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SCHWINGER-DYSON EQUATIONS IN REDUCED
TWISTED CHIRAL MODELS

B. Aneva, Y. Brihaye and P. Rossi
CERN -- Geneva

ABSTRACT

We have shown that the Schwinger-Dyson equations for the original (infinite volume) chiral model on the lattice and the reduced, twisted chiral model are equivalent in the large N limit. Our derivation is valid in any even dimension of space-time and does not depend on the coupling constant. Only a discrete symmetry, which is not broken, is invoked to establish the correspondence.

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After the important discovery of Eguchi and Kawai¹⁾, many investigations have been developed concerning the reduced version of large N matrix models. The hope is that the study of these new theories on a single space-time point (reduction) will provide, in the large N limit, examples of a soluble theory for $d > 2$. More generally, the development of both analytical and numerical investigations could improve our knowledge of non-perturbative phenomena in strong interaction physics.

A general reduction scheme for matrix models has been proposed by Parisi²⁾ and the formalism has been further developed by Gross and Kitazawa³⁾ who established a formal correspondence on a non-perturbative level between reduced and non-reduced gauge models on the lattice.

Chiral models have also been considered in their reduced version. Quenched chiral models have been studied earlier by Bars et al.⁴⁾ who obtained an approximate solution for the partition function and by Green⁵⁾ who established (under certain conditions) a non-perturbative proof of the equivalence between chiral quenched and chiral models on the lattice using the Schwinger-Dyson (SD) equation. On the other hand, the twisted chiral model on the lattice has also been studied by Eguchi and Nakayama⁶⁾. This model offers many advantages with respect to the quenched one: it is simpler and more suitable for numerical computations. In Refs. 6) and 7), it has been shown that the model reproduces the correct planar diagrams of the standard chiral model on the lattice.

To our knowledge, the equivalence between the reduced twisted and the original chiral model on the lattice has not yet been proven on a non-perturbative level. Therefore, it is the purpose of this paper to derive the SD equations for both models and to establish a one-to-one correspondence between the two sets of equations.

The chiral model on the lattice is described by the following partition function and action

$$Z(\beta) = \int \prod_{x \in L} dU_x \exp - S(\beta) \quad (1)$$

$$S(\beta) = -\beta N \sum_{x \in L} \sum_{\mu=1}^d \text{tr} (U_x U_{x+\mu}^\dagger + U_{x+\mu} U_x^\dagger) \quad (2)$$

where U_x denotes the $U(N)$ variable attached to the lattice site x and dU_x is the Haar measure on $U(N)$.

The most general Green function of this theory has the form:

$$\Gamma^{(n)} \{x_i, \kappa_i\} = \langle \text{tr} \left(\prod_{i=1}^n U_{x_i}^{\kappa_i} \right) \rangle, \quad (3)$$

where the expectation value is computed with the measure (1). The Schwinger-Dyson equations of the theory, established in detail in Ref. 5) relate several objects of the type (3). For instance, (for $\kappa_1 > 0$), they are of the form:

$$\begin{aligned} & \mathcal{N} \langle \text{tr} \prod_{i=1}^n U_{x_i}^{\kappa_i} \rangle + \beta \mathcal{N} \sum_{\mu=1}^d \langle \text{tr} U_{x_1} U_{x_1+\mu}^+ \prod_{i=1}^n U_{x_i}^{\kappa_i} \rangle \\ & - \beta \mathcal{N} \sum_{\mu=1}^d \langle \text{tr} U_{x_1+\mu} U_{x_1}^{\kappa_1-1} \prod_{i=2}^n U_{x_i}^{\kappa_i} \rangle + \langle \text{source terms} \rangle = 0 \end{aligned} \quad (4)$$

The source terms arise if $\kappa_1 > 1$ or if x_1 appears more than once in the product. If, for example, $x_\ell = x_1$ the corresponding source term is:

$$\left\langle \sum_{m=\frac{1-\varepsilon}{2}}^{\kappa_1 - \frac{1+\varepsilon}{2}} \text{tr} \left(\prod_{i=1}^{\ell-1} U_{x_i}^{\kappa_i} U_{x_1}^{\varepsilon m} \right) \text{tr} \left(U_{x_1}^{\varepsilon(\kappa_\ell - m)} \prod_{i=\ell+1}^n U_{x_i}^{\kappa_i} \right) \right\rangle, \quad \varepsilon = 5g\eta(\kappa_\ell) \quad (5)$$

Using the factorization properties of the trace in the large N limit, one can readily see that Eq. (4) relates only objects of the form (3).

