).$$

Note that the imaginary parts cancel for $0 \leq x \leq 1$. Further, since $S_{\nu,\rho}(1/x)$ is real for $x \geq 1$, it follows that the imaginary part of $S_{n,p}(x)$ for $x \geq 1$ comes only from $\log(-x)$. Using (1.4) in (5.13) and taking the imaginary part (with empty sums equal to zero) gives¹

(5.14)

$$\frac{1}{\pi} \operatorname{Im} S_{n,p}(x) = (-1)^n \sum_{k=1}^{p-1} (-1)^k \sum_{m=1}^k \binom{n+k-m-1}{k-m} S_{n+k-m,p-k} \left(\frac{1}{x} \right) \\ \times \sum_{j=0}^{[(m-1)/2]} (-1)^j \frac{\pi^{2j}}{(2j+1)!(m-2j-1)!} \log^{m-2j-1} x \\ + (-1)^p \sum_{m=1}^{n-1} C_{n-m,p} \sum_{j=0}^{[(m-1)/2]} (-1)^j \frac{\pi^{2j}}{(2j+1)!(m-2j-1)!} \log^{m-2j-1} x \\ + (-1)^p \sum_{j=0}^{[(n+p-1)/2]} (-1)^j \frac{\pi^{2j}}{(2j+1)!(n+p-2j-1)!} \log^{n+p-2j-1} x.$$

Equation (5.14) shows that, for $p > 1$, $\operatorname{Im} S_{n,p}(x)$ is expressible, apart from logarithms and (known) constants, in terms of $S_{\nu,\rho}(\xi)$ ($n \leq \nu \leq n+p-2$, $1 \leq \rho \leq p-1$) in the interval $0 < \xi \leq 1$.

For the polylogarithms, we obtain as a special case from (5.12), (5.9) and (2.22)

$$(5.15) \quad S_n(z) + (-1)^n S_n \left(\frac{1}{z} \right) \\ = -\frac{1}{n!} \log^n(-z) - \sum_{j=0}^{n-2} \frac{1}{j!} (1 + (-1)^{n-j}) (1 - 2^{1-n+j}) s_{n-j} \log^j(-z),$$

where for real $z=x$, $S_n(1/x)$ has to be replaced by $S_n^*(1/x)$.

As in the case of reflection, the inversion (5.12) has the remarkable property of remaining within the set of generalized polylogarithms. In contrast to the case of reflection, the inversion (5.15) for the polylogarithms also remains within the set of these functions.

¹ Note that this formula differs by a factor -1 from the expression given in [12], owing to the fact that, for real $z=x$, (5.12) had been erroneously used instead of (5.13), and a convention different from (1.4) had been introduced in order to overcome resulting contradictions. Because of this, the function computed by the Algol program for $S_{n,p}(x)$ in [12] is in fact $S_{n,p}^*(x)$ according to our present definition. For $n=p=1$, formula (5.14) agrees with Lewin [17, p. 2].

6. Bilinear transformations of the argument. If we consider the reflection $P_1(z) = 1 - z$ and the inversion $P_2(z) = 1/z$ as belonging to the group of bilinear transformations

$$P(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

with real coefficients and $\alpha\delta - \beta\gamma \neq 0$, we see that they generate a subgroup

$$P_j(z) = z, \quad 1 - z, \quad \frac{1}{z}, \quad \frac{1}{1 - z}, \quad \frac{z - 1}{z}, \quad \frac{z}{z - 1}$$

($j = 0, 1, \dots, 5$). By repeated use of (5.3) and (5.12) it is then possible to find formulae for

$$S_{n,p}\left(\frac{1}{1 - z}\right), \quad S_{n,p}\left(\frac{z - 1}{z}\right), \quad S_{n,p}\left(\frac{z}{z - 1}\right)$$

which will contain, apart from logarithms and known constants, only generalized polylogarithms. These formulae become soon very complicated, and we leave a more systematic investigation to further research. By the above procedure one can, in principle, find a (complicated) expression for $S_{2,2}(z)$ in terms of polylogarithms. Lewin [17, p. 204] has given a formula from which such an expression can be derived.

Nielsen also discusses the following bilinear transformations of the argument of the function $L_{n,p}(z)$.

6.1. The transformation $z \rightarrow 1/(1 + z)$. From the definition integral (2.13) we find, for $\text{Im } z \neq 0$,

(6.1)

$$\begin{aligned} L_{n,p}\left(\frac{1}{1 + z}\right) &= \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^{1/(1+z)} \log^{n-1} t \log^p(1-t) \frac{dt}{t} \\ &= s_{n,p} - \frac{1}{(n-1)!p!} \int_0^z \log^{n-1}(1+t) [\log(1+t) - \log t]^p \frac{dt}{1+t} \\ &= s_{n,p} - \frac{1}{(n-1)!p!} \sum_{k=0}^p (-1)^k \binom{p}{k} \int_0^z \log^{n+p-k-1}(1+t) \log^k t \frac{dt}{1+t}. \end{aligned}$$

Using (2.14), and taking into account that

$$\frac{\binom{p}{k}}{(n-1)!p!(n+p-k)} = \frac{\binom{n+p-k-1}{n-1} \binom{n+p}{k}}{(n+p)!},$$

yields, after integration by parts,

(6.2)

$$\begin{aligned} L_{n,p}\left(\frac{1}{1 + z}\right) &= s_{n,p} - \frac{1}{(n+p)!} \sum_{k=0}^p (-1)^k \binom{n+p-k-1}{n-1} \binom{n+p}{k} \log^k z \log^{n+p-k}(1-z) \\ &\quad - \sum_{k=1}^p \binom{n+p-k-1}{n-1} M_{k,n+p-k}(z) \quad (\text{Im } z \neq 0, \text{ or } \text{Im } z = 0 \text{ and } \text{Re } z \geq -1). \end{aligned}$$

Both sides of formula (6.2) are real for $z = x \geq 0$. Further, it follows from (1.4) and (3.3) that (6.2) remains valid for real $z = x$ if $-1 \leq x \leq 0$. For $z = x < -1$, however, we see from (3.2) that $L_{n,p}(1/(1+z))$ has to be replaced by $L_{n,p}^*(1/(1+z))$.

