Appendix 1
Variational Derivation of Differential Equations

Let us recall the definition of a critical point for a function \( f \) belonging to the class \( C^2 ((a, b) \subset \mathbb{R}; \mathbb{R}) \). Critical points are furnished by the roots of the equation \( f'(x) := \frac{df(x)}{dx} = 0 \) in the interval \((a, b)\). Local (or relative) extrema are usually given by a critical point \( x_0 \) such that \( f''(x_0) \neq 0 \). In case \( f'(x_0) = 0 \) and \( f''(x_0) > 0 \), the function \( f \) has a local (or relative) minimum. Similarly, for a critical point \( x_0 \) with \( f''(x_0) < 0 \), \( f \) has a local (or relative) maximum. In case a critical point is neither a maximum nor a minimum, it is called a stationary point (or a point of inflection). We will now provide some simple examples. (1) Consider \( f(x) := 1 \) for \( x \in (-1, 1) \). Every point in the interval \((-1, 1)\) is a critical point. (2) Consider \( f(x) := x \) in the open interval \((-2, 2)\). There exist no critical points in the open interval. However, if we extend the function into the closed interval \([-2, 2]\), there exists an extremum at the end points (which are not critical points). (3) The function \( f(x) := (x)^3 \), for \( x \in \mathbb{R} \), has a stationary point (or a point of inflection) at the origin. (4) The periodic function \( f(x) := \sin x \), \( x \in \mathbb{R} \), has denumerably infinite number of local extrema, but no stationary points. (See, e.g., [32].)

Now, we generalize these concepts to a function of \( N \) real variables \( x \equiv (x^1, \ldots, x^N) \). We assume that \( f \in C^2 (D \subset \mathbb{R}^N; \mathbb{R}) \). A local minimum \( x_0 \) is defined by the property \( f(x) \geq f(x_0) \) for all \( x \) in the neighborhood \( N_{\delta}(x_0) \). Similarly, a local maximum is defined by the property \( f(x) \leq f(x_0) \) for all \( x \) in the neighborhood. Consider \( f \) as a 0-form (see Sect. 1.2). The critical points are provided by the roots of the 1-form equation:

\[
\begin{align*}
d f(x) &= \tilde{\theta}(x), \\
\partial_i f &= 0, \quad i \in \{1, 2, \ldots, N\}. 
\end{align*}
\]  
(A1.1)

A local (strong) minimum at \( x_0 \) is implied by the inequality

\[
\begin{align*}
\sum_{i=1}^{N} \sum_{j=1}^{N} \left( x^i - x^i_0 \right) \left( x^j - x^j_0 \right) \left[ \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \right] \bigg|_{x_0} > 0. 
\end{align*}
\]  
(A1.2)

Similarly, a local (strong) maximum is guaranteed by
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} (x^i - x_0^i) (x^j - x_0^j) \left[ \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \right] |_{x_0} < 0. \]  
(A1.3)

**Example A1.1.** Consider the quadratic polynomial \( f(x) := \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij} x^i x^j \) for \( x \in \mathbb{R}^N \). There exists one critical point at the origin \( x_0 := (0, 0, \ldots, 0) \). Now,
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} x^i x^j \left[ \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \right] |_{(0,0,\ldots,0)} = 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij} x^i x^j > 0. \]
Therefore, the origin is a local, as well as a global (strong), minimum. \( \square \)

Consider a parametrized curve \( \mathcal{X} \) of class \( C^2 ([t_1, t_2] \subset \mathbb{R}; \mathbb{R}^N) \), as furnished in equation (1.19). A Lagrangian function \( L : [t_1, t_2] \times D_N \times D'_N \rightarrow \mathbb{R} \) is such that \( L(t; x; u) \) is twice differentiable. The action function (or action functional) \( J \) is a mapping from the domain set \( C^2 ([t_1, t_2] \subset \mathbb{R}; \mathbb{R}^N) \) into the range set \( \mathbb{R} \) and is given by:
\[ J(\mathcal{X}) := \int_{t_1}^{t_2} [L(t; x; u)]_{|x^i = x^i(t), u^i = \frac{dx^i(t)}{dt}} \cdot dt. \]  
(A1.4)

We assume that the function \( J \) is of class \( C^2 \), thus, totally differentiable in the abstract sense. (See (1.13).)

We consider a “slightly varied differentiable curve” \( \mathcal{\hat{X}} \in C^2 ([t_1, t_2] \subset \mathbb{R}; \mathbb{R}^N) \). (See Fig. A1.1.)

Two nearby curves are specified by the equations
\[ \chi^i = \chi^i(t), \]
\[ \bar{\chi}^i = \bar{\chi}^i(t) := \chi^i(t) + \varepsilon h^i(t), \]
\[ t \in [t_1, t_2] \subseteq \mathbb{R}. \] (A1.5)

Here, \(|\varepsilon| > 0\) is a small positive number and \(h^i(t)\) is arbitrary twice-differentiable function. The “slight variation” of the action function \(J\) in (A1.4) is furnished by

\[
\Delta J(\chi) := J\left(\bar{\chi}\right) - J(\chi)
\]
\[
= \varepsilon \int_{t_1}^{t_2} \left\{ h^i(t) \cdot \left[ \frac{\partial L(\cdot)}{\partial x^i} \right]_{..} + \frac{dh^i(t)}{dt} \cdot \left[ \frac{\partial L(\cdot)}{\partial u^i} \right]_{..} \right\} \, dt + 0(\varepsilon^2)
\]
\[
= \varepsilon \int_{t_1}^{t_2} \left\{ h^i(t) \cdot \left[ \frac{\partial L(\cdot)}{\partial x^i} \right]_{..} \right\} \, dt + \varepsilon \left\{ h^i(t) \cdot \left[ \frac{\partial L(\cdot)}{\partial u^i} \right]_{..} \right\} \Bigg|_{t_1}^{t_2}
\]
\[
- \varepsilon \int_{t_1}^{t_2} \left\{ h^i(t) \cdot \frac{d}{dt} \left[ \frac{\partial L(\cdot)}{\partial u^i} \right]_{..} \right\} \, dt + 0(\varepsilon^2). \] (A1.6)

The variational derivative of the function \(J\) is given by

\[
\frac{\delta J(\chi)}{\delta \chi} := \lim_{\varepsilon \to 0} \left\{ \frac{\Delta J(\chi)}{\varepsilon} \right\}
\]
\[
= \int_{t_1}^{t_2} \left\{ \left[ \frac{\partial L(\cdot)}{\partial x^i} \right]_{..} - \frac{d}{dt} \left[ \frac{\partial L(\cdot)}{\partial u^i} \right]_{..} \right\} \cdot h^i(t) \, dt
\]
\[
+ \left\{ \left[ \frac{\partial L(\cdot)}{\partial u^i} \right]_{..} \cdot h^i(t) \right\} \Bigg|_{t_1}^{t_2} + 0. \] (A1.7)

The critical functions or critical curves \(\chi(0)\) are defined by the solutions of the equation \(\frac{\delta J(\chi)}{\delta \chi} = 0\). We can prove from (A1.7) (with some more work involving the Dubois-Reymond lemma [224]), that critical functions \(\chi(0)\) must satisfy N Euler–Lagrange equations:

\[
\frac{\partial L(\cdot)}{\partial x^i} \Bigg|_{..} - \frac{d}{dt} \left\{ \left[ \frac{\partial L(\cdot)}{\partial u^i} \right]_{..} \right\} = 0. \] (A1.8)

Moreover, we must impose variationally permissible boundary conditions (consisting partly of prescribed conditions and partly of variationally natural boundary conditions [159]) so that the equations
must hold. Usually, $2N$ Dirichlet boundary conditions
\[ \mathcal{X}^i(t_1) = x^i(1) = \text{prescribed consts.,} \quad \mathcal{X}^i(t_2) = x^i(2) = \text{prescribed consts.} \]
are chosen. Such conditions imply, by (A1.5), that $h^i(t_1) = h^i(t_2) \equiv 0$. Therefore, (A1.9) will be validated. However, there are many other variationally permissible boundary conditions which imply (A1.9).

**Example A1.2.** Consider the Newtonian mechanics of a free particle in a Cartesian coordinate chart. The Lagrangian is furnished by
\[
L(v) := \frac{m}{2} \delta_{\alpha\beta} v^\alpha v^\beta, \quad m > 0,
\]
\[
\frac{\partial L(v)}{\partial v^\alpha} = m \delta_{\alpha\beta} v^\beta,
\]
\[
\frac{\partial^2 L(v)}{\partial v^\alpha \partial v^\beta} = m \delta_{\alpha\beta}. \tag{A1.10}
\]
The Euler–Lagrange equations (A1.8), with a slight change of notation and fixed boundary conditions $\mathcal{X}(t_1) = x(1)$ and $\mathcal{X}(t_2) = x(2)$, yield the unique critical curve
\[
\mathcal{X}^\alpha_{(0)}(t) = \left[ \frac{x^\alpha(2) - x^\alpha(1)}{t_2 - t_1} \right] (t - t_1) + x^\alpha(1). \tag{A1.11}
\]
This is a portion of a straight line or geodesic. Substituting the above solution into the finite variation in (A1.6), we derive that
\[
\left\{ \left[ \frac{\partial L}{\partial u^i} \right] \cdot h^i(t) \right\} \bigg|_{t_1}^{t_2} = \left\{ \left[ \frac{\partial L}{\partial u^i} \right] \cdot h^i(t) \right\} \bigg|_{t_1}^{t_2} - \left\{ \left[ \frac{\partial L}{\partial u^i} \right] \cdot h^i(t) \right\} \bigg|_{t_1}^{t_2} = 0. \tag{A1.9}
\]
Appendix 1 Variational Derivation of Differential Equations

Fig. A1.2 The mappings corresponding to a tensor field $y^{(r+s)} = \phi^{(r+s)}(x)$

\[ J (\mathcal{X}(0) + \varepsilon h) - J (\mathcal{X}(0)) \]

\[ = \varepsilon m \int_{t_1}^{t_2} \delta_{\alpha\beta} \left[ \frac{dh^\alpha(t)}{dt} \frac{d\mathcal{X}^\beta_{(0)}(t)}{dt} \right] dt + \frac{1}{2} \cdot \varepsilon^2 m \int_{t_1}^{t_2} \delta_{\alpha\beta} \frac{dh^\alpha(t)}{dt} \frac{dh^\beta(t)}{dt} dt + 0 \]

\[ = 0 + \frac{1}{2} \cdot \varepsilon^2 m \int_{t_1}^{t_2} \delta_{\alpha\beta} \frac{dh^\alpha(t)}{dt} \frac{dh^\beta(t)}{dt} dt > 0. \] (A1.12)

Therefore, the critical curve $\mathcal{X}(0)$, which is a part of a geodesic in the Euclidean space, constitutes a strong minimum compared to other neighboring smooth curves.

Remarks. (i) Local, strong minima provide stable equilibrium configurations.

(ii) Timelike geodesics in a pseudo-Riemannian space–time manifold constitute strong, local maxima!

Now we shall consider $(r + s)$th order, differentiable tensor fields in an $N$-dimensional manifold $M_N$, as discussed in (1.30). (More sophisticated treatments of tensor fields employ the concepts of tensor bundles [38,56].) For the sake of brevity, we denote an $(r + s)$th order tensor field by the symbol $\phi^{(r+s)}(x)$ (in this appendix). Various mappings, which are necessary for the variational formalism, are shown in Fig. A1.2.
According to the mapping diagram,
\[ y^{(r+s)} = \pi^{(r+s)} \circ F(x) =: \phi^{(r+s)}(x), \]
\[ x \in \mathcal{D}_N \subset \mathbb{R}^N. \]  
(A1.13)

The Lagrangian function \( \mathcal{L} : \mathcal{D}_N \times \mathbb{R}^{(N)^{r+s}} \times \mathbb{R}^{(N)^{r+s+1}} \rightarrow \mathbb{R} \) is such that \( \mathcal{L}(x; y^{(r+s)}; y^{(r+s)}_i) \) is twice differentiable. The action function (or functional) is defined by
\[ J(F) := \int_{\mathcal{D}_N} L\left(x; y^{(r+s)}; y^{(r+s)}_i\right) \bigg|_{y^{(r+s)}=\phi^{(r+s)}(x), y^{(r+s)}_i=\partial_i\phi^{(r+s)}} \cdot d^N x. \] (A1.14)

A “slightly varied function” is defined by
\[ \hat{y}^{(r+s)} = \pi^{(r+s)} \circ \hat{F}(x) = \phi^{(r+s)}(x) + \varepsilon h^{(r+s)}(x), \] (A1.15)
where \( |\varepsilon| \) is a small positive number. The variation in the totally differentiable action function is furnished by
\[ \Delta J(F) := J(\hat{F}) - J(F) \]
\[ = \varepsilon \int_{\mathcal{D}_N} h^{(r+s)}(x) \cdot \left[ \frac{\partial L(\cdot)}{\partial y^{(r+s)}} \right]_{\cdot} + \partial_i h^{(r+s)} \cdot \left[ \frac{\partial L(\cdot)}{\partial y^{(r+s)}_i} \right]_{\cdot} \cdot d^N x + 0(\varepsilon^2). \] (A1.16)

Here, indices \((r+s)\) are to be automatically summed appropriately.