The reduced, twisted chiral model is defined by the partition function

$$Z_R(\beta) = \int dU \exp - S(\beta) \quad (6)$$

$$- S(\beta) = \beta \mathcal{N} \sum_{\mu=1}^d \text{tr} (U \gamma_\mu U^\dagger \gamma_\mu^\dagger + U \gamma_\mu^\dagger U^\dagger \gamma_\mu) \quad (7)$$

where the γ_μ 's are twisting matrices. We consider here the case when these matrices obey⁷⁾

$$\gamma_\mu \gamma_\nu = \exp \frac{2i\pi}{N^{2/d}} \gamma_\nu \gamma_\mu \quad (8)$$

Another choice of the commutation relations [classified in Ref. 7)] is, of course, possible and will not affect the result. In the present case, the model is valid only for N 's such that $N^{2/d}$ is an integer, thus providing a particular way to approach the large N limit.

Thanks to the twisting matrices one is able to obtain a one-to-one correspondence between the Green functions of the reduced and non-reduced chiral models. Indeed if we identify

$$U_{x+\mu} = \gamma_\mu U_x \gamma_\mu^\dagger \quad (9)$$

we have as a consequence

$$U_x = V(x) U_0 V^\dagger(x) \quad (10)$$

where

$$V(x) = \prod_\mu \gamma_\mu^{x_\mu} \quad (11)$$

and the relation (8) guarantees Eq. (10) to be uniquely defined (i.e., path independently). Then, given a Green function (3), we can associate it to a reduced one using Eq. (10)

$$\Gamma_R^{(n)} \{x_i, \kappa_i\} = \langle \text{tr } V(x_1) U^{\kappa_1} V^\dagger(x_1) \dots V(x_n) U^{\kappa_n} V^\dagger(x_n) \rangle \quad (12)$$

which can be written as

$$\begin{aligned} \Gamma_R^{(n)} \{x_i, \kappa_i\} &= \langle \text{tr } U^{\kappa_1} \Gamma_1 U^{\kappa_2} \Gamma_2 \dots U^{\kappa_n} \Gamma_n \rangle \\ &\equiv \langle \text{tr } G(\kappa_i) \rangle \end{aligned} \quad (13)$$

with the condition

$$\prod_i \Gamma_i = 1, \quad \Gamma_i = V^\dagger(x_i) V(x_{i+1}) \quad (14)$$

and the expectation value is taken with respect to Eq. (6). While the invariance of the action (2) is known to be $U(N) \times U(N)/U(1)$, the reduced analogue [Eq. (7)] admits two kinds of invariance:

$$(i) \quad U \rightarrow e^{i\theta} U \quad [U(1) \text{ invariance}] \quad (15)$$

$$(ii) \quad U \rightarrow V(a) U V^\dagger(a) \quad \text{with} \quad a = (a_1, \dots, a_d), \quad a_i \in \mathbb{Z}_{N^{1/d}} \quad (16)$$

The first symmetry is a property of the original continuum model as well as of the reduced one. The second symmetry is discrete. Furthermore, anyone of the saddle points of the action (7) (i.e., the set $\{U = e^{i\omega} \mathbb{1}, \omega \in [0, 2\pi]\}$) is separately left invariant by all the transformations of the type (ii). Then, it can be argued⁸⁾ that this symmetry is not spontaneously broken. Moreover,

the symmetry (16) can be considered as the invariance of the model under translations in the original lattice which is actually the statement that the reduced model does not depend on the initial point U_0 in Eq. (10).

The following relation is easily verified using the definition (8) of the γ matrices:

$$V(a) V(x) V^+(a) = \exp \left\{ \frac{2i\tilde{\eta}}{N^{2/d}} \sum_{i \neq j} x_i a_j \right\} V(x) \quad (17)$$

As a consequence, the Green functions (12) are invariant under any transformation of the type (16). On the other hand, the Green functions (12) are invariant under the $U(1)$ transformation (15) only if $\sum_i k_i = 0$. If it is not the case, then the Green function is identically zero as long as the $U(1)$ symmetry is not spontaneously broken. This possibility is emphasized in detail in Green's paper⁵⁾. In what follows we shall use only the discrete symmetry in order to establish a one-to-one correspondence between the SD equations (4) and (5) and the one's in the reduced model.