6.2. The transformation $z \rightarrow z/(1+z)$. In this case, we have from (2.13) for $\text{Im } z \neq 0$ and $n > 1$

$$\begin{aligned} L_{n,p}\left(\frac{z}{1+z}\right) &= \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^{z/(1+z)} \log^{n-1} t \log^p(1-t) \frac{dt}{t} \\ &= \frac{1}{(n-1)!p!} \left\{ \int_0^z [\log(1+t) - \log t]^{n-1} \log^p(1+t) \frac{dt}{t} \right. \\ &\quad \left. - \int_0^z [\log(1+t) - \log t]^{n-1} \log^p(1+t) \frac{dt}{1+t} \right\}. \end{aligned}$$

Hence by comparison with (6.1),

$$\begin{aligned} L_{n,p}\left(\frac{z}{1+z}\right) &= L_{p+1,n-1}\left(\frac{1}{1+z}\right) - s_{p+1,n-1} \\ &\quad + \frac{1}{(n-1)!p!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^z \log^{n+p-k-1}(1+t) \log^k t \frac{dt}{t} \end{aligned}$$

and, using (2.14) and (2.21),

(6.3)

$$\begin{aligned} L_{n,p}\left(\frac{z}{1+z}\right) - L_{p+1,n-1}\left(\frac{1}{1+z}\right) &= -s_{n-1,p+1} + \sum_{k=0}^{n-1} \binom{n+p-k-1}{p} M_{k+1,n+p-k-1}(z) \\ &\quad (\text{Im } z \neq 0, \text{ or } \text{Im } z = 0 \text{ and } \text{Re } z \geq -1). \end{aligned}$$

For $n = 1$, we have directly from (2.13):

$$\begin{aligned} (6.4) \quad L_{1,p}\left(\frac{z}{1+z}\right) &= \frac{1}{p!} \left\{ \int_0^z \log^p(1+t) \frac{dt}{t} - \int_0^z \log^p(1+t) \frac{dt}{1+t} \right\} \\ &= M_{1,p}(z) - \frac{1}{(p+1)!} \log^{p+1}(1+z) \end{aligned}$$

or, by (2.19),

$$\begin{aligned} (6.5) \quad S_{1,p}\left(\frac{z}{1+z}\right) &= (-1)^p S_{1,p}(-z) - \frac{1}{(p+1)!} \log^{p+1}(1+z) \\ &\quad (\text{Im } z \neq 0, \text{ or } \text{Im } z = 0 \text{ and } \text{Re } z \geq -1). \end{aligned}$$

As in the case of formula (6.2), we see by setting $z = x + i\epsilon$ and using (1.4), (3.2) and (3.3) that the left-hand side of (6.2) must be replaced, for real $z = x < -1$, by

$$L_{n,p}^*\left(\frac{z}{1+z}\right) - L_{p+1,n-1}^*\left(\frac{1}{1+z}\right).$$

Since $L_{1,p}(z/(1+z)) = S_{1,p}(z/(1+z))$ is real for $x \geq -1$, the formula obtained by replacing the left-hand sides of (6.4) and (6.5) by their conjugates are valid for all real $z = x$.

6.3. The transformation $z \rightarrow (a+z)/(b+z)$. This transformation is a generalization of the transformation $z \rightarrow z/(1+z)$. The formula obtained by Nielsen [20, §13(7)] is incorrect, and, as we shall see in §9, it was this error which led him to wrong conclusions on the nature of the special values $\sigma_{n,p} = (-1)^p S_{n,p}(-1)$.

We have from the definition integral (2.13) for $a \neq b$, $\text{Im } z \neq 0$, and $n > 1$,

$$\begin{aligned} L_{n,p}\left(\frac{a+z}{b+z}\right) &= \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^{(a+z)/(b+z)} \log^{n-1} t \log^p(1-t) \frac{dt}{t} \\ &= s_{n,p} + \frac{(-1)^{n+p}}{(n-1)!p!} (b-a) \int_z^\infty \log^{n-1}\left(\frac{a+t}{b+t}\right) \log^p\left(\frac{b-a}{b+t}\right) \frac{dt}{(a+t)(b+t)} \\ &= s_{n,p} + \frac{(-1)^{n+p}}{(n-1)!p!} \left\{ \int_z^\infty \log^{n-1}\left(\frac{a+t}{b+t}\right) \log^p\left(\frac{b-a}{b+t}\right) \frac{dt}{a+t} \right. \\ &\quad \left. - \int_z^\infty \log^{n-1}\left(\frac{a+t}{b+t}\right) \log^p\left(\frac{b-a}{b+t}\right) \frac{dt}{b+t} \right\}. \end{aligned}$$

Denoting the integrals by J_1 and J_2 , and substituting $\tau = (b-a)/(a+t)$ and $\tau = (b-a)/(b+t)$ in J_1 and J_2 , respectively, we obtain, using (2.14),

$$\begin{aligned} J_1 &= (-1)^{n+p-1} \int_0^{(b-a)/(a+z)} \log^{n-1}(1+\tau) \log^p\left(\frac{1+\tau}{\tau}\right) \frac{d\tau}{\tau} \\ &= (-1)^{n+p-1} \sum_{k=0}^p (-1)^k \binom{p}{k} \int_0^{(b-a)/(a+z)} \log^{n+p-k-1}(1+\tau) \log^k \tau \frac{d\tau}{\tau} \\ &= (-1)^{n+p-1} p!(n-1)! \sum_{k=0}^p \binom{n+p-k-1}{n-1} M_{k+1, n+p-k-1} \left(\frac{b-a}{a+z}\right). \end{aligned}$$

Similarly, by (2.13),

$$\begin{aligned} J_2 &= \int_0^{(b-a)/(b+z)} \log^{n-1}(1-\tau) \log^p \tau \frac{d\tau}{\tau} \\ &= (-1)^{n+p-1} p!(n-1)! L_{p+1, n-1} \left(\frac{b-a}{b+z}\right). \end{aligned}$$

Therefore

(6.6)

$$L_{n,p}\left(\frac{a+z}{b+z}\right) = s_{n,p} - \sum_{k=0}^p \binom{n+p-k-1}{n-1} M_{k+1, n+p-k-1} \left(\frac{b-a}{a+z}\right) + L_{p+1, n-1} \left(\frac{b-a}{b+z}\right) \quad (\text{Im } z \neq 0).$$

This formula corrects Nielsen's formula [20, §13(7)], in which

$$\text{the term } L_{p+1, n-1} \left(\frac{b-a}{b+z}\right) \text{ reads } L_{n,p} \left(\frac{b-a}{b+z}\right).$$

For real $z = x$, formula (6.6) has to be adjusted in accordance with the results of §3. This will depend on the values of a and b .

Comparing formula (6.3) with (6.6) for $a=0$ and $b=1$, we obtain a relation between $M_{\nu,\rho}(z)$ and $M_{\nu,\rho}(1/z)$, namely ($n > 1$)

$$(6.7) \quad \sum_{k=0}^{n-1} \binom{n+p-k-1}{p} M_{k+1, n+p-k-1}(z) + \sum_{k=0}^p \binom{n+p-k-1}{n-1} M_{k+1, n+p-k-1}\left(\frac{1}{z}\right) = s_{n,p} + s_{n-1, p+1}.$$

($\operatorname{Im} z \neq 0$, or $\operatorname{Im} z = 0$ and $\operatorname{Re} z \geq 0$).

For real $z = x < 0$, $M_{k+1, n+p-k-1}(1/z)$ has to be replaced by $M_{k+1, n+p-k-1}^*(1/z)$.