Now, we introduce the notion of a total-partial derivative by
\[ \frac{d}{dx^i} \left[ f(x; y^{(r+s)}; y^{(r+s)}_j)\right]_{\cdot} \]
\[ := \left[ \frac{\partial f(\cdot)}{\partial x^i} \right]_{\cdot} + \partial_i \phi^{(r+s)} \cdot \left[ \frac{\partial f(\cdot)}{\partial y^{(r+s)}} \right]_{\cdot} + \partial_i \partial_j \phi^{(r+s)} \cdot \left[ \frac{\partial f(\cdot)}{\partial y^{(r+s)}_j} \right]_{\cdot}. \] (A1.17)

By (A1.17), we can express (A1.16) as
\[ \Delta J(F) = \varepsilon \int_{\mathcal{D}_N} h^{(r+s)}(x) \cdot \left[ \frac{\partial L(\cdot)}{\partial y^{(r+s)}} \right]_{\cdot}
+ \left( \frac{d}{dx^i} \left[ h^{(r+s)}(x) \cdot \frac{\partial L(\cdot)}{\partial y^{(r+s)}} \right]_{\cdot} \right) - h^{(r+s)}(x) \cdot \frac{d}{dx^i} \left[ \frac{\partial L(\cdot)}{\partial y^{(r+s)}_i} \right]_{\cdot} \cdot d^N x + 0(\varepsilon^2). \] (A1.18)
The variational derivative is provided by

\[
\frac{\delta J(F)}{\delta F} := \lim_{\epsilon \to 0} \left[ \frac{\Delta J(F)}{\epsilon} \right] = \int_{\mathcal{D}_N} \left\{ h^{(r+s)}(x) \left( \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial y^{(r+s)}} \right]_{..} - \frac{d}{dx^i} \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial y^{(r+s)}_{i}} \right]_{..} \right) \right\} d^N x
\]

\[
+ \int_{\partial \mathcal{D}_N} \left\{ h^{(r+s)}(x) \cdot \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial y^{(r+s)}_{i}} \right]_{..} \right\} n_i d^{N-1} x. \tag{A1.19}
\]

(We have used Gauss' theorem in (1.155) for the last term in (A1.19).) The critical or stationary functions \( \phi^{(r+s)}_0(x) \) are given by the solution of the equations \( \frac{\delta J(F)}{\delta F} = 0 \). We can derive, from (A1.19) [159], that critical functions must satisfy equations

\[
\frac{\partial \mathcal{L}(\cdot)}{\partial y^{(r+s)}}_{..} - \frac{d}{dx^i} \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial y^{(r+s)}_{i}} \right]_{..} = 0, \tag{A1.20i}
\]

and

\[
\int_{\partial \mathcal{D}_N} \left\{ h^{(r+s)}(x) \cdot \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial y^{(r+s)}_{i}} \right]_{..} \right\} n_i(x) d^{N-1} x = 0. \tag{A1.20ii}
\]

Here, (A1.20i) yields Euler–Lagrange equations for critical tensor fields, whereas (A1.20ii) provides the variationally admissible boundary conditions.

**Example A1.3.** Consider a background pseudo-Riemannian space–time and a scalar field \( y = \phi(x) \) for \( x \in D_4 \subset \mathbb{R}^4 \). The Lagrangian \( \mathcal{L} \), which is a scalar density, is furnished by

\[
\sqrt{-g(x)} \cdot L(x; y; y_i) = \mathcal{L}(x; y; y_i) := -\frac{\sqrt{-g(x)}}{2} \cdot g^{ij}(x) y_i y_j,
\]

\[
\frac{\partial \mathcal{L}(\cdot)}{\partial y} \equiv 0, \quad \frac{\partial \mathcal{L}(\cdot)}{\partial y_k} = -\sqrt{-g(x)} \cdot g^{kj}(x) y_j. \tag{A1.21}
\]

Euler–Lagrange equations (A1.21), using (A1.17), yield

\[
0 - \frac{d}{dx^k} \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial y_k} \right]_{..} = \partial_j \phi \cdot \partial_k \left[ \sqrt{-g} \cdot g^{kj} \right] + \partial_j \partial_k \phi \cdot \sqrt{-g} \cdot g^{kj}
\]

\[
= \sqrt{-g} \left[ g^{kj}(x) \nabla_k \nabla_j \phi \right] = \sqrt{-g} \Box \phi = 0,
\]

or \( \Box \phi = 0 \). \tag{A1.22}
Thus, the critical functions \( \phi_{(0)}(x) \) satisfy the wave equation in the prescribed curved space–time.

Consider the wave equation \((A1.22)\) in a domain \( D_4 := D_3 \times (0, T) \), \( 0 < T < T_1 \). Let the prescribed metric functions \( g_{ij}(x) \) be real-analytic and \( g^{44}(x) < 0 \) in the domain \( D_4 \). Consider the following initial value problem for the wave equation:

\[
\phi_{(0)}(x, 0) = p(x), \quad \partial_4 \phi_{(0)}|_{(x, 0)} = q(x), \quad x \in D_3.
\]

Here, the prescribed functions \( p(x) \) and \( q(x) \) are assumed to be real-analytic. This initial value problem, by the Cauchy-Kowaleski Theorem 2.4.7, admits a unique, real-analytic solution \( \phi_{(0)}(x) \) for a small positive number \( T \). The “varied function” \( \phi_{(0)}(x) + \varepsilon h(x) \), which respects the initial values, must have \( h(x, 0) = 0 = \partial_4 h|_{(x, 0)} \).

However, other than being real-analytic, there are no restrictions on \( h(x) \) for \( 0 < x^4 < T \). Thus, the integral in \((A1.20\text{ii})\),

\[
- \int_{\partial D_4} h(x) \cdot g^{ij}(x) \cdot \partial_j \phi \cdot n_i(x) \sqrt{-g} \, d^3x \neq 0.
\]

Therefore, in general, initial value problems are not variationally admissible (due to the fact that on the final time hypersurface, the function \( h(x) \) is completely arbitrary).

Now, we shall investigate the variational derivation of the gravitational field equations. However, this process demands a slightly different mathematical approach. We explain the methodology by first exploring the following toy model.

Example A1.4. Consider the flat space–time \( M_4 \) and a Minkowskian coordinate chart. Therefore, the metric tensor field is given by \( g_{ij}(x) = d_{ij} = \text{consts.} \), and \( \sqrt{-g}(x) = 1 \). We explore a Lagrangian function

\[
\mathcal{L}(x; y, y', y_{i}, y'_{j}) := \left[ d^{ij} \partial_i \partial_j W(x) \right] \cdot y - y_{i}',
\]

\[
\mathcal{L}(x; y, y', y_{i}, y'_{j}) \bigg|_{y=\phi(x), \, y_{i} = \partial_{i} \phi, \, y'_{i} = \Gamma_{i}^{j}, \, y' = \Gamma(x)} = (\Box W) \cdot \phi(x) - \partial_i \Gamma^i.
\]

\[
(A1.23)
\]

The slightly varied functions are denoted by \( \tilde{y} = \phi(x) + \varepsilon h(x) \) and \( \tilde{y}_{i} = \Gamma_{i}^j(x) + \varepsilon h^j(x) \). By \((A1.16)\), we obtain that

\[
\frac{\Delta J(F)}{\varepsilon} = \int_{D_4} \left[ \Box W \cdot h(x) \right] d^4x - \int_{\partial D_4} \partial_i h^i \, d^4x
\]

\[
= \int_{D_4} \left[ \Box W \cdot h(x) \right] d^4x - \int_{\partial D_4} h^i(x) n_i \, d^3x.
\]
Therefore, the critical functions, obeying \( \frac{\delta J(F)}{\delta F} = 0 \), must satisfy:

\[
\Box W(x) = 0,
\]

and,

\[
\int_{\partial D_4} h^i(x) n_i \, d^3x = 0. \tag{A1.24}
\]

Here, the coefficient function \( W(x) \) need not satisfy any boundary condition and the critical functions \( \Gamma^i_{(0)}(x) \) need not satisfy any differential equation! \( \Box \)

Now, we shall take up the case of general relativity. Recall that in (2.146i–iii) for relativistic mechanics, position variables and momentum variables are treated on the same footing. Similarly, in what is known as the Hilbert-Palatini approach of variation, metric functions \( y^{ij} = g^{ij}(x) \) and connection coefficients \( \gamma^k_{ij} \) are both treated as independent variables. We choose the following invariant Lagrangian function \( L(\cdot) \):

\[
L \left( x; y^{ij}, \gamma^k_{ij}, \gamma^k_{ijl} \right) := y^{ij} \left[ \gamma^k_{kij} - \gamma^k_{ijk} - \gamma'^l_{ikl} \gamma^l_{ij} + \gamma'^l_{ik} \gamma^l_{ij} \right] =: y^{ij} \rho_{ij} \left( \gamma^c_{ab}; \gamma^c_{abd} \right),
\]

\[
L(\cdot) = \sqrt{-g}(x) \, g^{ij}(x) \cdot \left[ \partial_j \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} - \partial_k \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} - \left\{ \begin{array}{c} l \\ i \\ k \end{array} \right\} \cdot \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} + \left\{ \begin{array}{c} l \\ i \\ j \end{array} \right\} \cdot \left\{ \begin{array}{c} k \\ i \\ l \end{array} \right\} \right];
\]

\[
\mathcal{L}(\cdot) = \sqrt{-g}(x) \, g^{ij}(x) \cdot \left[ \partial_j \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} - \partial_k \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} - \left\{ \begin{array}{c} l \\ i \\ k \end{array} \right\} \cdot \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} + \left\{ \begin{array}{c} l \\ i \\ j \end{array} \right\} \cdot \left\{ \begin{array}{c} k \\ i \\ l \end{array} \right\} \right]. \tag{A1.25}
\]

From the equations above, we can deduce the following partial derivatives:

\[
\left[ y \cdot y^{ab} \right] \cdot y_{ac} = y \cdot \delta^b_c, \quad \frac{\partial(y)}{\partial y^{ab}} = y \cdot y^{ab},
\]

\[
\frac{\partial \left( \sqrt{-y} \right)}{\partial y^{ab}} = + \frac{1}{2} \sqrt{-y} \cdot y^{ab}, \quad \frac{\partial \left( \sqrt{-y} \right)}{\partial y^{ab}} = - \frac{1}{2} \sqrt{-y} \cdot y^{ab},
\]

\[\text{Caution: Here, } \gamma^c_{ab} \text{ is not related to Ricci rotation coefficients as such.}\]
\[
\frac{\partial}{\partial y_{ab}} \left( \sqrt{-g} \cdot y^{ij} \right) = \sqrt{-g} \left[ \delta_a^i \cdot \delta_b^j - \frac{1}{2} y_{ab} y^{ij} \right];
\]
\[
\frac{\partial \rho_{ij} (\cdots)}{\partial y_{a \cdots c}} = -\delta_a^b \cdot \gamma_{ij} - \delta_a^b \cdot \delta_i^c \cdot \gamma_{ja} + \delta_a^b \cdot \gamma_{aj} + \delta_i^c \cdot \gamma_{ja} - \delta_i^c \cdot \gamma_{ja},
\]
\[
\frac{\partial \rho_{ij} (\cdots)}{\partial y_{a \cdots c d}} = \lambda_{a d} \cdot \delta_i^c \cdot \delta_j^d - \lambda_{j d} \cdot \delta_i^c \cdot \delta_j^d.
\]
(A1.26)

We denote the slightly varied functions by \( \tilde{\gamma}^{ij} = g^{ij}(x) + \varepsilon h^{ij}(x) \) and \( \tilde{\gamma}^{ij} = \{ k \}_{ij} + \varepsilon h^{ij}(x) \). (Note that \( h^{ij}(x) \) and \( h^{ij}(x) \) are \((2 + 0)\)th order and \((1 + 2)\)th order tensor fields, respectively, even though \( \{ k \}_{ij} \) is not!)\(^2\) Using (A1.16) for the Lagrangian in (A1.25), we derive that

\[
\frac{\Delta J(F)}{\varepsilon} = \int_{D_4} \left\{ \left[ \rho_{ij} (\cdots) \cdot \frac{\partial \left( \sqrt{-g} \cdot y^{ij} \right)}{\partial y_{ab}} \right] \cdot h_{ab} (x) \right\} d^4 x
\]
\[
+ \int_{D_4} \sqrt{-g} (x) \cdot g^{ij} (x) \cdot \left\{ \left[ \frac{\partial \rho_{ij} (\cdots)}{\partial y_{a \cdots c}} \right] \cdot h_{a \cdots c} (x) \right\} d^4 x + 0(\varepsilon).
\]
(A1.27)

Therefore, for critical functions, (A1.25)–(A1.27), and a shortcut notation in regard to \( G_{ab} (x) \) yield

\[
0 = \int_{D_4} G_{ab} (x) \cdot h_{ab} (x) \cdot \sqrt{-g} (x) d^4 x
\]
\[
+ \int_{D_4} \nabla_j \left[ g^{ij} (x) \cdot h^{k_{ki}} (x) - g^{ki} (x) \cdot h^{k_{ki}} (x) \right] \cdot d^4 v.
\]
(A1.28)

Thus, the critical functions \( g_{ij}^{(0)} (x) \) and \( \{ k \}_{ij} \) must satisfy

\[
G_{ab} (x) = 0 \quad \text{in} \quad D_4,
\]
(A1.29i)

\(^2\)The popular way of writing one of the variations in (A1.26) is to put \( \delta \left( \sqrt{|g|} g^{ij} \right) = \sqrt{|g|} \left[ \delta g^{ij} - \frac{1}{2} g_{kl} \cdot g^{ij} \cdot \delta g^{kl} \right]. \) (This equation holds in any Riemannian or pseudo-Riemannian manifold.)
and (via the divergence theorem)

$$
\int_{\partial D_4} \left[ g^{ij}(x) \cdot h^k_{ki}(x) - g^{kl}(x) \cdot h^j_{ki}(x) \right] n_j \, d^3v = 0. \quad (A1.29ii)
$$

The variables $\sqrt{-y} y^{ij}$ and $\gamma_k^k$ are analogous to the variables $y$ and $\gamma^i_j$ respectively of Example A1.4. Equations (A1.29i) and (A1.29ii) are exactly analogous to the equations in (A1.4).

Equation (A1.29i) is obviously equivalent to the vacuum equations of (2.160i). Moreover, (A1.29ii) provides implicitly the variationally admissible boundary conditions for the vacuum equations.