The SD equations of the reduced twisted chiral model^{6),7)} are obtained in the standard way:

- (i) first one replaces U^{K_1} in G [see Eq. (13)] by $\lambda^a U^{K_1}$, where λ^a is an $U(N)$ generator,
- (ii) performs in the integral (12) the change of variables $U \rightarrow (1 + i\alpha\lambda^b)U$, using the invariance of the Haar measure,
- (iii) then contracting the indices a, b and extracting only the linear part in the parameter α , one obtains the set of equations:

$$N \langle \text{tr } G \rangle + \beta N \sum_{\mu=1}^d \left\{ \langle \text{tr } U \delta_\mu U^\dagger \delta_\mu^\dagger G \rangle - \langle \text{tr } \delta_\mu^\dagger U \delta_\mu U^\dagger G \rangle \right\} \quad (18)$$

$$+ \langle \text{source terms} \rangle = 0$$

The source terms can be written as:

$$\sum_{i=1}^{K_1} \langle \text{tr } U^i \text{tr } U^{\dagger i} G \rangle + \sum_{\kappa_\ell \neq K_1} \sum_{i=\frac{1-\varepsilon}{2}}^{\kappa_\ell - (\frac{1+\varepsilon}{2})} \langle \text{tr } (U^{\kappa_1} \Gamma_1 U^{\kappa_2} \Gamma_2 \dots \Gamma_{\ell-1} U^{\varepsilon i}) (\text{tr } U^{\varepsilon(i\kappa_\ell - i)} \Gamma_\ell \dots \Gamma_n) \rangle \quad (19)$$

with $\varepsilon = \text{sign}(\kappa_\ell)$.

The correspondence of the "non-source" terms in Eqs. (18) and (4) is straightforward. In order to establish the full correspondence, we have to identify which of the source terms in Eq. (19) are not contained among the source terms given in Eq. (5), and to prove that these additional source terms (arising due to the reduction scheme) are identically zero in the large N limit. It is generally assumed that factorization in the large N limit is a property of the traces in Eq. (19), so that it is sufficient to deal with objects like:

$$\langle \text{tr } U^{k_1} \Gamma_1 U^{k_2} \Gamma_2 \dots U^{k_\ell} \Gamma_\ell \rangle \quad (20)$$

The net effect of performing an arbitrary discrete transformation $V(a)$ in the integral (20) is a multiplication by a phase which after using Eq. (17) appears to be:

$$\begin{aligned} \exp \frac{2i\gamma}{N^{2/d}} \sum_{k \neq j} p_k (x_2 - x_1 + x_3 - x_2 + \dots + x_{\ell+1} - x_\ell)_j \\ = \exp \frac{2i\gamma}{N^{2/d}} \sum_{k \neq j} p_k (-x_1 + x_{\ell+1})_j \end{aligned} \quad (21)$$

So, if $x_1 \neq x_{\ell+1}$ the phase (21) depends on p and the integral (20) vanishes due to the discrete invariance of the measure density. On the other hand, if $x_1 = x_{\ell+1}$, nothing can be said about the expectation value (20), but then, a source term of analogous structure appears in the SD equation of the chiral model, as stressed in Ref. 5). Thus, the correspondence between the source terms in the non-reduced and reduced chiral models is achieved.

We want to point out once again, that we have only used a discrete symmetry of the model, which leaves the vacuum invariant, to establish the equivalence of both sets of SD equations. The present situation is different from what happens in the Eguchi-Kawai reduced model of $SU(N)$ gauge theory when a Z_N^d symmetry was invoked to establish the equivalence of the SD equations. However, the latter symmetry does not leave a given saddle point configuration invariant and it was shown to be spontaneously broken⁹⁾.

If the $U(1)$ symmetry is invoked in our case and if it is assumed not to be spontaneously broken, then, only the Green functions with $\sum_{i=1}^d k_i = 0$ will survive in the SD equation and the correspondence between the source terms (5) and (19) still works. This assumption, however, corresponds to the requirement that the $U(1)$ symmetry of the infinite lattice model is unbroken - we do not expect this to be true a priori when $d > 2$.

In conclusion, we stress that it is not necessary to invoke any continuous symmetry of the models (2) and (7) to establish the equivalence between the corresponding SD equations. However, like Green, we are unable to say whether the SD equations completely define the theory. The uniqueness of the solution can be argued to be proven for $d \leq 2$, but it is an open problem for larger dimensions, where it is related to the breakdown of the $U(1)$ symmetry.

REFERENCES

- 1) T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063.
- 2) G. Parisi, Phys. Lett. 112B (1982) 463.
- 3) D. Gross and Y. Kitazawa, Princeton Univ. preprint (1982).
- 4) I. Bars, M. Günaydin and S. Yankielowicz, CERN preprint TH.3379 (1982).
- 5) F. Green, NUB preprint N2584 (1982).
- 6) T. Eguchi and R. Nakayama, Phys. Lett. 122B (1983) 59.
- 7) Y. Brihaye and P. Rossi, CERN preprint TH.3590 (1983).
- 8) H. Georgi and A. Pais, Phys. Rev. D10 (1974) 1246.
- 9) G. Bhanot, U. Heller and H. Neuberger, Phys. Lett. 113B (1982) 47.