6.4. A sum containing $L_{n,p}(z/(1+z))$. Nielsen closes his section on bilinear transformations of the argument by considering a sum containing $L_{\nu,\rho}(z/(1+z))$. This sum can be expressed in terms of (ordinary) polylogarithms. To avoid divergent integrals, we introduce the functions ($0 < \varepsilon < 1$)

$$(6.8) \quad \left. \begin{aligned} &L_{n,p}^\varepsilon(z) \\ &(-1)^p M_{n,p}^\varepsilon(z) \end{aligned} \right\} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_\varepsilon^z \log^{n-1} t \log^p(1 \mp t) \frac{dt}{t}$$

and write

$$(-1)^j k! L_{k-j+1,j}^\varepsilon(\zeta) = (-1)^j k! \frac{(-1)^k}{(k-j)!j!} \int_\varepsilon^\zeta \log^{k-j} t \log^j(1-t) \frac{dt}{t}.$$

Thus, by summation,

$$\begin{aligned} \sum_{j=0}^k (-1)^j L_{k-j+1,j}^\varepsilon(\zeta) &= \frac{(-1)^k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} \int_\varepsilon^\zeta \log^{k-j} t \log^j(1-t) \frac{dt}{t} \\ &= \frac{(-1)^k}{k!} \int_\varepsilon^\zeta \log^k \left(\frac{t}{1-t} \right) \frac{dt}{t}. \end{aligned}$$

Setting $\zeta = z/(1+z)$, and making the substitution $\tau = t/(1-t)$, we obtain, with $\varepsilon' = \varepsilon/(1-\varepsilon)$,

$$\begin{aligned} \sum_{j=0}^k (-1)^j L_{k-j+1,j}^\varepsilon\left(\frac{z}{1+z}\right) &= \frac{(-1)^k}{k!} \int_{\varepsilon'}^z \log^k \tau \frac{d\tau}{\tau(1+\tau)} \\ &= \frac{(-1)^k}{(k+1)!} [\log^{k+1} z - \log^{k+1} \varepsilon'] - \frac{(-1)^k}{k!} \int_{\varepsilon'}^z \log^k \tau \frac{d\tau}{1+\tau}. \end{aligned}$$

By partial integration, using (6.8)

$$\int_{\varepsilon'}^z \log^k \tau \frac{d\tau}{1+\tau} = \log^k z \log(1+z) - \log^k \varepsilon' \log(1+\varepsilon') + (-1)^k k! M_{k,1}^{\varepsilon'}(z).$$

Further, from (6.8),

$$L_{k+1,0}^\varepsilon\left(\frac{z}{1+z}\right) = \frac{(-1)^k}{(k+1)!} \left[\log^{k+1}\left(\frac{z}{1+z}\right) - \log^{k+1} \varepsilon \right].$$

Collecting these results, we see that the divergent parts cancel for $\varepsilon \rightarrow 0$, and we obtain

$$(6.9) \quad \sum_{j=1}^k (-1)^j L_{k-j+1,j} \left(\frac{z}{1+z} \right) = \frac{(-1)^k}{(k+1)!} \left[\log^{k+1} z - \log^{k+1} \left(\frac{z}{1+z} \right) \right] \\ - \frac{(-1)^k}{k!} \log^k z \log(1+z) - M_{k,1}(z)$$

or, using (2.18) and replacing k by $n-1$,

$$(6.10) \quad \sum_{m=0}^{n-2} (-1)^m L_{n-m-1,m+1} \left(\frac{z}{1+z} \right) = \frac{(-1)^n}{n!} \left[\log^n z - \log^n \left(\frac{z}{1+z} \right) \right] \\ - \sum_{j=0}^{n-1} \frac{(-1)^j \log^j z}{j!} S_{n-j}(-z).$$

As in the case of formula (6.2), the left-hand side of (6.10) has to be replaced by its conjugate for real $z = x < -1$.

7. The constants $C_{n,p}$. We now return to the constants $C_{n,p}$ defined by (5.9). Because no relation seems to be known which expresses $\sigma_{n,p} = (-1)^p S_{n,p}(-1)$ for arbitrary $n \geq 1$, $p \geq 1$ in terms of known constants, Nielsen did not, apart from giving a formula for $C_{2n-1,2}$ in terms of $\zeta(q)$, investigate relation (5.9) further. For some small values of n or p , however, expressions for $C_{n,p}$ are known [12] in addition to $C_{2n-1,2}$, namely

$$(7.1) \quad \begin{aligned} C_{1,p} &= \zeta(p+1), & (p=1,2,3,4), \\ C_{2n,1} &= 0, & (n \geq 1), \\ C_{3,1} &= -C_{2,2} = \frac{7}{4} \zeta(4) = \frac{7}{360} \pi^4, \\ C_{2,3} &= -C_{3,2} = -\zeta(2)\zeta(3) - \zeta(5). \end{aligned}$$

The question arises whether it is possible to express $C_{n,p}$ for all n and p , in terms of known constants. This is indeed the case, as is shown by the following

THEOREM 1. *Let δ_{ij} be the Kronecker symbol; let $\varepsilon_m = 1$ if m is even, $\varepsilon_m = 0$ if m is odd, and let $s_{\nu,p} = S_{\nu,p}(1)$. Then, for integer n, p ($n \geq 1, p \geq 1$)*

$$(7.2) \quad C_{n,p} = (-1)^{n+p-1} \sum_{k=0}^{p-1} (-1)^k (1 - (-1)^n \delta_{k0}) \binom{n+k-1}{k} \\ \times \sum_{j=0}^{[(n+k-1)/2]} (-1)^j \frac{\pi^{2j}}{(2j)!} s_{n+k-2j,p-k} \\ + (-1)^{[(n+p)/2]+n} \varepsilon_{n+p} \frac{\pi^{n+p}}{(n+p)(n-1)!p!}.$$

Proof. We set $x = -1$ in (5.8) and obtain, using (1.4),

$$C_{n,p} = M_{n,p}(-1) - (-1)^n \sum_{k=0}^{p-1} \binom{n+k-1}{k} M_{n+k,p-k}^*(-1) + (-1)^n \frac{(i\pi)^{n+p}}{(n+p)(n-1)!p!}.$$

From (2.18), we have

$$(7.3) \quad M_{n,p}(-1) = (-1)^p \sum_{j=0}^{n-1} \frac{(-i\pi)^j}{j!} s_{n-j,p}.$$

Therefore

$$(7.4) \quad \begin{aligned} C_{n,p} = & (-1)^{n+p-1} \left\{ \sum_{k=0}^{p-1} (-1)^k \binom{n+k-1}{k} \sum_{j=0}^{n+k-1} \frac{(i\pi)^j}{j!} s_{n+k-j,p-k} \right. \\ & \left. - (-1)^n \sum_{j=0}^{n-1} (-1)^j \frac{(i\pi)^j}{j!} s_{n-j,p} \right\} \\ & + (-1)^n \frac{(i\pi)^{n+p}}{(n+p)(n-1)!p!}. \end{aligned}$$