There is an unsatisfactory aspect in the derivation of (A1.29i,ii). In that process, we have varied the metric tensor and connection coefficients independently. However, the variational derivation does not reveal the actual relationship between these two. We can rectify this logical gap by augmenting the Lagrangian in (A1.25) with Lagrange multipliers $\lambda^k_j$ to incorporate the required constraints. Defining the unique entries $[y_{ij}] := [y^{ij}]^{-1}$, we furnish the augmented Lagrangian $\tilde{L}$ as the following function of (4)\textsuperscript{16} variables (without assuming any symmetry):

$$
\tilde{L}(x; y^{ij}, \gamma^k_j, \lambda^k_j, y_{ij}, \gamma^k_{ij}) := y^{ij} \left[ y^k_{kij} - \gamma^k_{ijk} - \gamma^l_{ik} \cdot \gamma^j_{lj} + \gamma^l_{ik} \cdot \gamma^k_{lj} \right]
$$

$$
+ \lambda^k_j \left[ y^k_{ij} - \frac{1}{2} y^{kl} (y_{jli} + y_{lij} - y_{ijl}) \right],
$$

$$
\tilde{L}(\cdot) \bigg|_{y^{ij} = g^{ij}(x), \gamma^k_j = \partial_\kappa g^{ij}, \lambda^k_j = \Lambda^k_j(x)}.
$$

$$
= g^{ij}(x) \left[ \partial_j \left\{ \begin{array}{c} k \\ k \\ i \end{array} \right\} \left\{ \begin{array}{c} k \\ k \\ i \end{array} \right\} - \partial_k \left\{ \begin{array}{c} k \\ i \end{array} \right\} - \left\{ \begin{array}{c} l \\ l \end{array} \right\} \left\{ \begin{array}{c} k \\ i \end{array} \right\} \right] \left( x \right) \cdot \left\{ \begin{array}{c} k \\ i \end{array} \right\} \left( x \right)
$$

$$
+ \left\{ \begin{array}{c} l \\ i \end{array} \right\} \left( x \right) \cdot \left\{ \begin{array}{c} k \\ i \end{array} \right\} \left( x \right)
$$

$$
+ \Lambda^k_j(x) \left[ \left\{ \begin{array}{c} k \\ i \end{array} \right\} \left( x \right) - \frac{1}{2} g^{kl}(x) \left( \partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right) \right],
$$

$$
\tilde{L}(\cdot) |_{\cdot} := \sqrt{-y} \tilde{L}(\cdot) |_{\cdot}.
$$

(A1.30)

The Euler–Lagrange equation $\frac{\partial \tilde{L}}{\partial (\lambda^k_j)} = 0$ yield the Christoffel symbols in terms of metric tensor components.
Now, the boundary term in (A1.29ii) is related to the trace of the exterior curvature of the hypersurface \( \partial D_4 \). To show this, consider the projection operator of p. 191, viz.,

\[
P_{ij}(x) := \delta_{ij} - \varepsilon(n) n^i(x)n_j(x).
\]

We define the extended extrinsic curvature \([56, 126]\) by

\[
\chi_{ij}(x) := -\frac{1}{2} P^k_{ij} P^l_{jk} (\nabla_k n_l + \nabla_l n_k), \quad (A1.31i)
\]

\[
K_{\mu\nu}(u) = \partial_\mu \xi^i \cdot \partial_\nu \xi^j \cdot \chi_{ij}(\xi(u)). \quad (A1.31ii)
\]

(Here, the extrinsic curvature, \( K_{\mu\nu}(u) \), was defined in (1.234). The vector \( \xi^i \) was defined in the same section.) We can derive the trace from (A1.31ii) as

\[
\chi^i_i(\cdot) = -\nabla_i n^i \equiv -\nabla^i n_i. \quad (A1.32)
\]

The variation of \( \chi^i_i(\cdot) \) is denoted by (Caution: Note that \( \varepsilon \) is different from \( \varepsilon(n) \)):

\[
\dot{\chi}^i_i(\cdot) - \chi^i_i(\cdot) = :\varepsilon h(\cdot). \quad (A1.33)
\]

Now, (A1.32) yields two expressions:

\[
-\chi^i_i(\cdot) = \partial_i n^i + \left\{ \begin{array}{ll} i \\ j \end{array} \right\} n^j, \quad (A1.34i)
\]

\[
-\chi^i_i(\cdot) = g^{ij}(\cdot) \left[ \partial_j n_i - \left\{ \begin{array}{ll} k \\ i \\ j \end{array} \right\} n_k \right]. \quad (A1.34ii)
\]

For a fixed boundary \( \partial D_4 \), the variations of \( n_i \) are exactly zero. In consideration of the boundary term in (A1.29ii), only the variation of \( \left\{ \begin{array}{ll} k \\ i \\ j \end{array} \right\} \) is allowed. The variation of \( g_{ij}(\cdot) \) and \( n^i(\cdot) \) is set to zero for this boundary term. Therefore, (A1.34i,ii) provide

\[
-\varepsilon h(\cdot) = 0 + \varepsilon h^i_{ij}(\cdot)n^j(\cdot), \quad (A1.35i)
\]

\[
-\varepsilon h(\cdot) = g^{ij}(\cdot) \left[ 0 - \varepsilon h^k_{ij}(\cdot)n_k(\cdot) \right]. \quad (A1.35ii)
\]

Adding the two equations above, we obtain

\[
-2h(\cdot) = \left[ g^{ik}(\cdot)h^i_{ij}(\cdot) - g^{ij}(\cdot)h^k_{ij}(\cdot) \right] n_k(\cdot). \quad (A1.36)
\]
Therefore, the boundary term in (A1.29ii) is furnished by
\[
\int_{\partial D_4} \left[ g^{ij}(x) h^k_{\ i}(x) - g^{ki}(x) h^i_{\ k}(x) \right] n_j \, d^3v = -2 \int_{\partial D_4} h(x) \, d^3v. \tag{A1.37}
\]

A popular way to write the above is to express it as
\[
\int_{\partial D_4} \left[ g^{ij}(x) \delta \left( \begin{array}{c} k \\ k i \\ j \\ k i \\ n_j \end{array} \right) - g^{ki}(x) \delta \left( \begin{array}{c} j \\ k i \\ i \\ n_j \end{array} \right) \right] n_j \, d^3v = -2 \int_{\partial D_4} \delta \left[ \chi'(x) \right] \, d^3v. \tag{A1.38}
\]

(We have tacitly assumed here that \( \delta g^{ij}(x) \equiv 0 \) on the boundary.)

Example A1.5. The invariant Lagrangians (A1.25) and (A1.30) contain second-order derivatives of the metric tensor components. In general, for Lagrangians involving second-order derivatives of the metric, we would have obtained fourth-order field equations as the corresponding Euler–Lagrange equations. However, for the Lagrangian density \( \sqrt{-g(x)} R(x) \), the higher order terms are transformable into a boundary term in (A1.29ii) and thus do not contribute to the field equations. Einstein investigated a Lagrangian \([84]\) which has only first-order derivatives of the metric tensor components. It is furnished by
\[
\mathcal{L}(\cdot) := \sqrt{-y} y^{ij} \left[ \gamma^l_{\ ik} \gamma^k_{\ lj} - \gamma^l_{\ ik} \gamma^k_{\ ij} \right] \\
\equiv Y^{ij} \cdot F_{ij} \left( Y^{kl}_{\ m} \right), \\
Y^{ij} := \sqrt{-y} y^{ij}, \\
\mathcal{L}(\cdot) \mid_{y^{ij} = g^{ij}(x), \gamma^l_{\ ij} = \left( \begin{array}{c} k \\ i j \end{array} \right)} = \sqrt{-g(x)} \cdot g^{ij}(x) \cdot \left[ \left( \begin{array}{c} l \\ i k \end{array} \right) \cdot \left( \begin{array}{c} k \\ l j \end{array} \right) - \left( \begin{array}{c} l \\ i k \end{array} \right) \cdot \left( \begin{array}{c} k \\ l j \end{array} \right) \right]. \tag{A1.39}
\]

Note that the above Lagrangian is only a part of \( \sqrt{-g(x)} R(x) \), and it is not a scalar density (of weight +1). We can compute the partial derivatives as
\[
\frac{\partial \mathcal{L}(\cdot)}{\partial Y^{ij}} \mid_{Y^{ij} = \sqrt{-g} g^{ij}, \gamma^l_{\ ij} = \delta_k(\sqrt{-g} g^{ij})} \equiv \left( \begin{array}{c} l \\ i k \end{array} \right) \cdot \left( \begin{array}{c} k \\ i j \end{array} \right) - \left( \begin{array}{c} l \\ i k \end{array} \right) \cdot \left( \begin{array}{c} k \\ l j \end{array} \right),
\]
\[
\frac{\partial \mathcal{L}(\cdot)}{\partial Y^{ij} \mid_{k}} = \delta^l_{i j} \left( \begin{array}{c} l \\ j l \end{array} \right) - \left( \begin{array}{c} k \\ i j \end{array} \right). \tag{A1.40}
\]

3In the metric variation approach, it is assumed that the connection is the metric connection and that only the metric is to be varied. In this scenario, the boundary term is much more complicated. The general case (i.e., not assuming that \( \delta g_{ij}(x) \equiv 0 \) on the boundary) yields
\[-2 \int_{\partial D_4} \delta \left[ \chi'(x) \right] \, d^3v + \int_{\partial D_4} \nabla_i \left[ P^{ij}(x) n^k \delta g_{jk}(x) \right] \, d^3v - \int_{\partial D_4} g^{ik}(x) \nabla_i n^j \delta g_{jk}(x) \, d^3v \]
for the boundary term.
The corresponding variational equations \((A1.20i,ii)\) yield
\[
\frac{\partial \mathcal{L}(\cdot)}{\partial \gamma_{ij}} .. - \frac{d}{dx^k} \left[ \frac{\partial \mathcal{L}(\cdot)}{\partial \gamma_{ij}^k} \right] .. = -R_{ij}(x) = 0, \tag{A1.41i}
\]
\[
\int_{\partial D_4} h^{kj}(x) \left\{ l \atop j \right\} - h^{ij}(x) \left\{ k \atop i \right\} n_k \, d^3x = 0. \tag{A1.41ii}
\]
Thus, \((A1.41i,ii)\) provide the gravitational field equations and variationally admissible boundary conditions outside material sources.

**Example A1.6.** We shall now derive the Arnowitt–Deser–Misner (ADM) action integral \([7], [184]\). Assuming \(g_{44}(x) < 0\), we express the metric tensor in the following equations:
\[
g_{ij}(x) = g_{\alpha\beta}(x) \left[ dx^\alpha + N^\alpha(x) \, dx^4 \right] \otimes \left[ dx^\beta + N^\beta(x) \, dx^4 \right] - [N(x)]^2 \, dx^4 \otimes dx^4;
\]
\[
\left[ g_{ij} \right] = \begin{bmatrix}
g_{\alpha\beta} & g_{\mu\nu} N^\nu \\
g_{\mu\nu} N^\nu & g_{\rho\sigma} N^\rho N^\sigma - N^2
\end{bmatrix},
\]
\[
\left[ g^{ij} \right] = \begin{bmatrix}
g^{\alpha\beta} - N^{-2} N^\alpha N^\beta & N^{-2} N^\mu \\
N^{-2} N^\mu & -N^{-2}
\end{bmatrix};
\]
\[
\sqrt{-\det \left[ g_{ij} \right]} = N \sqrt{\det \left[ g_{\alpha\beta} \right]}.
\tag{A1.42}
\]
Suppose that the space–time locally admits a one-parameter family of three-dimensional spacelike hypersurfaces. On the spacelike hypersurface characterized by \(x^4 = T\), the intrinsic metric is furnished as
\[
g_{..}(x, T) = g_{\alpha\beta}(x, T) \, dx^\alpha \otimes dx^\beta
\]
\[
= g_{\alpha\beta}(x, T) \, dx^\alpha \otimes dx^\beta. \tag{A1.43}
\]
There is a slight notational difference between this equation and (1.223) because we choose here one member of the infinitely many hypersurfaces. We show the ADM decomposition schematically in Fig. A1.3.

**Gauss’ equation** (1.242i) for a hypersurface yields in this example, with the (usual) notation \(\overline{A}^\alpha(\cdot, \cdot) := g^{\alpha\beta}(\cdot, \cdot) A_\beta(\cdot, \cdot),\)
The last term in (A1.44) can be expressed as a divergence term analogous to \( \nabla_j A^j \) [184]. Therefore, the Hilbert action integral goes over into

\[
\int_{\mathcal{D}_4} R(x) \sqrt{-g(x)} \, d^4x = \int_{\mathcal{D}_4} \left\{ \overline{R}(x, T) + \overline{K}_\beta^\alpha(\cdot) \cdot \overline{K}_\alpha^\beta(\cdot) - \left[ \overline{K}_\alpha^\alpha(\cdot) \right]^2 + 2 R^{\alpha\beta}_{\cdot4\cdot} \right\} \cdot N(x, T) \sqrt{g(x, T)} \cdot d^3x \, dT + \text{(boundary term)}. \tag{A1.45}
\]

The above action integral is useful in several approaches to quantum gravity [10, 199, 221, 246].

Finally, we would like to make the following two mathematical comments. Firstly, consider a nonconstant Lagrangian density function \( \mathcal{L} (y_{ij}; y_{ijk}) \) such that the action functional is given by

\[
J(F) := \int_{\mathcal{D}_4} \mathcal{L} (y_{ij}; y_{ijk}) \, d^4x,
\]

\[
y_{ij} = \pi_{ij} \circ F(x) = g_{ij} (x),
\]

\[
y_{ijk} = \pi_{ijk} \circ F'(x) = \partial_k g_{ij} \cdot \tag{A1.46}
\]
It can be rigorously proved [171] that such an integral cannot be tensorially invariant. That is why, in Example A1.5, we had to deal with a noninvariant action integral.

The second comment is about a Lagrangian scalar density of the type $\mathcal{L}(y_{ij}; y_{ijk}; y_{ijkl})$. It can be rigorously proved [170,171], that in a four-dimensional manifold, the most general Lagrangian density (made up of curvature invariants), which yields second-order Euler–Lagrange equations, must be of the form

$$
\mathcal{L}(\cdot) := c_{(1)} \sqrt{-g(x)} \, R(x) + c_{(2)} \sqrt{-g(x)}
$$

$$
+ c_{(3)} \epsilon^{ijkl} R^{mn}_{\ ij} \cdot R_{mnkl}(x) + c_{(4)} \sqrt{-g(x)}
$$

$$
\cdot \left[ (R(x))^2 - 4R^i_i(x) \cdot R^i_j(x) + R^{kl}_{ij}(x) \cdot R^{ij}_{kl}(x) \right].
$$

(A1.47)

Here, $c_{(1)}, c_{(2)}, c_{(3)}, c_{(4)}$ are four arbitrary constants with just one constraint $[c_{(1)}]^2 + [c_{(3)}]^2 + [c_{(4)}]^2 > 0$. The first term is simply proportional to the Einstein-Hilbert Lagrangian density, giving rise to the equations of motion of general relativity (2.161i). The second term gives rise to a cosmological constant (proportional to $c_{(2)}$) in the equations of motion (see (2.158)). The third term has been shown to yield trivial equations of motion in four dimensions [157]. Finally, each term in the last expression in (A1.47) produces fourth-order equations. However, their combination yields second-order equations and, in four dimensions, its integral is simply a topological number proportional to the Euler characteristic. This expression is commonly referred to as the Gauss-Bonnet term.

Outside this appendix, we shall refrain from adding subscripts (0) to critical functions $\phi_{(0)}^{(r+s)}(x)$ satisfying the Euler–Lagrange equations (A1.20i).
Appendix 2
Partial Differential Equations

In general relativity, a considerable amount of effort is expended in attempting to solve partial differential equations. Therefore, we are including here an extremely brief review of the subject. (For extensive study, we suggest [43, 77, 94].)

Consider the simplest partial differential equation known to mankind:

\[ u = U(x, y), \quad (x, y) \in D \subset \mathbb{R}^2; \]
\[ \partial_x u = \frac{\partial U(x, y)}{\partial x} = 0. \]  \hspace{1cm} (A2.1)

The general solution of equation above is furnished by

\[ u = U(x, y) = f(y), \quad y_1 < y < y_2. \]  \hspace{1cm} (A2.2)

Here, \( f(y) \) is an arbitrary differentiable function. Equation (A2.2) comprises of infinitely many solutions, and it is aptly called the most general solution of the partial differential equation (p.d.e.) (A2.1).

Remarks. (i) The function \( f(y) \) in (A2.2) may be misconstrued as a function of a single variable. Strictly speaking, it is a shortcut notation for a function \( F(c, y) \) of two variables such that the first variable is restricted to a constant \( c \).

(ii) The function \( f(y) \) can be chosen to be discontinuous in the interval \( (y_1, y_2) \), and still it will satisfy the p.d.e. (A2.1) exactly!

(iii) In case the domain of validity \( D \subset \mathbb{R}^2 \) in (A2.1) is nonconvex, the set of solutions in (A2.2) is not the most general. (See [112] for counter-examples.)
Now, let us consider the two-dimensional (one space and one time) wave equation given by the second order, linear p.d.e.:

$$w = W(x, t), \quad (x, t) \in D \subset \mathbb{R}^2, \quad (x, 0) \subset D \text{ for } x_1 < x < x_2;$$

$$\partial_x \partial_x w - \partial_t \partial_t w = \frac{\partial^2 W(x, t)}{\partial x^2} - \frac{\partial^2 W(x, t)}{\partial t^2} = 0. \quad (A2.3)$$

The most general solution of the equation above is furnished by

$$W(x, t) = f(u) + g(v), \quad f, g \in C^2(D \subset \mathbb{R}^2; \mathbb{R}),$$

but otherwise arbitrary. Thus, there are infinitely many solutions of the p.d.e. (A2.3). A class of general solutions containing infinitely many solutions is provided by

$$W(x, t) = x - t + \exp(x + t), \quad \frac{\partial W(x, t)}{\partial t} \big|_{t=0} = e^x - 1.$$

Let us employ a doubly null coordinate system $u = x - t, \quad v = x + t$. (Compare with the Example 2.1.17.) The p.d.e. in (A2.3) goes over into

$$w = \hat{W}(u, v) := W(x, y), \quad (u, v) \in \hat{D} \subset \mathbb{R}^2;$$

$$\frac{\partial^2 \hat{W}(u, v)}{\partial u \partial v} = 0. \quad (A2.4)$$

The most general solution is provided by

$$w = \hat{W}(u, v) = f(u) + g(v), \quad \text{where } f, g \in C^2(\hat{D} \subset \mathbb{R}^2; \mathbb{R}).$$

However, we notice that the condition of $C^2$ differentiability on $f(u)$ can be completely relaxed. Discontinuous functions $f(u)$ can yield exact solutions $\hat{W}(u, v) = f(u) + g(v)$ for the p.d.e. (A2.4), but not for the p.d.e. $\frac{\partial^2 \hat{W}(u, v)}{\partial u \partial v}$. Physically speaking, such solutions yield shock waves not revealed in the usual $x - t$ coordinate system.

However, the definition of a solution can be generalized to a weak solution [43], which can extract rigorously discontinuous solutions in any coordinate system. Instead of the p.d.e. (A2.3), a weak solution has to satisfy the integral condition:

$$\int_D W(x, t)(\partial_x \partial_x f - \partial_t \partial_t f) \, dx \, dt = 0, \quad (A2.5)$$

for every $C^\infty$-function $f$ with compact support in $D$. (Such functions are called distributions [270].)

The second-order p.d.e. (A2.3) is exactly equivalent to the following linear, first order system:

$$\partial_x w = P_x(x, t),$$

$$\partial_t w = P_t(x, t),$$

$$\partial_x P_x - \partial_t P_t = 0,$$

$$\partial_t P_x - \partial_x P_t = 0. \quad (A2.6)$$
The last of the above equations is the integrability condition. (We have already discussed a system of \((N - 1)\), first-order, linear p.d.e.s in (1.236).)

In a general system of p.d.e.s in \(D \subset \mathbb{R}^N\), the highest derivatives that occur determine the order of the system. In a nonlinear system, the highest power (or exponent) of highest derivatives, is called the degree of the system.

An \(N\)-dimensional first-order p.d.e. is called a linear equation if it has the form:

\[
\sum_{i=1}^{N} \Gamma^i(x) \partial_i w + \Gamma(x) w = h(x), \quad x \in D \subset \mathbb{R}^N. \tag{A2.7}
\]

Here, \(\Gamma^i(x)\), \(\Gamma(x)\), and \(h(x)\) are prescribed continuous functions. (For a homogeneous p.d.e., \(h(x) \equiv 0\).)

An \(N\)-dimensional first-order equation is called a semilinear p.d.e. provided it is given by

\[
\sum_{i=1}^{N} \Gamma^i(x) \partial_i w + h(w, x) = 0. \tag{A2.8}
\]

A first-order p.d.e. is called quasilinear p.d.e. if it has the form:

\[
\sum_{i=1}^{N} \Gamma^i(w, x) \partial_i w + h(w, x) = 0. \tag{A2.9}
\]

A second-order p.d.e. in an \(N\)-dimensional domain is called a linear p.d.e. provided it is furnished by

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma^{ij}(x) \partial_i \partial_j w + \sum_{i=1}^{N} \Gamma^i(x) \partial_i w + \Gamma(x) w = h(x). \tag{A2.10}
\]

(The linear equation (A2.10) is homogeneous provided \(h(x) \equiv 0\).)

A second-order p.d.e. is called a semilinear p.d.e. provided it is expressible as

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma^{ij}(x) \partial_i \partial_j w + h(\partial_i w, w, x) = 0. \tag{A2.11}
\]

A second-order p.d.e. is called a quasilinear p.d.e. if it has the form:

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma^{ij}(\partial_k w, x) \partial_i \partial_j w + h(\partial_i w, w, x) = 0. \tag{A2.12}
\]

A system of p.d.e.s in an \(N\)-dimensional domain is a collection of coupled p.d.e.s.
We classify systems of p.d.e.s in an $N$-dimensional domain in Fig. A2.1.

We shall now sketch very briefly some of the solution procedures for each of the type of systems mentioned in the figure.

Linear systems with constant coefficients are soluble usually by employing Fourier series, Fourier integrals, or Laplace integrals. Special classes of linear p.d.e.s admit separable solutions of the type $W(x) = \prod_{i=1}^{N} W_i(x_i)$.

Consider a single quasilinear first-order p.d.e. (A2.9). Lagrange’s solution method involves $(N + 1)$ characteristic ordinary differential equations (o.d.e)s:

$$
\frac{dx^1}{\Gamma^{1}(w,x)} = \frac{dx^2}{\Gamma^{2}(w,x)} = \cdots = \frac{dx^N}{\Gamma^{N}(w,x)} = -\frac{dw}{h(w,x)} = dt,
$$

or,

$$
\frac{d\chi^i(t)}{dt} = \Gamma^{i}(w,x)_{|_{t}}, \quad \frac{d\mathcal{W}(t)}{dt} = -h(w,x)_{|_{t}}.
$$

\[
\mathcal{W}(t) := W[\chi(t)] = W[\chi^1(t), \ldots, \chi^N(t)]. 
\] (A2.13)

The general solution of (A2.13) consists of $(N + 1)$ arbitrary constants. Alternatively, the solution curve for (A2.13) may be furnished by the intersections of the following differentiable hypersurfaces:

$$
\phi_{(i)}(x;w) = c_{(i)} = \text{const.},
$$

$$
\frac{\partial (\phi_{(1)}, \ldots, \phi_{(N)})}{\partial (x^1, \ldots, x^N)} \neq 0. 
\] (A2.14)

Consider an arbitrary nonconstant function $F \in C^1(D_{(\phi)} \subset \mathbb{R}^N; \mathbb{R})$. The most general solution of the quasilinear p.d.e. in (A2.9) is implicitly provided by

$$
F[\phi_{(1)}(x;w), \ldots, \phi_{(N)}(x;w)] = c = \text{const.} 
\] (A2.15)
**Remark.** Lagrange’s method is valid for a linear or a semilinear p.d.e. also.

**Example A2.1.** Consider the following first-order, quasilinear p.d.e.:

\[
w = W(x), \quad x \in D \subset \mathbb{R}^2; \quad w(\partial_1 w + \partial_2 w) - (w)^3 = 0.
\]

The characteristic o.d.e.s (A2.13) reduce to

\[
\frac{d\lambda^1(t)}{dt} = \mathcal{W}(t) = \frac{d\lambda^2(t)}{dt}, \quad \frac{d\mathcal{W}(t)}{dt} = [\mathcal{W}(t)]^3.
\]

Solving the above equations, we obtain

\[
\frac{d}{dt} \left[ \lambda^1(t) - \lambda^2(t) \right] \equiv 0,
\]

\[
\phi_1(x; w) := x^1 - x^2 = c_1(1),
\]

\[
\phi_2(x; w) := x^1 + (w)^{-1} = c_2(1),
\]

\[
\frac{\partial (\phi_1, \phi_2)}{\partial (x^1, x^2)} = 1.
\]

Therefore, by (A2.14), the most general solution of the p.d.e. is implicitly provided by

\[
F \left( x^1 - x^2, x^1 + (w)^{-1} \right) = c = \text{const.}
\]

(However, there is an additional singular solution, \( W(x^1, x^2) \equiv 0 \).) In case \( \frac{\partial F(\phi_1, \phi_2)}{\partial \phi_2} \neq 0 \), the above yields, by the implicit function theorem [32],

\[
x^1 + (w)^{-1} = g(x^1 - x^2; c),
\]

or,

\[
w = W(x^1, x^2) = \left[ g(x^1 - x^2; c) - x^1 \right]^{-1},
\]

or,

\[
w(\partial_1 w + \partial_2 w) - w^3 = [-\cdot]^{-3} \cdot [(-g' + 1) + g'] - [-\cdot]^{-3} \equiv 0.
\]

Here, \( g \) is an arbitrary differentiable function. The solution above furnishes a general class of infinitely many explicit solutions. (The domain of validity \( D \) must avoid the curve given by \( g(x^1 - x^2; c) - x^1 = 0 \).)

The quasilinear p.d.e. of this example is a disguised linear equation. By the transformation of the variable \( s := -(w)^{-1} \), the p.d.e. is transformed into the linear equation:

\[
\partial_1 s + \partial_2 s - 1 = 0.
\]
The characteristic o.d.e.s are
\[ \frac{dX^1(t)}{dt} = \frac{dX^2(t)}{dt} = 1 = \frac{dS(t)}{dt}. \]
The most general solution is implicitly provided by (A2.15) as
\[ F(x^1 - x^2, x^1 - s) = F(x^1 - x^2, x^1 + (w)^{-1}) = c. \]
Here, \( F \) is an arbitrary, nonconstant function of class \( C^1 \).

Now we consider the nonlinear first-order p.d.e.:
\[ G(x_1, \ldots, x_N; w; p(1), \ldots, p(N)) = 0, \]
\[ p(i) : = \partial w = \frac{\partial W(x)}{\partial x^i}, \quad D \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N. \quad \text{(A2.16)} \]
Here, \( G \) is assumed to be a nonconstant differentiable function such that at least one of \( p(i) \) occurs as \([p(i)]^{n(i)} \), \( n(i) \geq 2 \). The associated \((2N + 1)\) characteristic o.d.e.s are furnished by [43, 94]
\[ \frac{dX^i(t)}{dt} = \frac{\partial G(\cdot)}{\partial p(i)} \bigg|_{\cdot} \] \[ \frac{dP_i(t)}{dt} = -\frac{\partial G(\cdot)}{\partial x^i} \bigg|_{\cdot} - \left[ p(i) \frac{\partial G(\cdot)}{\partial w} \right] \bigg|_{\cdot}, \]
\[ \frac{dW(t)}{dt} = \sum_{i=1}^{N} \left[ p(i) \frac{\partial G(\cdot)}{\partial p(i)} \right] \bigg|_{\cdot}. \quad \text{(A2.17)} \]
Note that along each of the characteristic curves,
\[ \frac{d[G(\cdot)]}{dt} \bigg|_{\cdot} = \frac{\partial G(\cdot)}{\partial x^i} \bigg|_{\cdot} \frac{dX^i(t)}{dt} + \frac{\partial G(\cdot)}{\partial p(i)} \bigg|_{\cdot} \frac{dP_i(t)}{dt} \]
\[ + \frac{\partial G(\cdot)}{\partial w} \bigg|_{\cdot} \frac{dW(t)}{dt} \equiv 0. \quad \text{(A2.18)} \]
Therefore, the solution hypersurface for the p.d.e. (A2.16) is spanned by the congruence of characteristic curves.