On taking the real part of (7.4), Theorem 1 follows. In addition, we obtain from the imaginary part a relation between the $s_{\nu,\rho}$, namely

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k (1 + (-1)^n \delta_{k0}) \binom{n+k-1}{k} \sum_{j=0}^{[(n+k-2)/2]} (-1)^j \frac{\pi^{2j+1}}{(2j+1)!} s_{n+k-2j-1,p-k} \\ + (-1)^{[(n+p-1)/2]+p} (\varepsilon_{n+p} - 1) \frac{\pi^{n+p}}{(n+p)(n-1)!p!} = 0. \end{aligned}$$

It has been shown in [12], [13], [20] that $s_{\nu,\rho}$ can be expressed by a homogeneous polynomial in $s_q = \zeta(q)$ ($2 \leq q \leq \nu + \rho$), with rational coefficients, where $\zeta(q)$ is the Riemann zeta function. This polynomial is of degree $\nu + \rho$, if $\zeta(q)$ is defined to be of degree q and if the degree of a product is the sum of the degree of its factors. Since even powers of π can, through [8, No. 9.616]

$$\pi^{2m} = \frac{(2m)! \zeta(2m)}{2^{2m-1} |B_{2m}|}$$

be expressed in terms of $\zeta(q)$ and the Bernoulli numbers B_{2m} , one has the remarkable fact that $C_{n,p}$ can be expressed by a polynomial in $\zeta(q)$ ($2 \leq q \leq n+p$) which is of the same type as the polynomial for $s_{n,p}$.

A short table of $C_{n,p}$ ($1 \leq n \leq 5, 1 \leq p \leq 5$) is given in Table 1. This table has been computed by REDUCE [10] from the expressions for $s_{r,m}$ given in [13]. For the special case $n = 1$, we find from (5.9) and Theorem 3 in §9 that

$$(7.5) \quad C_{1,p} = 2\sigma_{1,p} + \sum_{k=1}^{p-1} \sigma_{k+1,p-k} = \zeta(p+1), \quad (p \geq 1),$$

which gives, with (7.2), another relation between the $s_{\nu,\rho}$.

TABLE 1

$$\begin{aligned}
 C_{1,1} &= \zeta(2) = \frac{\pi^2}{6} & C_{2,1} &= 0 \\
 C_{1,2} &= \zeta(3) & C_{2,2} &= -6\zeta^2(2) + \frac{53}{4}\zeta(4) = -\frac{7}{360}\pi^4 \\
 C_{1,3} &= \zeta(4) = \frac{\pi^4}{90} & C_{2,3} &= -\zeta(2)\zeta(3) - \zeta(5) \\
 C_{1,4} &= \zeta(5) & C_{2,4} &= -\frac{25}{6}\zeta^3(2) + \frac{21}{2}\zeta(2)\zeta(4) - \frac{1}{2}\zeta^2(3) - \frac{163}{48}\zeta(6) \\
 C_{1,5} &= \zeta(6) = \frac{\pi^6}{945} & C_{2,5} &= -\frac{5}{2}\zeta^2(2)\zeta(3) - \zeta(2)\zeta(5) + \frac{21}{4}\zeta(3)\zeta(4) - 2\zeta(7) \\
 C_{3,1} &= 6\zeta^2(2) - \frac{53}{4}\zeta(4) = \frac{7}{360}\pi^4 \\
 C_{3,2} &= \zeta(2)\zeta(3) + \zeta(5) \\
 C_{3,3} &= \frac{25}{6}\zeta^3(2) - 12\zeta(2)\zeta(4) + \frac{1}{2}\zeta^2(3) + \frac{191}{24}\zeta(6) \\
 C_{3,4} &= \frac{17}{2}\zeta^2(2)\zeta(3) + 2\zeta(2)\zeta(5) - \frac{37}{2}\zeta(3)\zeta(4) + 3\zeta(7) \\
 C_{3,5} &= \frac{23}{8}\zeta^4(2) - \frac{17}{2}\zeta^2(2)\zeta(4) + \frac{1}{2}\zeta(2)\zeta^3(3) + \frac{531}{16}\zeta(2)\zeta(6) + 2\zeta(3)\zeta(5) - \frac{175}{8}\zeta^2(4) - \frac{939}{64}\zeta(8) \\
 C_{4,1} &= 0 \\
 C_{4,2} &= 3\zeta(2)\zeta(4) - \frac{73}{8}\zeta(6) = -\frac{31}{7560}\pi^6 \\
 C_{4,3} &= -12\zeta^2(2)\zeta(3) - 2\zeta(2)\zeta(5) + \frac{53}{2}\zeta(3)\zeta(4) - 2\zeta(7) \\
 C_{4,4} &= -2\zeta^4(2) + \frac{35}{4}\zeta^2(2)\zeta(4) - \zeta(2)\zeta^2(3) - \frac{205}{4}\zeta(2)\zeta(6) - 2\zeta(3)\zeta(5) + \frac{131}{4}\zeta^2(4) + \frac{1357}{64}\zeta(8) \\
 C_{4,5} &= -\frac{25}{3}\zeta^3(2)\zeta(3) - 23\zeta^2(2)\zeta(5) + \frac{45}{2}\zeta(2)\zeta(3)\zeta(4) - 5\zeta(2)\zeta(7) - \frac{1}{3}\zeta^3(3) - \frac{545}{48}\zeta(3)\zeta(6) + \frac{201}{4}\zeta(4)\zeta(5) - \frac{23}{3}\zeta(9) \\
 C_{5,1} &= -\frac{3}{2}\zeta(2)\zeta(4) + \frac{73}{16}\zeta(6) = \frac{31}{15120}\pi^6 \\
 C_{5,2} &= 6\zeta^2(2)\zeta(3) + \zeta(2)\zeta(5) - \frac{53}{4}\zeta(4) + \zeta(7) \\
 C_{5,3} &= \zeta^4(2) - \frac{35}{8}\zeta^2(2)\zeta(4) + \frac{1}{2}\zeta(2)\zeta^2(3) + \frac{755}{16}\zeta(2)\zeta(6) + \zeta(3)\zeta(5) - \frac{281}{8}\zeta^2(4) - \frac{1261}{64}\zeta(8) \\
 C_{5,4} &= \frac{25}{6}\zeta^3(2)\zeta(3) + \frac{53}{2}\zeta^2(2)\zeta(5) - 15\zeta(2)\zeta(3)\zeta(4) + 5\zeta(2)\zeta(7) + \frac{1}{6}\zeta^3(3) + \frac{205}{12}\zeta(3)\zeta(6) - \frac{233}{4}\zeta(4)\zeta(5) + \frac{19}{3}\zeta(9) \\
 C_{5,5} &= \frac{73}{120}\zeta^5(2) + \frac{109}{24}\zeta^3(2)\zeta(4) + \frac{41}{4}\zeta^2(2)\zeta^2(3) + \frac{4405}{96}\zeta^2(2)\zeta(6) + 4\zeta(2)\zeta(3)\zeta(5) - 62\zeta(2)\zeta^2(4) \\
 &\quad + \frac{1659}{32}\zeta(2)\zeta(8) - \frac{45}{2}\zeta^2(3)\zeta(4) + 5\zeta(3)\zeta(7) - \frac{3385}{96}\zeta(4)\zeta(6) + \frac{5}{2}\zeta^2(5) - \frac{22177}{640}\zeta(10)
 \end{aligned}$$