Example A2.2. Consider the (truly nonlinear) p.d.e.:
\[ G(x^1, x^2; w; p(1), p(2)) := p(1) \cdot p(2) - w = 0, \]
\[ \partial_1 w \cdot \partial_2 w = w, \]
\[ W(0, x^2) := (x^2)^2. \]
Appendix 2 Partial Differential Equations

(This is an initial value problem of a nonlinear p.d.e.) The characteristic o.d.e.s (A2.17) reduce to

\[
\frac{d\chi^1(t)}{dt} = P(2)(t), \quad \frac{d\chi^2(t)}{dt} = P(1)(t),
\]
\[
\frac{dP(1)(t)}{dt} = P(1)(t), \quad \frac{dP(2)(t)}{dt} = P(2)(t), \quad \frac{dW(t)}{dt} = 2P(1)(t) \cdot P(2)(t).
\]

The general solutions of the system of o.d.e.s are given by

\[
P(1)(t) = c_1e^t, \quad P(2)(t) = c_2e^t,
\]
\[
\chi^1(t) = c_2e^t + c_3, \quad \chi^2(t) = c_1e^t + c_4,
\]
\[
W(t) = c_1c_2e^{2t} + c_5.
\]

Here, \(c_1, \ldots, c_5\) are arbitrary constants of integration with \(c_2 \neq 0\). Inserting initial values \(\chi^1(0) = 0, W(0) = W[\chi^1(0), \chi^2(0)] = [\chi^2(0)]^2\), etc., we finally obtain

\[
(\chi^1(t), \chi^2(t)) = (c_2 \cdot (e^t - 1), (4)^{-1}c_2 \cdot (e^t + 1)),
\]
\[
W[\chi^1(t), \chi^2(t)] = W(t) = \left[(c_2/2)e^t\right]^2 = \left[\frac{\chi^1(t) + 4\chi^2(t)}{4}\right]^2.
\]

From equation above, we conclude that

\[
w = W(x^1, x^2) = \left[\frac{x^1 + 4x^2}{4}\right]^2
\]
is the unique solution of the present p.d.e. under the imposed initial values. \(\square\)

Example A2.3. Consider a domain of pseudo-Riemannian space–time and the relativistic Hamilton–Jacobi equation (for \(m > 0\)) furnished by

\[
G(x^1, x^2, x^3, x^4; s; p_1, p_2, p_3, p_4) := (2m)^{-1}\left[g^{ij}(x)p_ip_j + m^2\right] = 0, \quad (A2.19i)
\]
\[
g^{ij}(x) \cdot \frac{\delta S(x)}{\delta x^i} \cdot \frac{\delta S(x)}{\delta x^j} + m^2 = 0 \quad (A2.19ii)
\]

or

\[
g^{ij}(x) \cdot \nabla_i S \cdot \nabla_j S + m^2 = 0. \quad (A2.19iii)
\]

Compare (A2.19i) with the constraint of the mass shell in (2.33). (Caution: Here the parameter \(s\) is not the proper time parameter of (2.146i–iii).)
From (A2.19i), we deduce that
\[ \frac{\partial G(\cdot)}{\partial p_i} = (m)^{-1} \cdot g^{ij}(x) \cdot p_j, \quad \frac{\partial G(\cdot)}{\partial x^i} = (2m)^{-1} \cdot \left[ \partial_i g^{jk} \right] \cdot p_k \cdot p_j, \]
\[ \frac{\partial G(\cdot)}{\partial s} \equiv 0. \]

Therefore, the characteristic equations (A2.17) yield
\[ \frac{d\lambda^i(t)}{dt} = (m)^{-1} \cdot g^{ij}(x) \cdot p_j, \]
\[ \frac{dP_i(t)}{dt} = -(2m)^{-1} \cdot \left[ \partial_i g^{jk} \right] \cdot p_j \cdot p_k, \]
\[ \frac{dS(t)}{dt} = m^{-1} \cdot g^{ij}(x) \cdot p_i \cdot p_j = -m. \quad (A2.20) \]

Identifying the parameter \( t \) with the proper time parameter, we have just re-derived the relativistic canonical equations (2.146i–iii) for a timelike geodesic in space–time.

**Example A2.4.** Consider the (truly) nonlinear, first-order p.d.e.:
\[ G \left( x^1, x^2; w; p_{(1)}, p_{(2)} \right) := p_{(1)} \cdot p_{(2)} - x^1 = 0, \]

or,
\[ \partial_1 w \cdot \partial_2 w - x^1 = 0, \quad x = (x^1, x^2) \in D \subset \mathbb{R}^2. \]

We shall solve the p.d.e. by a different technique. Note that the nonzero Hessian is furnished by
\[ \det \left[ \frac{\partial^2 G(\cdot)}{\partial p_{(i)} \partial p_{(j)}} \right] = -1 \neq 0. \]

We make a Legendre transformation (as we have done in (2.138)), by
\[ p_{(i)} = \frac{\partial W(x)}{\partial x^i}, \]
\[ \frac{\partial}{\partial x^k} \left[ x^i p_{(i)} - W(x) \right] = \delta^i_k \cdot p_{(i)} - \frac{\partial W(x)}{\partial x^k} \equiv 0, \]
\[ \sigma \left( p_{(1)}, p_{(2)} \right) := x^i p_{(i)} - W(x^1, x^2). \]
\[
\frac{\partial \sigma(\cdot)}{\partial p_{(i)}} = x^i, \\
W(x^1, x^2) = x^i p_{(i)} - \sigma(p_{(1)}, p_{(2)}).
\]

The p.d.e. under consideration reduces to
\[
\frac{\partial \sigma(\cdot)}{\partial p_{(1)}} = p_{(1)} \cdot p_{(2)}.
\]

The most general solution of the above is provided by
\[
\sigma(p_{(1)}, p_{(2)}) = (1/2) \left( p_{(1)} \right)^2 \cdot p_{(2)} + h(p_{(2)}).
\]

Here, \( h(p_{(2)}) \) is an arbitrary differentiable function. Thus, the most general solution of the original p.d.e. is implicitly given by
\[
W(x^1, x^2) = \frac{\partial \sigma(\cdot)}{\partial p_{(1)}} \cdot p_{(1)} + \frac{\partial \sigma(\cdot)}{\partial p_{(2)}} \cdot p_{(2)} - \sigma(\cdot)
\]
\[
= \left( p_{(1)} \right)^2 \cdot p_{(2)} - h(p_{(2)}) + p_{(2)} \cdot h'(p_{(2)}).
\]

Choosing \( h(p_{(2)}) \equiv 0 \), we obtain a particular solution \( W(x^1, x^2) = x^1 \cdot \sqrt{2x^2} \) for \( x^2 > 0 \).

**Remark.** The technique of Legendre transformations is often employed in attempts at the canonical quantization of gravitational fields [10, 221, 246].

We shall now investigate second-order p.d.e.s in \( \mathbb{R}^2 \). We start with a second-order, semilinear p.d.e. (in a two-dimensional domain) expressed as
\[
\sum_{i=1}^{2} \sum_{j=1}^{2} \Gamma^{ij}(x) \partial_i \partial_j w + h(x; w; \partial_i w) = 0,
\]
\[
\Gamma^{11}(x) \partial_1 \partial_1 w + 2\Gamma^{12}(x) \partial_1 \partial_2 w + \Gamma^{22}(x) \partial_2 \partial_2 w + h(x; w; \partial_i w) = 0,
\]
\[
\left[ \Gamma^{11}(x) \right]^2 + \left[ \Gamma^{12}(x) \right]^2 + \left[ \Gamma^{22}(x) \right]^2 > 0
\]
\( x \in D \subset \mathbb{R}^2 \).
The coefficients $\Gamma^{ij}(x)$ are of class $C^1$. The classification of the p.d.e. (A2.21) is based on the determinant

$$\Delta(x) := \det [\Gamma^{ij}(x)].$$

(A2.22)

The p.d.e. (A2.21) is said to be

I. *Hyperbolic p.d.e.* in $D$, provided $\Delta(x) < 0$

II. *Elliptic p.d.e.* in $D$, provided $\Delta(x) > 0$

III. *Parabolic p.d.e.* in $D$, provided $\Delta(x) = 0$

*Example A2.5.* Consider the Tricomi’s p.d.e. [43]:

$$\partial_1 \partial_1 w + x^1 \cdot \partial_2 \partial_2 w = 0.$$ 

The p.d.e. is *elliptic in the half-plane $x^1 > 0$*, and it is *hyperbolic in the other half-plane $x^1 < 0$*.

Now, we would like to simplify the p.d.e. (A2.21) for the purpose of solving it. We explore a possible coordinate transformation of class $C^2$ and (1.107i) to derive the following equations:

$$\hat{x}^i = \hat{X}^i(x) = \hat{X}^i(x^1, x^2),$$

(A2.23i)

$$w = W(x) = \hat{W}(\hat{x}),$$

(A2.23ii)

$$\hat{\Gamma}^{ij}(\hat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \hat{X}^i(x)}{\partial x^k} \cdot \frac{\partial \hat{X}^j(x)}{\partial x^l} \cdot \Gamma^{kl}(x),$$

(A2.23iii)

$$\hat{\Delta}(\hat{x}) = \left[\frac{\partial (x^1, x^2)}{\partial (x^1, x^2)}\right]^2 \cdot \Delta(x),$$

(A2.23iv)

$$\text{sgn} \left[\hat{\Delta}(\hat{x})\right] = \text{sgn} \left[\Delta(x)\right].$$

(A2.23v)

By (A2.23ii) and (A2.23iv), it is clear that *the classification of a p.d.e. remains intact under a coordinate transformation*.

The p.d.e. (A2.21) goes over into

$$\hat{\Gamma}^{11}(\hat{x}) \hat{\partial}_1 \hat{\partial}_1 w + 2 \hat{\Gamma}^{12}(\hat{x}) \hat{\partial}_1 \hat{\partial}_2 w + \hat{\Gamma}^{22}(\hat{x}) \hat{\partial}_2 \hat{\partial}_2 w + \cdots = 0,$$

(A2.24i)

$$\hat{\Gamma}^{11}(\hat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \hat{X}^1(\cdot)}{\partial x^k} \cdot \frac{\partial \hat{X}^1(\cdot)}{\partial x^l} \cdot \Gamma^{kl}(x),$$

(A2.24ii)
\[ \hat{T}^{22}(\hat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \hat{X}^{2}(\cdot)}{\partial x^{k}} \cdot \frac{\partial \hat{X}^{2}(\cdot)}{\partial x^{l}} \cdot \Gamma^{kl}(x), \quad (A2.24iii) \]

\[ \hat{T}^{12}(\hat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \hat{X}^{1}(\cdot)}{\partial x^{k}} \cdot \frac{\partial \hat{X}^{2}(\cdot)}{\partial x^{l}} \cdot \Gamma^{kl}(x) = \hat{T}^{21}(\hat{x}). \quad (A2.24iv) \]

The characteristic surface \( \phi(x) \) over \( D \subset \mathbb{R}^{2} \) of p.d.e. (A2.21) is defined by the first-order, nonlinear p.d.e. of degree two:

\[ \sum_{i=1}^{2} \sum_{j=1}^{2} \Gamma^{ij}(x) \cdot \frac{\partial \phi(x)}{\partial x^{i}} \cdot \frac{\partial \phi(x)}{\partial x^{j}} = 0. \tag{A2.25} \]

In case both the coordinate functions \( \hat{X}^{1}(x) \) and \( \hat{X}^{2}(x) \) satisfy the characteristic p.d.e. (A2.25), the original p.d.e. (A2.21) reduces to

\[ \hat{T}^{12}(\hat{x}) \cdot \hat{\partial}_{1}\hat{\partial}_{2}w + \cdots = 0, \]

or,

\[ \frac{\partial^{2} \hat{W}(\hat{x}^{1}, \hat{x}^{2})}{\partial \hat{x}^{1}\partial \hat{x}^{2}} + \cdots = 0. \tag{A2.26} \]

The normal forms of the various p.d.e.s are listed below:

I(i). Hyperbolic p.d.e.:

\[ \frac{\partial^{2} \hat{W}(\hat{x}^{1}, \hat{x}^{2})}{\partial \hat{x}^{1}\partial \hat{x}^{2}} + \hat{h}(\cdot) = 0. \tag{A2.27i} \]

I(ii). Hyperbolic p.d.e.:

\[ \frac{\partial^{2} \hat{W}(\hat{x}^{1}, \hat{x}^{2})}{(\partial \hat{x}^{1})^{2}} - \frac{\partial^{2} \hat{W}(\hat{x}^{1}, \hat{x}^{2})}{(\partial \hat{x}^{2})^{2}} + \hat{h}(\cdot) = 0. \tag{A2.27ii} \]

II. Elliptic p.d.e.:

\[ \frac{\partial^{2} \hat{W}(\hat{x}^{1}, \hat{x}^{2})}{(\partial \hat{x}^{1})^{2}} + \frac{\partial^{2} \hat{W}(\hat{x}^{1}, \hat{x}^{2})}{(\partial \hat{x}^{2})^{2}} + \hat{h}(\cdot) = 0. \tag{A2.27iii} \]
III. Parabolic p.d.e.:

\[
\frac{\partial^2 \tilde{W}(\tilde{x}^1, \tilde{x}^2)}{(\partial \tilde{x}^1)^2} + \tilde{h}\left(\tilde{x}^1, \tilde{x}^2; w; \tilde{\partial}_1 w, \tilde{\partial}_2 w\right) = 0.
\]  

(A2.27iv)

Example A2.6. Consider the homogeneous, linear, second-order p.d.e.:

\[
\begin{align*}
\partial_1 \partial_1 w - (c^{-2}) \cdot \partial_2 \partial_2 w + \partial_1 w - (c^{-1}) \cdot \partial_2 w &= 0, \\
\Delta(x) &= -(c^{-2}) < 0.
\end{align*}
\]

(A2.28)

The parameter \( c \) is assumed to be nonzero, and the p.d.e. (A2.28) is obviously hyperbolic.

The characteristic surface of p.d.e. (A2.28) is governed by the nonlinear first-order p.d.e.:

\[
(\partial_1 \phi)^2 - (c^{-2}) \cdot (\partial_2 \phi)^2 = 0,
\]

(A2.29i)

\[
G\left(x^1, x^2; \phi; p_{(1)}, p_{(2)}\right) := (1/2) \left[ (p_{(1)})^2 - (c^{-2}) \cdot (p_{(2)})^2 \right] = 0.
\]

(A2.29ii)

The characteristic curves for equations above (which are bicharacteristic curves for the p.d.e. (A2.28)) are furnished by (A2.17) as

\[
\frac{d\chi^1(t)}{dt} = \frac{\partial G(\cdot)}{\partial p_{(1)}} \bigg|_{\cdot} = p_{(1)}|_{\cdot}, \quad \frac{d\chi^2(t)}{dt} = -c^{-2} \cdot p_{(2)}|_{\cdot},
\]

\[
\frac{dP_{(1)}(t)}{dt} = \frac{dP_{(2)}(t)}{dt} \equiv 0.
\]

(A2.30)

The general solutions of (A2.30) and (A2.29ii) are given by

\[
\begin{align*}
P_{(1)}(t) &= c_{(1)}, \quad P_{(2)}(t) = c_{(2)}, \quad c_{(1)}^2 = (c^{-2}) \cdot c_{(2)}^2, \quad c_{(1)} = \pm c^{-1} \cdot c_{(2)}, \\
\chi^1(t) &= -c^{-1}c_{(2)}t + c_{(3)} = c \chi^2(t) + \tilde{c}_{(3)}, \\
\text{or else,} \quad \chi^1(t) &= c^{-1}c_{(2)}t + c_{(3)} = -c \chi^2(t) + \tilde{c}_{(4)},
\end{align*}
\]

thus, \( \phi_{(1)}(x) := x^1 - c x^2 = \tilde{c}_{(3)}, \quad \phi_{(2)}(x) := x^1 + c x^2 = \tilde{c}_{(4)} \).

(A2.31)

Making a coordinate transformation

\[
\begin{align*}
\hat{x}^1 &= \phi_{(1)}(x) = x^1 - c x^2, \quad \hat{x}^2 = \phi_{(2)}(x) = x^1 + c x^2, \\
w &= W(x) = \tilde{W}(\hat{x}),
\end{align*}
\]
the p.d.e. (A2.28) reduces to
\[ \hat{\partial}_1 \left[ \hat{\partial}_2 w + (1/2)w \right] = 0. \] (A2.32)

The above second-order p.d.e. is equivalent to a pair of first-order equations
\[ \hat{\partial}_2 w + (1/2)w = \hat{g}(\hat{x}), \]
\[ \hat{\partial}_1 \hat{g} = 0. \] (A2.33)

(Compare the above equations with (A2.6).) The most general solution of (A2.33) and (A2.28), putting \( \hat{g}(\hat{x}^2) = e^{-\hat{x}^2/2}g'(\hat{x}^2) \), is provided by
\[
w = \hat{W}(\hat{x}) = e^{-\hat{x}_1^2/2}g(\hat{x}^2) + e^{-\hat{x}_1^2/2} \cdot f(\hat{x}_1)
\]
\[
= e^{-(x^1+cx^2)/2}g(x^1 + cx^2) + e^{-(x^1+cx^2)/2} \cdot f(x^1 - cx^2) = W(x). \] (A2.34)

Here, \( f \) and \( g \) are of class \( C^2 \) and otherwise arbitrary.

Consider now a related nonhomogeneous, linear p.d.e.: \[
\partial_1 \partial_1 w - \left( c^{-2} \right) \cdot \partial_2 \partial_2 w + \partial_1 w - \left( c^{-1} \right) \cdot \partial_2 w = 4. \] (A2.35)

A particular solution of this equation is given by \[
w(p) = 2 \left( x^1 - cx^2 \right). \]

Therefore, denoting solution (A2.34) of the homogeneous equation by \( w(h) \), the most general solution of (A2.35) is furnished by the superposition:
\[
w = w(p) + w(h) = 2 \left( x^1 - cx^2 \right) + e^{-(x^1+cx^2)/2}g(x^1 + cx^2)
\]
\[
+ e^{-(x^1+cx^2)/2} \cdot f(x^1 - cx^2). \]

Example A2.7. Consider the elliptic Liouville equation [191]:
\[
\Delta w \equiv \nabla^2 w = \partial_1 \partial_1 w + \partial_2 \partial_2 w = 4e^{2w}, \quad x \in D \subset \mathbb{R}^2. \] (A2.36)

The above is a semilinear, second-order, elliptic p.d.e. in the normal form.

The corresponding characteristic surface, as given by (A2.25), reduces to the p.d.e.
\[
(\partial_1 \phi)^2 + (\partial_2 \phi)^2 = 0. \] (A2.37)
Obviously, only real-valued solutions\(^1\) of (A2.37) are provided by constant-valued functions \(\phi(x)\). Therefore, nondegenerate, characteristic curves, analogous to the preceding hyperbolic example, do not exist in an elliptic case. However, there is an interesting device for treating elliptic equations by complex conjugate coordinates (which are formally analogous to the characteristic coordinates of hyperbolic cases). (See [191].)

Let us introduce complex conjugate coordinates and complex derivatives by the following equations:

\[
\begin{align*}
\zeta & := x^1 + ix^2, \quad \bar{\zeta} := x^1 - ix^2, \\
\nu^1 & = \text{Re}(\zeta) = (1/2) \left( \zeta + \bar{\zeta} \right), \quad \nu^2 = \text{Im}(\zeta) = (1/2i) \left( \zeta - \bar{\zeta} \right), \\
\widehat{F} \left( \zeta, \bar{\zeta} \right) & := F(\nu^1, \nu^2), \\
\partial_{\zeta} \widehat{F} & = \frac{\partial}{\partial \zeta} \widehat{F} \left( \zeta, \bar{\zeta} \right) := (1/2) \left[ \frac{\partial}{\partial \nu^1} - i \frac{\partial}{\partial \nu^2} \right] F(\nu^1, \nu^2), \\
\partial_{\bar{\zeta}} \widehat{F} & = \frac{\partial}{\partial \bar{\zeta}} \widehat{F} \left( \zeta, \bar{\zeta} \right) := (1/2) \left[ \frac{\partial}{\partial \nu^1} + i \frac{\partial}{\partial \nu^2} \right] F(\nu^1, \nu^2), \\
\Delta F & \equiv \nabla^2 F = \left[ \left( \frac{\partial}{\partial \nu^1} \right)^2 + \left( \frac{\partial}{\partial \nu^2} \right)^2 \right] F(\nu^1, \nu^2) = 4 \cdot \frac{\partial^2 \widehat{F}(\zeta, \bar{\zeta})}{\partial \zeta \partial \bar{\zeta}}. \\
\end{align*}
\]

\[\text{(A2.38)}\]

A holomorphic function \(f(\zeta)\) and a conjugate holomorphic function \(\overline{g(\bar{\zeta})}\) satisfy respectively the p.d.e.s:

\[
\begin{align*}
\frac{\partial f(\zeta)}{\partial \overline{\zeta}} & = (1/2) \left[ \frac{\partial}{\partial \nu^1} + i \frac{\partial}{\partial \nu^2} \right] \left[ \text{Re} \left( f(\nu^1 + i\nu^2) \right) + i \text{Im} \left( f(\nu^1 + i\nu^2) \right) \right] = 0, \\
\frac{\partial \overline{g(\bar{\zeta})}}{\partial \zeta} & = (1/2) \left[ \frac{\partial}{\partial \nu^1} - i \frac{\partial}{\partial \nu^2} \right] \left[ \text{Re} \left( g(\nu^1 + i\nu^2) \right) - i \text{Im} \left( g(\nu^1 + i\nu^2) \right) \right] = 0.
\end{align*}
\]

\[\text{(A2.39i)} \quad \text{(A2.39ii)}\]

(The complex equation (A2.39i) is exactly equivalent to the Cauchy-Riemann equations.)

\(^1\)A nonlinear (real) differential equation of degree greater than one may or may not admit any solution. The p.d.e. \((\partial_1 \phi)^2 + (\partial_2 \phi)^2 + \cosh[\phi(x)] = 0\) does not admit any real-valued solution function \(\phi(x)\). However, the p.d.e. \((\partial_1 \phi)^2 + (\partial_2 \phi)^2 + \cosh[\phi(x) - \sqrt{2}] = 1\) admits a single, particular solution \(\phi(x^1, x^2) = \sqrt{2}\).
Consider the real-valued harmonic function \( h(x^1, x^2) = \hat{h}(\zeta, \bar{\zeta}) \) satisfying
\[
\Delta h \equiv \nabla^2 h = \partial_1 \partial_1 h + \partial_2 \partial_2 h = 0.
\]
or,
\[
\partial_\zeta \partial_{\bar{\zeta}} \hat{h} = 0.
\]  
(A2.40)

It can be proved [191] that the most general solution of (A2.40) is given by
\[
\hat{h}(\zeta, \bar{\zeta}) = (1/2) \left[ f(\zeta) + \overline{f(\zeta)} \right],
\]
\[
h(x^1, x^2) = \text{Re} \left[ f(x^1 + ix^2) \right].
\]  
(A2.41)

Here, \( f(\zeta) \) is an arbitrary holomorphic function in the domain of consideration.

Now let us go back to the Liouville equation (A2.36) again. The most general solution of this equation is provided by [191]
\[
w = W(x^1, x^2) = \hat{W}(\zeta, \bar{\zeta}) = \log \left[ \frac{|f'(\zeta)|}{1 - |f(\zeta)|^2} \right],
\]
\[
D := \{ \zeta \in \mathbb{C} : f'(\zeta) \neq 0, |f(\zeta)| < 1 \}.
\]  
(A2.42)

Here, \( f'(\zeta) \neq 0 \), and \( |f(\zeta)| < 1 \), but the holomorphic function \( f(\zeta) \) is otherwise arbitrary.

Now, we consider second-order, semilinear p.d.e.s in an \( N \)-dimensional domain. Such p.d.e.s need not be tensor field equations. (That is why we suspended the summation convention in preceding discussions!) Let us now revert back to the usual summation convention for tensorial, as well as nontensorial p.d.e.s. Recall that the semilinear, second-order p.d.e. (A2.11), reinstating the summation convention, can be expressed as
\[
\Gamma^{ij}(x) \cdot \partial_i \partial_j w + h(x; w; \partial_i w) = 0; \quad x \in D \subset \mathbb{R}^N.
\]  
(A2.43)

Here, the coefficients \( \Gamma^{ji}(x) \equiv \Gamma^{ij}(x) \) are continuous functions such that at least one of them is nonzero. Therefore, the symmetric matrix \( [\Gamma^{ij}(x_0)] \) has \( N \) real (usual) eigenvalues so that at least one of them is nonzero. The eigenvalues can be always arranged in the order:
\[
\lambda_{(1)}(x_0) > 0, \ldots, \lambda_{(p)}(x_0) > 0; \quad \lambda_{(p+1)}(x_0) < 0, \ldots, \lambda_{(p+n)}(x_0) < 0; \quad \lambda_{(p+n+1)}(x_0) = \cdots = \lambda_{(p+n+n)}(x_0) = 0.
\]  
(A2.44)

Here, \( p + n + \nu = N \), \( r := p + n \) is the rank of the matrix \( [\Gamma^{ij}(x_0)] \), and \( \nu = N - r \) is the nullity of the matrix. Thus, \( \nu > 0 \) if and only if the matrix \( [\Gamma^{ij}(x_0)] \)

\[
\begin{align*}
\text{Appendix 2} & \quad \text{Partial Differential Equations} & 599
\end{align*}
\]
is singular. (Without loss of generality, we can always arrange \( p - n \geq 0 \). Compare with the metric equation (1.90).) According to Sylvester’s law of inertia [177, 240], the numbers \( p, n, \nu \) remain invariant under a (real) coordinate transformation of the type (A2.23iii) in (A2.43). Thus, it is logical to choose the classification of the general semilinear p.d.e. (A2.43) according to the following criteria:

I. Elliptic p.d.e. \( \quad \text{if } \nu = 0 = n, \quad p > 0 \)
II. Hyperbolic p.d.e. \( \quad \text{if } \nu = 0, \quad n = 1, \quad p = N - 1 \)
III. Ultrahyperbolic p.d.e. \( \quad \text{if } \nu = 0, \quad 1 < n < p < N - 1 \)
IV. Parabolic p.d.e. \( \quad \text{if } \nu > 0 \)

Note that the \((N - 1)\)-dimensional characteristic hypersurface of equation (A2.43) is governed by the first-order, nonlinear p.d.e.:

\[
\Gamma^{ij}(x) \cdot \partial_i \phi \cdot \partial_j \phi = 0. \quad \text{(A2.45)}
\]

Example A2.8. Consider the following semilinear, second-order p.d.e.:

\[
\partial_1 \partial_1 w + \partial_2 \partial_2 w - \partial_3 \partial_3 w + \left[ \frac{2w}{1 - w^2} \right] \cdot \left[ (\partial_1 w)^2 + (\partial_2 w)^2 - (\partial_3 w)^2 \right] = 0; \quad x \in D \subset \mathbb{R}^3; \quad |w| < 1. \quad \text{(A2.46)}
\]

In this case, \( N = 3, \quad p = 2, \quad n = 1, \quad \nu = 0, \) and \( r = 3 \). Therefore, the p.d.e. is hyperbolic in the domain of consideration. The corresponding characteristic surface is furnished by

\[ (\partial_1 \phi)^2 + (\partial_2 \phi)^2 - (\partial_3 \phi)^2 = 0. \]

The bicharacteristic curves are (null) straight lines in the three-dimensional domain.