8. Some integrals involving $S_{n,p}(z)$. We now consider some integrals containing $S_{n,p}(z)$ in the integrand which were discussed by Nielsen. From the differentiation rule (2.11) for $S_{n,p}(x)$ we see immediately that

$$(8.1) \quad \int S_{n-1,p}(z) S_{n,p}(z) \frac{dz}{z} = \frac{1}{2} S_{n,p}^2(z), \quad (n \geq 2).$$

By repeated partial integration we obtain for $k < n-1$, (note the misprints in Nielsen's formulae [20, p. 190])

$$\begin{aligned}
 \int S_{n-1,p}(z) S_{n,p}(z) \frac{dz}{z} &= \sum_{j=1}^k (-1)^{j-1} S_{n-j,p}(z) S_{n+j,p}(z) \\
 &\quad + (-1)^k \int S_{n-k-1,p}(z) S_{n+k,p}(z) \frac{dz}{z}.
 \end{aligned}$$

Therefore

(8.2)

$$\begin{aligned} (-1)^k \int S_{n-k-1,p}(z) S_{n+k,p}(z) \frac{dz}{z} &= \frac{1}{2} S_{n,p}^2(z) + \sum_{j=1}^k (-1)^j S_{n-j,p}(z) S_{n+j,p}(z) \\ &= \frac{1}{2} (-1)^k \sum_{j=0}^{2k} (-1)^j S_{n-k+j,p}(z) S_{n+k-j,p}(z). \end{aligned}$$

Using the definitions (2.6) of $S_0(z)$ and $S_1(z)$, and rule (2.12), we obtain, by repeated partial integration, for $\alpha \neq 1$ and $|z| < 1$,

$$\begin{aligned} (8.3) \quad \int z^\alpha S_n(z) dz &= z^{\alpha+1} \sum_{j=0}^{n-1} (-1)^j \frac{S_{n-j}(z)}{(\alpha+1)^{j+1}} + \frac{(-1)^n}{(\alpha+1)^n} \int \frac{z^{\alpha+1}}{1-z} dz \\ &= z^{\alpha+1} \sum_{j=0}^{n-2} \frac{(-1)^j S_{n-j}(z)}{(\alpha+1)^{j+1}} + \frac{(-1)^n}{(\alpha+1)^n} (z^{\alpha+1} - 1) \log(1-z) \\ &\quad + \frac{(-1)^{n-1}}{(\alpha+1)^n} \int \frac{1-z^{\alpha+1}}{1-z} dz, \end{aligned}$$

and hence, using [8, No. 8.361 7], we obtain the definite integral

(8.4)

$$\int_0^1 x^{\alpha-1} S_n(x) dx = \sum_{j=0}^{n-2} \frac{(-1)^j S_{n-j}}{\alpha^{j+1}} + \frac{(-1)^{n-1}}{\alpha^n} [\psi(\alpha+1) + \gamma], \quad (\operatorname{Re} \alpha > -1),$$

where $\psi(x)$ is the logarithmic derivative of the gamma function and γ is Euler's constant. In particular, for $\alpha = m > 0$ an integer,

$$(8.5) \quad \int_0^1 x^{m-1} S_n(x) dx = \sum_{j=0}^{n-2} \frac{(-1)^j S_{n-j}}{m^{j+1}} + \frac{(-1)^{n-1}}{m^n} \sum_{k=1}^m \frac{1}{k}.$$

Lewin [17, p. 308] gives (8.4) and (8.5) for $n = 2$.

Equation (8.5), together with the power series expansion (2.5), allows us to compute

$$(8.6) \quad \phi(x) = \int_0^1 S_q(tx) S_n(t) \frac{dt}{t} = \sum_{m=1}^{\infty} \frac{x^m}{m^q} \int_0^1 t^{m-1} S_n(t) dt.$$

Using (8.5) and the series (2.5) and (2.9), we obtain

$$\begin{aligned} (8.7) \quad \phi(x) &= \sum_{j=0}^{n-2} (-1)^j S_{n-j} \sum_{m=1}^{\infty} \frac{x^m}{m^{q+j+1}} - (-1)^n \sum_{m=1}^{\infty} \left(\sum_{k=1}^m \frac{1}{k} \right) \frac{x^m}{m^{q+n}} \\ &= \sum_{j=0}^{n-2} (-1)^j S_{n-j} S_{q+j+1}(x) - (-1)^n (S_{n+q+1}(x) + S_{n+q-1,2}(x)). \end{aligned}$$

Some other integrals containing $S_n(x)$ can be found in [12] and [15].

9. Values of $S_{n,p}(z)$ for special arguments z .

9.1. The integrals $s_{n,p} = S_{n,p}(1)$. As has been shown in [13], the homogeneous polynomial in $\zeta(q)$ for the integrals

$$(9.1) \quad s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \log^{n-1} t \log^p(1-t) \frac{dt}{t}$$

can be written in the form

$$s_{n,p} = (-1)^{n+p-1} \sum_{\nu=0}^{n-1} b_{n-\nu-1} \sum_{\rho=0}^p \binom{\nu+\rho+1}{\rho} b_{p-\rho} a_{\nu+\rho+1}$$

where [8, No. 8.321]

$$a_k = -\frac{1}{k} \sum_{m=1}^k (-1)^m \zeta(m) a_{k-m}, \quad b_k = \frac{1}{k} \sum_{m=1}^k (-1)^m \zeta(m) b_{k-m},$$

are the coefficients in the power series for $1/\Gamma(1+x)$ and $\Gamma(1+x)$ respectively, and where $a_0 = b_0 = 1$ and $\zeta(1) \equiv \gamma$ by definition. In order to derive this expression, the well-known relation [8, No. 9.122 1]

$$(9.2) \quad {}_2F_1(\alpha, \beta; \beta+1; 1) = \frac{\Gamma(1-\alpha)\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}$$

is essential.