Now, we make the following transformation:

\[ g(w) := \frac{1}{2} \ln \left| \frac{1 + w}{1 - w} \right|, \quad w = \tanh(g), \quad |w| < 1, \]

\[ W(x^1, x^2, x^3) := \tanh \left[ g \left( x^1, x^2, x^3 \right) \right]. \]

The p.d.e. (A2.46) reduces to the linear p.d.e.:

\[ \partial_1 \partial_1 g + \partial_2 \partial_2 g - \partial_3 \partial_3 g = 0. \quad \text{(A2.47)} \]

Thus, the original p.d.e. (A2.46) is a disguised linear p.d.e.
A class of general solutions of (A2.47) and (A2.46) is provided by

\[ g(x^1, x^2, x^3) = \int f \left( k_1 x^1 + k_2 x^2 + \sqrt{k_1^2 + k_2^2} \cdot x^3 \right) dk_1 dk_2, \]

\[ W(x^1, x^2, x^3) = \tanh \left[ \int f \left( k_1 x^1 + k_2 x^2 + \sqrt{k_1^2 + k_2^2} \cdot x^3 \right) dk_1 dk_2 \right]. \]

Here, \( f \) is a twice-differentiable function such that the integrals converge uniformly. The function \( f \) is otherwise arbitrary. (Uniform convergences are needed to commute differentiations and the integration [32].) \( \Box \)

Let us go back to the semilinear, second-order p.d.e. (A2.43). It is equivalent to the first-order system:

\[ \partial_i w = p(i), \]
\[ \partial_j p(i) = \partial_i p(j), \]
\[ \Gamma^{ij}(x) \partial_i p(j) + h(x; w; p(i)) = 0. \] (A2.48)

(Compare (A2.6) and (A2.43).)

In general, a system of semilinear second-order p.d.e.s (and possibly another system of first-order p.d.e.s) can be expressed equivalently as a single first order system:

\[ \Gamma^{i}_{AB}(x) \partial_i w^B + h_A(x; w) = 0. \] (A2.49)

Here, the summation convention is also carried on capital Roman indices which take values from \( \{1, 2, \ldots, d\} \).

We construct the characteristic matrix [43] by the following:

\[ [\Gamma(\cdot)] = [\Gamma^{i}_{AB}(x)] := [\Gamma^{i}_{AB}(x) \partial_i \phi]. \] (A2.50)

The characteristic hypersurface is furnished by the first-order p.d.e.:

\[ \det [\Gamma(\cdot)] = \det [\Gamma^{i}_{AB}(x) \partial_i \phi] = 0. \] (A2.51)

Example A2.9. Consider Maxwell’s equations for electromagnetic fields in flat space–time. (See (2.54i–iv).) With the notation

\[ (w^1, w^2, w^3) := (E^1, E^2, E^3), \quad (w^4, w^5, w^6) := (H^1, H^2, H^3), \]
six of the Maxwell’s equations can be expressed as
\[
\begin{align*}
\partial_3 w^5 - \partial_2 w^6 + \partial_4 w^1 &= 0, \\
\partial_1 w^6 - \partial_3 w^4 + \partial_4 w^2 &= 0, \\
\partial_2 w^4 - \partial_1 w^5 + \partial_4 w^3 &= 0, \\
\partial_3 w^2 - \partial_2 w^3 - \partial_4 w^4 &= 0, \\
\partial_1 w^3 - \partial_3 w^1 - \partial_4 w^5 &= 0, \\
\partial_2 w^1 - \partial_1 w^2 - \partial_4 w^6 &= 0.
\end{align*}
\]
(A2.52)

By (A2.50) and (A2.51), we derive
\[
\begin{align*}
\det[\Gamma] &= - (\partial_4 \phi)^2 \cdot \left[ \left( (\partial_1 \phi)^2 + (\partial_2 \phi)^2 + (\partial_3 \phi)^2 - (\partial_4 \phi)^2 \right)^2 \right]. \\
\end{align*}
\]
(A2.53ii)

It is clear that for \( \partial_4 \phi \neq 0 \), the system of p.d.e.s in (A2.52) is hyperbolic. □

(We have obtained classifications of p.d.e.s in general relativity on page 202.)

Now, we shall very briefly touch upon the topic of nonunique (or chaotic) solutions. Consider a system of o.d.e.s:
\[
\begin{align*}
\frac{d\chi^i(t)}{dt} &= F^i(t; x)_{|x^i = \chi^i(t)}, \\
D &:= \{(t; x) \in \mathbb{R} \times \mathbb{R}^N : |t - t(0)| < A, \|x - x(0)\| < B\}, \\
\|x - x(0)\|^2 &:= \delta_{ij} \cdot \left[ \chi^i - \chi^i(0) \right] \cdot \left[ \chi^j - \chi^j(0) \right].
\end{align*}
\]
(A2.54)

(Compare the equation above with (1.75).) Suppose that the functions \( F^i(\cdot) \) are continuous over \( \overline{D} := D \cup \partial D \). However, the entries of the Jacobian matrix
\[
\begin{bmatrix}
\partial_1 F^1 & \partial_2 F^1 & \cdots & \partial_N F^1 \\
\vdots & \vdots & \ddots & \vdots \\
\partial_1 F^N & \partial_2 F^N & \cdots & \partial_N F^N
\end{bmatrix}
\]
are continuous in \( D \), but not continuous on \( \partial D \). Then, the initial value problem
Appendix 2  Partial Differential Equations 603

Fig. A2.2 Graphs of nonunique solutions

\[ x^i(t_0) = x^i_0, \]  
\[ \frac{dX^i(t)}{dt} \bigg|_{t=t_0} = c^i_0, \]

has nonunique (or chaotic) solutions\(^2\) [151].

Example A2.10. Consider the single o.d.e.:

\[ \frac{dX(t)}{dt} = F(t; x) := \sqrt{X(t)}, \]
\[ D = \{(t, x) \in \mathbb{R} \times \mathbb{R} : 0 < t < 1, 0 < x < 1\}. \]

The function \( F(t; x) = \sqrt{x} \) is continuous over \( D \). However, \( \frac{\partial F(x)}{\partial x} = \frac{1}{2\sqrt{x}} \) is not continuous on \( \partial D \). Therefore, the solution for the specific I.V.P.

\[ X(0) = 0, \quad \frac{dX(t)}{dt} \bigg|_{t=0} = 0 \]

has (infinitely) many answers! We furnish these solutions explicitly in the following:

\[ X(t) := \begin{cases} 
0 & \text{for } t \leq k < 1; \\
(1/4)(t - k)^2 & \text{for } 0 < k < t.
\end{cases} \]

The functions \( X(t) \) are shown graphically in the Fig. A2.2.

\(^2\)Moreover, bifurcation points will satisfy simultaneous equations: \( F^i(t; x) = 0 \), and \( \det\left[ \frac{\partial F(x)}{\partial x} \right] = 0 \). (See [138].)
Appendix 3
Canonical Forms of Matrices

We shall start this appendix with some very simple models. Consider $2 \times 2$ matrices with real entries (or elements). (See [5].)

Example A3.1. Consider the symmetric matrix $[S] := \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The characteristic polynomial is given by $p(\lambda) := \det [S - \lambda I] = \lambda^2 - 6\lambda + 8$. Therefore, the usual eigenvalues are $\lambda_{(1)} = 4$ and $\lambda_{(2)} = 2$. The corresponding eigenvectors are

$$\begin{bmatrix} [\mathbf{e}_{(1)}] \\ [\mathbf{e}_{(2)}] \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} [\mathbf{e}_{(1)}] \\ [\mathbf{e}_{(2)}] \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}. $$

These column vectors are orthonormal in the usual Euclidean sense. We construct a matrix $[P]$ (with help of eigenvectors) as

$$[P] := \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. $$

By the similarity transformation

$$[P]^{-1} [S] [P] = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, $$

and the symmetric matrix is diagonalized.

Example A3.2. Consider the symmetric matrix $[S] := \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. The usual eigenvalue $\lambda_{(1)} = \sqrt{2}$ has the multiplicity 2. Moreover, every nonzero $2 \times 1$ column vector is an eigenvector.
Example A3.3. Consider the matrix \( [M] := \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \). The (usual) characteristic polynomial is furnished by \( p(\lambda) = (\lambda + 1)^2 \). The eigenvalue \( \lambda_{(1)} = -1 \) is of multiplicity 2. The eigenvectors are of the form \( \begin{bmatrix} t \\ t \end{bmatrix}, \ t \neq 0 \). Therefore, there is only one eigendirection and the matrix is nondiagonalizable. (This matrix has a nonlinear or nonsimple elementary divisor \( E_{(2)}(\lambda) = (\lambda + 1)^2 \) according to (A3.3).) However, consider the matrix

\[
[P] := \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}.
\]

By the similarity transformation

\[
[P]^{-1} [M] [P] = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Thus, the matrix \([M]\) is reducible to an upper triangular form. Moreover, the triangular form is a diagonal matrix plus a nilpotent matrix. □

Example A3.4. Consider the antisymmetric matrix \([A] := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\). The characteristic polynomial \( p(\lambda) = \lambda^2 + 1 \). Thus, there exist no real eigenvalues. Consequently, there are no (real) eigenvectors. However, extending into the (algebraically closed) complex field, we have two complex-conjugate eigenvalues \( \lambda_{(1)} = i, \lambda_{(2)} = -i = \bar{\lambda}_{(1)} \). Consider the complex matrix \([P] := \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\). The similarity transformation \([P]^{-1} [A] [P]\) reduces \([A]\) into the complex diagonal matrix \( \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \). □

Example A3.5. Consider the \(2 \times 2\) matrix generated by the Lorentz metric (in Example 1.3.2) \([D] := [d_{(\alpha)(\beta)}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). The Lorentz-invariant characteristic polynomial of a matrix \([M]\) is defined by \( p^\#(\lambda) := \det[M - \lambda D] \). The invariant eigenvalues are furnished by the roots of \( p^\#(\lambda) = 0 \). (Compare with (1.213).) A symmetric matrix can be expressed as \([S] = \begin{bmatrix} 2a & a + b \\ a + b & 2b \end{bmatrix}\). The corresponding invariant eigenvalues are given by \( p^\#(\lambda) = -[\lambda - (a - b)]^2 = 0 \). There exists one (real) invariant eigenvalue \( \lambda_{(1)} = (a - b) \) of multiplicity 2. There exists one eigendirection \( \begin{bmatrix} t \\ -t \end{bmatrix} (t \neq 0) \). The Lorentz-invariant separations (of (1.86)) are provided by \([t, -t][1 & 0 \\ 0 & -1][t & -t] \equiv 0 \). Thus, the eigendirection is along a double
null vector. Moreover, this matrix is associated with nonsimple elementary divisor $E_{(2)}(\lambda) = (\lambda - a + b)^2$ according to (A3.3).

Example A3.6. Consider the symmetric matrix $[S] := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (This happens to be one of the Pauli matrices.) The usual eigenvalues are given by $\lambda_{(1)} = 1, \lambda_{(2)} = -1$. The Lorentz-invariant eigenvalues are furnished by $p^\#(\lambda) = -(\lambda^2 + 1) = 0$. Therefore, we have complex-conjugate invariant eigenvalues $\lambda_{(1)} = i, \lambda_{(2)} = -i = \bar{\lambda}_{(1)}$. Therefore, a symmetric matrix, with real entries, can have complex, Lorentz-invariant eigenvalues!

Now we shall investigate the canonical classification of an $N \times N$ matrix $[M]$ with real entries $M_{(a)(b)}$. The (usual) characteristic polynomial is furnished by

$$ p(\lambda) := \det \left[ M_{(a)(b)} - \lambda \delta_{(a)(b)} \right] = (-\lambda)^N + c_{(1)} \lambda^{N-1} + \cdots + c_{(N)} = 0, $$

or,

$$ \det \left[ \delta^{(a)(c)} M_{(c)(b)} - \lambda \delta^{(a)(b)} \right] = p(\lambda) = 0. $$

The equation above admits $2s \geq 0$ complex roots and $N - 2s \geq 0$ real roots [21].

Now consider the same real $N \times N$ matrix $[M]$, together with an $N \times N$ metric matrix $[D]$ of (1.90) (which is not positive definite). The corresponding invariant, characteristic polynomial equation is provided by

$$ p^\#(\lambda) := \det \left[ M_{(a)(b)} - \lambda \delta_{(a)(b)} \right] = (-\lambda)^N + c_{(1)}^{\#} \lambda^{N-1} + \cdots + c_{(N)}^{\#} = 0, $$

or,

$$ \det \left[ \delta^{(a)(c)} M_{(c)(b)} - \lambda \delta^{(a)(b)} \right] = 0. $$

Suppose that the above equation admits $2s^\# > 0$ complex roots and $N - 2s^\# > 0$ real roots. These invariant roots must differ from the roots of the usual characteristic equation $p(\lambda) = 0$. (Compare with Example A3.6.) However, the subsequent general theorems involving any real $N \times N$ matrix $[M] = [M^i_j]$ apply to both distinct cases $p(\lambda) = 0$ and $p^\#(\lambda) = 0$. In the case where there exist only real roots of $\det[M^i_j - \lambda \delta^i_j] = 0$ as eigenvalues, the Jordan decomposition theorem is stated as follows:

**Theorem A3.7.** Let $[M] = [M^i_j]$ be an $N \times N$ matrix with real entries and real eigenvalues as roots of $\det[M^i_j - \lambda \delta^i_j] = 0$. Then the matrix can be transformed by a similarity transformation into the following block diagonal form:
Here, \([A_{(l)}]\), for \(l \in \{1, \ldots, k\}\), is an \(n_{(l)} \times n_{(l)}\) matrix given by

\[
\begin{bmatrix}
A_{(1)} & 0 \\
& \ddots \\
0 & 0 & A_{(k)}
\end{bmatrix}.
\] (A3.1)

Proof of the above theorem is available in [133, 177].