9.2. The integrals $\sigma_{n,p} = (-1)^n S_{n,p}(-1)$. The problem of evaluating

$$(9.3) \quad \sigma_{n,p} = \frac{(-1)^{n-1}}{(n-1)!p!} \int_0^1 \log^{n-1} t \log^p(1+t) \frac{dt}{t}$$

seems to be far more difficult. This difficulty is related to the apparent nonexistence of a relation for ${}_2F_1(\alpha, \beta; \beta+1; -1)$ similar to (9.2). Nielsen claimed that he had succeeded in representing $\sigma_{n,p}$ by $\zeta(q)$ if $n+p$ is odd, but, as we shall see, his proof is erroneous. Those relations between $\sigma_{n,p}$ and $s_{n,p}$ which can be deduced from relations given in the previous sections are of such a special nature that it is not easy to see how they could be used, for example, to derive an expression for $\sigma_{n,p}$ in terms of $\zeta(k)$. On the contrary, it seems likely that, in the general case, no such relations exist. (Barlow [4] states, without giving a reference, that " $S_{n,p}(-1)$, $S_{n,p}(\frac{1}{2})$ and $S_{n,p}(1)$ are easily evaluated analytically from equation [(2.1)]".)

Nielsen did, however, find a closed formula for the case $p=2$, n odd, which seems to be the most general result known for $\sigma_{n,p}$. He proved the following

THEOREM 2. *Let $\zeta(k)$ be the Riemann zeta function for integer argument. Then for $n \geq 0$*

$$(9.4) \quad \begin{aligned} \sigma_{2n+1,2} &= \frac{1}{2(2n)!} \int_0^1 \log^{2n} t \log^2(1+t) \frac{dt}{t} \\ &= \frac{1}{2} [1 - (2n+1)(1-2^{-2n-2})] \zeta(2n+3) \\ &\quad + \sum_{j=1}^n (1-2^{1-2j}) \zeta(2j) \zeta(2n-2j+3). \end{aligned}$$

The sum is zero if $n=0$.

Proof. Nielsen's proof, which is full of misprints, starts from the inversion formula (5.12) for $z=1$ and $p=2$. After applying Lemma (4.1a), he obtains an expression for $C_{2n-1,2}$. He then takes the real part and compares his result with the power series expansion of both sides of the relation

$$\pi \csc \pi x = \pi \cot \pi x \cos \pi x + \pi \sin \pi x.$$

Theorem 1 allows us to shorten this proof. From (5.9), we have

$$\sigma_{2n-1,2} = \frac{1}{2} [C_{2n-1,2} - (2n-1)\sigma_{2n+1}].$$

Hence, for $n=1$, using (7.1) and (2.22),

$$\sigma_{1,2} = \frac{1}{2} (C_{1,2} - \sigma_3) = \frac{1}{2} (s_3 - \sigma_3) = \frac{1}{8} \zeta(3).$$

For $n \geq 2$, we obtain, using (7.2),

$$(9.5) \quad \sigma_{2n-1,2} = \frac{1}{2} \left\{ 2 \sum_{j=0}^{n-1} (-1)^j \frac{\pi^{2j}}{(2j)!} s_{2n-2j-1,2} \right. \\ \left. - (2n-1) \sum_{j=0}^{n-1} (-1)^j \frac{\pi^{2j}}{(2j)!} s_{2n-2j+1} - (2n-1)\sigma_{2n+1} \right\}.$$

With the help of [13], [20],

$$(9.6) \quad s_{m,2} = \frac{1}{2} \left[(m+1)s_{m+2} - \sum_{k=2}^m s_k s_{m-k+2} \right], \quad (m \geq 2) \\ s_{1,2} = s_3,$$

(9.5) can be written as

$$\sigma_{2n-1,2} = \frac{1}{2} \left\{ - \sum_{j=0}^{n-1} (-1)^j \frac{\pi^{2j}}{2j(2j-2)!} s_{2n-2j+1} \right. \\ \left. - \sum_{j=0}^{n-2} (-1)^j \frac{\pi^{2j}}{(2j)!} \sum_{k=2}^{2n-2j-1} s_k s_{2n-2j-k+1} + s_{2n+1} - (2n-1)\sigma_{2n+1} \right\}.$$

Noting the symmetry in the double sum, and reordering, gives

$$\sigma_{2n-1,2} = \frac{1}{2} \left\{ \sum_{j=1}^{n-1} \left[(-1)^{j-1} \frac{\pi^{2j}}{2j(2j-2)!} - 2 \sum_{k=0}^{j-1} (-1)^k \frac{\pi^{2k}}{(2k)!} s_{2j-2k} \right] \right. \\ \left. \times s_{2n-2j+1} + s_{2n+1} - (2n-1)\sigma_{2n+1} \right\}.$$

Referring to (7.2), we see that the expression in square brackets equals $C_{2j-1,1} = 2\sigma_{2j}$, hence

$$(9.7) \quad \sigma_{2n-1,2} = \sum_{j=1}^{n-1} \sigma_{2j} s_{2n-2j+1} + \frac{1}{2} [s_{2n+1} - (2n-1)\sigma_{2n+1}],$$

and Theorem 2 follows on replacing n by $n+1$ and using (2.22).

Formula (9.4) shows that $\sigma_{2n+1,2}$ can be expressed by a homogeneous polynomial of degree $2n+3$ in $\zeta(q)$ ($2 \leq q \leq 2n+3$), with rational coefficients. In contrast to the expressions for $s_{n,p}$ and $C_{n,p}$, however, all coefficients of terms which contain more than two factors vanish. A short table of $\sigma_{2n+1,2}$ is given in Table 2.

TABLE 2

Let

$$I_n = \sigma_{2n+1,2} = \frac{1}{2(2n)!} \int_0^1 \log^{2n} t \log^2(1+t) \frac{dt}{t}.$$

Then

$$\begin{aligned} I_0 &= \frac{1}{8} \zeta(3) \\ I_1 &= \frac{1}{2} \zeta(2) \zeta(3) - \frac{29}{32} \zeta(5) \\ I_2 &= \frac{1}{2} \zeta(2) \zeta(5) + \frac{7}{8} \zeta(3) \zeta(4) - \frac{251}{128} \zeta(7) \\ I_3 &= \frac{1}{2} \zeta(2) \zeta(7) + \frac{31}{32} \zeta(3) \zeta(6) + \frac{7}{8} \zeta(4) \zeta(5) - \frac{1529}{512} \zeta(9) \\ I_4 &= \frac{1}{2} \zeta(2) \zeta(9) + \frac{127}{128} \zeta(3) \zeta(8) + \frac{7}{8} \zeta(4) \zeta(7) + \frac{31}{32} \zeta(5) \zeta(6) - \frac{8183}{2048} \zeta(11) \\ I_5 &= \frac{1}{2} \zeta(2) \zeta(11) + \frac{511}{512} \zeta(3) \zeta(10) + \frac{7}{8} \zeta(4) \zeta(9) + \frac{127}{128} \zeta(5) \zeta(8) + \frac{31}{32} \zeta(6) \zeta(7) - \frac{40949}{8192} \zeta(13) \end{aligned}$$

For a discussion of Nielsen's work on $\sigma_{n,p}$ for other values of n and p , it is perhaps useful to quote his monograph [20, pp. 199–202, translated from the German]. He starts with formula (2.22) for σ_n and continues that

a similar result is not available for the more general numbers $\sigma_{n,p} \cdots$, if $p > 1$. On the contrary, it seems that these numbers cannot, in general, be expressed in closed form by the numbers s_k . In any case, I have found such an expression only when $n+p$ is odd.

After having proved Theorem 2, he continues

On the values $\sigma_{n,p}$, we still have to prove that, for $n \geq 2$,

$$(N1) \quad \sigma_{p,n} = (-1)^{p-1} \binom{n+p-1}{n} \sigma_{1,n+p-1} + \sum_{q=0}^{p-2} (-1)^q \binom{n+q}{n} s_{n+q+1,p-q-1}.$$

For this purpose, we set $x=1$ in $(^2)$ and obtain

$$(N2) \quad s_{n,p} = \sum_{q=0}^p \binom{n+p-q-1}{n-1} \sigma_{q+1,n+p-q-1},$$

which easily gives formula (N1) by induction. Setting $n=2$ and $p=2n-1$ in (N1) yields

$$(N3) \quad \sigma_{2n-1,2} = \binom{2n+1}{2} \sigma_{1,2n} + \sum_{q=0}^{2n-3} (-1)^q \binom{q+2}{q} s_{q+3,2n-q-2}.$$

Therefore, because of [(9.4)] and (N1), we can deduce the following theorem:

For $n+p$ odd, the value $\sigma_{n,p}$ is a homogeneous polynomial of degree $n+p$ in $s_2, s_3, \dots, s_{n+p}$ with rational coefficients, provided s_r is of degree r .

In order to find the numbers $\sigma_{n,p}$ also for $n+p$ even, one needs only to express the numbers $\sigma_{1,2n-1}$ by s_r , in which I did not succeed.

A numerical check for small n and p shows, however, that (N2) cannot be correct as it stands. In addition, formula (N2), which represents, for given $n+p$, a system of $n+p-1$ linear equations in the $n+p-1$ unknowns $\sigma_{v,p}$, is not symmetric with respect

² There is an obvious misprint in the equation number referenced.

to n and p , as it should be according to (2.21). Further, for $n=1$ it contains a quantity $\sigma_{n+p,0}$ which is undefined. However, it is easy to see the origin of (N2) in spite of the misprint in the equation number referenced. It lies in Nielsen's incorrect equation corresponding to (6.6), which gives, for $a=0$, $b=1$, $z=1$, the relation (N2) as a special case.

A correct relation between $s_{n,p}$ and $\sigma_{v,\rho}$ is given by the following

THEOREM 3. *The numbers $s_{n,p} = S_{n,p}(1)$ and $\sigma_{v,\rho} = (-1)^\rho S_{v,\rho}(-1)$ are linearly related through*

$$(9.8) \quad s_{n,p} = \sum_{j=1}^n \binom{n+p-j-1}{p-1} \sigma_{j,n+p-j} + \sum_{j=1}^p \binom{n+p-j-1}{n-1} \sigma_{j,n+p-j}.$$

Proof. From (5.1), we obtain for $z = \frac{1}{2}$

$$L_{n,p}\left(\frac{1}{2}\right) + L_{p,n}\left(\frac{1}{2}\right) = s_{n,p} - \frac{\log^{n+p} 2}{n!p!}$$

and from (6.2) for $z=1$, using (2.21),

$$\begin{aligned} L_{n,p}\left(\frac{1}{2}\right) + L_{p,n}\left(\frac{1}{2}\right) &= 2s_{n,p} - \left[\binom{n+p-1}{n-1} + \binom{n+p-1}{p-1} \right] \frac{\log^{n+p} 2}{(n+p)!} \\ &\quad - \sum_{j=1}^n \binom{n+p-j-1}{p-1} \sigma_{j,n+p-j} - \sum_{j=1}^p \binom{n+p-j-1}{n-1} \sigma_{j,n+p-j}. \end{aligned}$$

Theorem 3 follows by comparison.

Note that (9.8) is symmetric in n and p . This relation for $s_{n,p}$ was given in [12] without proof. Note also that setting $z=1$ in formula (6.7), derived from the correct equation (6.6) merely yields an equation equivalent to equation (9.8), as can be seen from elementary properties of binomial coefficients.

It seems puzzling that Nielsen should have published such an obviously incorrect equation as (N2), when the correct relation follows easily from his relations for $L_{n,p}(z)$.³ However, by introducing (N1) into (N2) and performing some manipulations with binomial coefficients, it can be shown that (N1) is indeed the solution of the linear system (N2) for $\sigma_{v,\rho}$. This suggests that the more special formula (N3) for $\sigma_{1,2n}$ is also incorrect, a fact which is easily verified by a numerical check using small values of n , after correcting an obvious misprint in (N3).

By comparing (N2) and (9.8), we see that Nielsen was misled by the (wrong) triangular structure of (N2), which enabled him to solve his system in closed form. On the other hand, the correct system (9.8) does not allow a solution $\sigma_{n,p}$ in terms of $s_{v,\rho}$ (and therefore of $\zeta(q)$). For $n+p \geq 4$, the system (9.8) consists of $[(n+p)/2]$ equations with either $n+p-2$ ($n+p$ even) or $n+p-3$ ($n+p$ odd) unknowns, so that for $n+p \geq 6$ there are more unknowns than equations. The cases $n+p=4$ and $n+p=5$, where one has as many equations as unknowns, namely two, result in a vanishing determinant, yielding the single equations

$$(9.9) \quad 2\sigma_{1,3} + \sigma_{2,2} = \frac{\pi^4}{720}, \quad 2\sigma_{1,4} + \sigma_{2,3} = \frac{31}{32}\zeta(5) - \frac{1}{2}\zeta(2)\zeta(3).$$

The cases $n+p=2$ and $n+p=3$ are trivial.

³ One could imagine that Nielsen, or somebody else at the time might have recognized this error later. However, a search through the volumes of *Nova Acta Leopoldina* and *Oversigt Danske Vidensk. Selsk. Forh.* up to 1918 shows no published erratum or remark relating to Nielsen's paper. As far as the author knows, no collected papers of Nielsen exist (cf. the list of known collected papers in Struik [22, p. 196]).

When examining other relations of §§5 and 6, one realizes that there seems to be an inherent difficulty in getting relations which could be used to compute $\sigma_{n,p}$ in terms of $s_{v,\rho}$. This is true in particular for the (singular) system (5.9) of linear equations for the unknowns $\sigma_{v,\rho}$. Therefore, it seems likely that, not only are Nielsen's relations (N1) and (N3) incorrect, but that no such relation exists. An unsuccessful computer search through relations of the form

$$\left\{ \begin{array}{l} |\sigma_{1,4}| \\ |\sigma_{2,3}| \end{array} \right\} = \frac{p_1}{p_2} \zeta(2) \zeta(3) + \frac{p_3}{p_4} \zeta(5),$$

with integers p_i in the range $-50 \leq p_1 \leq 50$, $1 \leq p_2 \leq 50$, $0 \leq p_3 \leq 50$, $1 \leq p_4 \leq 50$ supports this conclusion.

9.3. The integrals $a_{n,p} = S_{n,p}(\frac{1}{2})$. The problem of evaluating

$$(9.10) \quad a_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \log^{n-1} t \log^p \left(1 - \frac{1}{2}t\right) \frac{dt}{t}$$

apparently faces obstacles similar to those found for the evaluation of $\sigma_{n,p}$; in particular, because no closed expression seems to be known for ${}_2F_1(\alpha, \beta; \beta+1; \frac{1}{2})$. A result which can easily be obtained from formulae treated in the previous sections is the following

THEOREM 4. For $n \geq 1$ and $p \geq 1$, the numbers $a_{n,p}$ can be expressed in terms of $\sigma_{v,\rho}$ and $\log 2$, through the relation

$$(9.11) \quad a_{n,p} = \sum_{j=0}^{n-1} (-1)^j \frac{\log^j 2}{j!} \sum_{k=1}^{n-j} \binom{n+p-j-k-1}{p-1} \sigma_{k,n+p-j-k} + (-1)^n \frac{\log^{n+p} 2}{(n+p)!}.$$

Proof. From (6.2) with $z = 1$ and (2.16) with $z = \frac{1}{2}$ it follows that

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{\log^j 2}{j!} a_{n-j,p} &= s_{n,p} - \frac{1}{(n+p)!} \binom{n+p-1}{n-1} \log^{n+p} 2 \\ &\quad - \sum_{k=1}^{\infty} \binom{n+p-k-1}{n-1} \sigma_{k,n+p-k}. \end{aligned}$$

Applying Lemma (4.1a) and using (5.11) yields

$$(9.12) \quad a_{n,p} = \sum_{j=0}^{n-1} \frac{(-1)^j \log^j 2}{j!} \left\{ s_{n-j,p} - \sum_{k=1}^p \binom{n+p-j-k-1}{n-j-1} \sigma_{k,n+p-j-k} \right\} + (-1)^n \frac{\log^{n+p} 2}{(n+p)!}.$$

Replacing $s_{n-j,p}$ by (9.8) gives Theorem 4.

As a special case, we obtain for $p = 1$,

$$(9.13) \quad a_n = \sum_{j=0}^{n-2} \frac{(-1)^j \log^j 2}{j!} \sum_{k=1}^{n-j-1} \sigma_{k,n-j-k} - (-1)^n \frac{\log^n 2}{n!}, \quad (n \geq 2).$$

In particular, for $n=2$,

$$(9.14) \quad a_2 = - \int_0^1 \log\left(1 - \frac{1}{2}t\right) \frac{dt}{t} = \sigma_{1,1} - \frac{1}{2} \log^2 2 = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2,$$

which is a well-known formula due to Euler (attributed to Landen by Lewin [17, p. 6]); and for $n=3$,

$$(9.15) \quad a_3 = \int_0^1 \log t \log\left(1 - \frac{1}{2}t\right) \frac{dt}{t} = \frac{7}{8} \zeta(3) - \frac{\pi^2}{12} \log 2 + \frac{1}{6} \log^3 2.$$

For $n=4$, we have, using (9.9),

$$(9.16) \quad a_4 = \frac{\pi^4}{90} + \frac{\pi^2}{24} \log^2 2 - \frac{7}{8} \zeta(3) \log 2 - \frac{1}{24} \log^4 2 - \sigma_{1,3}.$$

The appearance of the quantity $\sigma_{1,3}$, for which no closed formula seem to be known, makes it impossible to compute

$$a_4 = - \frac{1}{2} \int_0^1 \log^2 t \log\left(1 - \frac{1}{2}t\right) \frac{dt}{t}$$

from (9.13).

From the power series (2.5) we have

$$(9.17) \quad a_n = \sum_{j=1}^{\infty} \frac{1}{2^j j^n}.$$

Therefore

$$(9.18) \quad a_2 = \sum_{j=1}^{\infty} \frac{1}{2^j j^2} = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2$$

$$(9.19) \quad a_3 = \sum_{j=1}^{\infty} \frac{1}{2^j j^3} = \frac{7}{8} \zeta(3) - \frac{\pi^3}{12} \log 2 + \frac{1}{6} \log^3 2$$

and, trivially, $a_0 = 1$, $a_1 = \log 2$. Formula (9.19) had been found by Legendre.

It seems puzzling that (9.17) should exist in closed form only for $n=0, 1, 2, 3$. Knowing a_4 for example would allow us to obtain, from (9.16) and (9.9), expressions for $\sigma_{1,3}$ and $\sigma_{2,2}$. This in turn, together with a knowledge of a_5 , would make possible a computation of $\sigma_{1,4}$ and $\sigma_{2,3}$ from (9.12), (9.4) and (9.9). Levine et al. [15] and Gastmans and Troost [7] therefore consider a_4 and a_5 as “new” constants. This makes it possible to express more kinds of logarithmic integrals in “closed” form.

Another special case is $n=1$. Here we find from (9.11) that

$$(9.20) \quad a_{1,p} = \frac{(-1)^p}{p!} \int_0^1 \log^p\left(1 - \frac{1}{2}t\right) \frac{dt}{t} = \sigma_{1,p} - \frac{\log^{p+1} 2}{(p+1)!}.$$

Note that Nielsen's remark [20, p. 203] that $a_{1,2p}$ can be expressed in terms of $\log 2$ and $\zeta(q)$ ($2 \leq q \leq 2p+1$), is probably incorrect, being a consequence of the incorrect formula (N3).

From (2.7), (9.4) and (9.20), one finds

$$\begin{aligned} a_{1,2} &= \sum_{j=0}^{\infty} \frac{(-1)^j S_{j+2}^{(2)}}{(j+2)!(j+2)} \frac{1}{2^{j+2}} \\ &= \sum_{j=2}^{\infty} \frac{1}{2^j j^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{j-1} \right) = \frac{1}{8} \zeta(3) - \frac{1}{6} \log^3 2 \end{aligned}$$

or

$$(9.21) \quad \int_0^1 \log^2 \left(1 - \frac{1}{2} t \right) \frac{dt}{t} = \frac{1}{4} \zeta(3) - \frac{1}{3} \log^3 2.$$

Nielsen also gave two relations between $a_{1,n}$ and $a_{n,1}$. From (5.1) and (2.16), we obtain for $z = \frac{1}{2}$,

$$(9.22) \quad a_{1,n} = s_{n+1} - \sum_{j=0}^{n-1} \frac{\log^j 2}{j!} a_{n-j+1} - \frac{1}{n!} \log^{n+1} 2.$$

With the help of Lemma (4.1a) and a well-known relation for the binomial coefficients, (9.22) can be inverted to give

$$(9.23) \quad a_n = \sum_{j=0}^{n-2} \frac{(-1)^j \log^j 2}{j!} (s_{n-j} - a_{1,n-j-1}) - (-1)^n \frac{\log^n 2}{(n-1)!}.$$

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