Remarks.  
(i) It is usually assumed that \(n_{(1)}^{(1)} \geq n_{(1)}^{(2)} \geq \cdots \geq n_{(1)}^{(q_1)} \geq 1\).

(ii) In the case \(n_{(1)}^{(i)} = 1\), the corresponding \(1 \times 1\) matrix \(J_{(l)}^{(i)}\) has the single entry \(\lambda_{(l)}\). It stands for a diagonal element.

(iii) Some authors write \(J_{(l)}^{(i)}\) as a lower triangular form.

(iv) In relativity theory, after solving \(p^\#(\lambda) = 0\), many authors use the inequalities in the reversed order as

\[
n_{(2)}^{(q_1)} \geq \cdots \geq n_{(1)}^{(2)} \geq n_{(1)}^{(1)} \geq 1.
\]
(v) The Segre characteristic of the matrix \([M]_{(J)}\) is denoted ([90, 210]) by the symbol \([n_1^{(1)}, n_2^{(1)}, \ldots], (n_1^{(2)}, n_2^{(2)}, \ldots), \ldots\].

The matrices \([M]_{(J)}\) in (A3.1) and (A3.2i,ii) define a hierarchy of elementary divisors [113]. It is provided by the following string of equations:

\[
E_{(N)}(\lambda) := (\lambda - \lambda_1^{(1)})^{n_1^{(1)}} \cdot (\lambda - \lambda_2^{(2)})^{n_2^{(2)}} \cdots ,
\]

\[
E_{(N-1)}(\lambda) := (\lambda - \lambda_1^{(2)})^{n_1^{(2)}} \cdot (\lambda - \lambda_2^{(2)})^{n_2^{(2)}} \cdots ,
\]

............... (A3.3)

**Example A3.8.** We shall discuss an example from the general theory of relativity. Consider the orthonormal components

\[
t_{(a)(b)} := T_{(a)(b)}(x_0) = t_{(b)(a)}
\]

of the energy–momentum–stress tensor in (2.45) and (2.161ii) (at a particular event \(x_0 \in D \subset \mathbb{R}^4\)). The usual eigenvalues of the symmetric matrix \([t_{(a)(b)}]\) are all real, and the matrix is always diagonalizable. However, the Lorentz invariant eigenvalues of \([t_{(a)(b)}]\) are, physically speaking, much more relevant. Since \(\det [t_{(a)(b)} - \lambda \delta_{(a)(b)}] = 0 \iff \det [d^{(a)(c)}t_{(c)(b)} - \lambda \delta^{(a)}_{(b)}] = 0\), these invariant eigenvalues are identical with the usual eigenvalues of the 4 × 4 matrix \([\Theta] := [d^{(a)(c)}t_{(c)(b)}]\) which need not be symmetric. (See Examples A3.5 and A3.6.) Assuming that the invariant eigenvalues are all real, we classify the allowable types of canonical forms of \([\Theta]\) in the following:

I: Type-I(a):

\[
[\Theta]_{(J)} = \begin{bmatrix}
\lambda_1^{(1)} & 0 & 0 & 0 \\
0 & \lambda_2^{(2)} & 0 & 0 \\
0 & 0 & \lambda_3^{(3)} & 0 \\
0 & 0 & 0 & -\lambda_4^{(4)}
\end{bmatrix}.
\]

(A3.4)

Here, if \(\lambda_1^{(1)}, \lambda_2^{(2)}, \lambda_3^{(3)}, -\lambda_4^{(4)}\) are all distinct. The Segre characteristic in (A3.4) is specified by [1, 1, 1, 1].

Type-I(b): If \(\lambda_1^{(1)} = \lambda_2^{(2)}\) and \(\lambda_1^{(1)}, \lambda_3^{(3)}, -\lambda_4^{(4)}\) are distinct. The Segre characteristic is specified by [(1, 1), 1, 1].

Type-I(c): If \(\lambda_1^{(1)} = \lambda_2^{(2)} = -\lambda_4^{(4)}\) and \(\lambda_1^{(1)} \neq \lambda_3^{(3)}\). The Segre characteristic is provided by [(1, 1), (1, 1)].

Type-I(d): If \(\lambda_1^{(1)} = \lambda_2^{(2)} = \lambda_3^{(3)}\) and \(\lambda_1^{(1)} \neq -\lambda_4^{(4)}\). The Segre characteristic is furnished by [(1, 1, 1, 1)].
Type-I: If $\lambda_1 = \lambda_2 = \lambda_3 = -\lambda_4$. The Segre characteristic is specified by $[(1, 1, 1, 1)]$.

II: Type-II(a):

$$\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 1 \\
0 & 0 & 0 & \lambda_3
\end{bmatrix}.$$

(A3.5)

If $\lambda_1, \lambda_2$, and $\lambda_3$ are all distinct. The Segre characteristic is given by $[1, 1, 2]$.

Type-II(b): If $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_3$ in (A3.5). The Segre characteristic is provided by $[(1, 1), 2]$.

Type-II(c): Here, $\lambda_1 \neq \lambda_2$ and $\lambda_2 = \lambda_3$. The Segre characteristic is furnished by $[1, (1, 2)]$.

Type-II(d): If $\lambda_1 = \lambda_2 = \lambda_3$. The Segre characteristic is specified by $[(1, 1, 2)]$.

III: Type-III(a):

$$\begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 1 \\
0 & 0 & 0 & \lambda_2
\end{bmatrix}.$$

(A3.6)

Here, we assume $\lambda_1 \neq \lambda_2$. The Segre characteristic is $[2, 2]$.

Type-III(b): If $\lambda_1 = \lambda_2$ in (A3.6). The Segre characteristic is provided by $[(2, 2)]$.

IV: Type-IV(a):

$$\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 1 & 0 \\
0 & 0 & \lambda_2 & 1 \\
0 & 0 & 0 & \lambda_2
\end{bmatrix}.$$

(A3.7)

Here, we assume that $\lambda_1 \neq \lambda_2$. The Segre characteristic is $[1, 3]$. 

Appendix 3  Canonical Forms of Matrices

Type-IV\(_{(b)}\): Here, we assume that \(\lambda_{(1)} = \lambda_{(2)}\). The corresponding Segre characteristic is \([(1, 3)]\).

\[
[\Theta](J) = \begin{bmatrix}
\lambda_{(1)} & 1 & 0 & 0 \\
0 & \lambda_{(1)} & 1 & 0 \\
0 & 0 & \lambda_{(1)} & 1 \\
0 & 0 & 0 & \lambda_{(1)}
\end{bmatrix}.
\]  

(A3.8)

The Segre characteristic is furnished by [4]. The corresponding eigenvectors are quadrupole null vectors along a single null eigendirection.

Since the signature of the Lorentz metric \([d_{(a)(b)}]\) is +2 and \(t_{(b)(a)} = t_{(a)(b)}\), the 14 cases in the equations above in this example contain all possible Segre characteristics of the energy–momentum–stress tensor (with real Lorentz-invariant eigenvalues). (See [90, 210].) Furthermore, it can be noted that an eigenvector associated with a nonsimple, elementary divisor is null.

Now we shall deal with a real \(N \times N\) matrix \([M] = [M^T]\), which possesses only \(2s = N\) complex conjugate roots of the characteristic equation. (Thus, in this case, there are no real roots.) The following theorem elaborates the canonical form of such a matrix:

**Theorem A3.9.** Let \([M]\) be a real \(N \times N\) matrix such that the (usual) characteristic equation admits \(2s = N\) complex conjugate roots \(\lambda_{(l)} = a_{(l)} + ib_{(l)}, \quad \bar{\lambda}_{(l)} = a_{(l)} - ib_{(l)}, \quad l \in \{1, \ldots, s\}\). Then, the Jordan canonical form of the matrix is furnished by:

\[
[M](J) = \begin{bmatrix}
B_{(1)} & \quad & \\
& \ddots & \quad \\
& & \quad \\
\vdots & & \ddots \end{bmatrix},
\]

\[
[B_{(l)}] := \begin{bmatrix}
C_{(l)}^{(1)} & \quad & \\
& \ddots & \quad \\
& & \quad \\
\vdots & & \ddots \end{bmatrix},
\]

\[(2n_{(l)} \times 2n_{(l)})\]
The proof of this theorem can be found in [133, 177]. The Segre characteristic of the matrix \( [M]_{(j)} \), in (A3.9) is given by

\[
\begin{bmatrix}
    n_{(1)}^{(1)}, n_{(1)}^{(2)}, \ldots, \left(n_{(1)}^{(i)}, n_{(1)}^{(2)}, \ldots\right); \left(n_{(2)}^{(1)}, n_{(2)}^{(2)}, \ldots\right); \ldots
\end{bmatrix}.
\]

Finally, we consider an \( N \times N \) real matrix \( [M] \) which admits both real and complex conjugate (usual) eigenvalues.

The canonical or block diagonal form is provided by

\[
[M]_{(j)} = \begin{bmatrix}
    A_{(1)} & & & & \\
    & A_{(2)} & & & \\
    & & \ddots & & \\
    & & & A_{(k)} & \\
    & & & & B_{(1)} \\
    & & & & & B_{(2)} \\
    & & & & & & \ddots \\
    & & & & & & & & B_{(r)}
\end{bmatrix}.
\] (A3.10)

Here, matrices \( [A_{(i)}] \) and \( [B_{(i)}] \) are furnished by (A3.2i) and (A3.9), respectively.
**Example A3.10.** Consider the $4 \times 4$ matrix

\[
[M] := \begin{bmatrix}
-1/2 & -1/2 & 0 & 0 \\
1/2 & -3/2 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}.
\]

The usual characteristic polynomial equation is furnished by

\[
p(\lambda) = (\lambda + 1)^2 \cdot (\lambda^2 - 4\lambda + 5) = (\lambda + 1)^2 \cdot (\lambda - 2 - i) \cdot (\lambda - 2 + i) = 0.
\]

Thus, the distinct eigenvalues are

\[
\lambda_{(1)} = -1, \quad a_{(1)} + ib_{(1)} = 2 + i, \quad a_{(1)} - ib_{(1)} = 2 - i
\]

with multiplicities 2, 1, and 1, respectively. Therefore, the Jordan canonical form is given by the block diagonal form

\[
[M]_{(J)} = \begin{bmatrix}
A_{(1)} \\
B_{(1)}
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
0 & -1 \\
2 & 1 \\
-1 & 2
\end{bmatrix}.
\]

The Segre characteristic is $[2; 1, \bar{1}]$. The elementary divisor for this matrix is $E_{(4)}(\lambda) = (\lambda + 1)^2 \cdot (\lambda - 2 - i) \cdot (\lambda - 2 + i)$. □

**Example A3.11.** Consider a domain of the space–time manifold and orthonormal components of the energy–momentum–stress tensor field provided by \[126\]:

\[
[T_{(a)(b)}(x)] := \begin{bmatrix}
\Lambda_{(1)}(x) & 0 & 0 & 0 \\
0 & \Lambda_{(2)}(x) & 0 & 0 \\
0 & 0 & \Sigma(x) & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \tag{A3.11i}
\]

\[
[t_{(a)(b)}] := [T_{(a)(b)}(x_0)] = \begin{bmatrix}
\lambda_{(1)} & 0 & 0 & 0 \\
0 & \lambda_{(2)} & 0 & 0 \\
0 & 0 & \sigma & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \tag{A3.11ii}
\]
Four eigenvalues (invariant values for the matrix in (A3.11iii) or usual values of the matrix in (A3.11iii)) are furnished by $\lambda_{(1)}$, $\lambda_{(2)}$, $\lambda_{(3)} = (1/2) \left[ \sigma + \sqrt{\sigma^2 - 4} \right]$, and $\lambda_{(4)} = (1/2) \left[ \sigma - \sqrt{\sigma^2 - 4} \right]$. Here, $\lambda_{(1)}$ and $\lambda_{(2)}$ are real and $\lambda_{(3)}, \lambda_{(4)}$ are real provided $|\sigma| \geq 2$. However, the eigenvectors in this case can be spacelike! Therefore, for the case $\lambda_{(1)} \neq \lambda_{(2)}$ and $|\sigma| > 2$, we have the Segre characteristic $[1, 1, 1, 1]$. In the case $\lambda_{(1)} \neq \lambda_{(2)}$ and $|\sigma| = 2$, the Segre characteristic is $[1, 1, (1, 1)]$. But for $\lambda_{(1)} \neq \lambda_{(2)}$ and $|\sigma| < 2$, the Segre characteristic is $[1, 1; 1, 1]$. Moreover, there are corresponding Segre characteristics for the coincidence $\lambda_{(1)} = \lambda_{(2)}$. This example illustrates that the energy–momentum–stress tensor of an exotic material can change Segre characteristics from domain to domain!

**Example A3.12.** In Example A3.8, we considered invariant eigenvalues of the symmetric energy–momentum–stress matrix $[\Theta]$ := $[T_{(a)(b)}] := [T_{(a)(b)}(x_0)]$. In that example, we restricted the classification only to the various cases of real, invariant eigenvalues. In this example, we extend the mathematical investigation to cases of complex, invariant eigenvalues of $[T_{(a)(b)}]$. (Recall that these eigenvalues are identical to the usual eigenvalues of the nonsymmetric matrix $[\Theta] = [\Theta_{(a)}] := [T^{(a)(c)}t_{(c)(b)}].$

The Jordan canonical forms of various types follow from (A3.9). The type VI is furnished below:

**VI**: Type-VI$_{(a)}$: 

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & a_{(1)} & b_{(1)} \\ 0 & 0 & -b_{(1)} & a_{(1)} \end{bmatrix}. \quad (A3.12)$$

Here, we assume that real eigenvalues $\lambda_{(1)} \neq \lambda_{(2)}$ and the real number $b_{(1)} \neq 0$. The corresponding Segre characteristic is provided by $[1, 1; 1, 1]$. Type-VI$_{(b)}$: If $\lambda_{(1)} = \lambda_{(2)}$ and $b_{(1)} \neq 0$, the Segre characteristic is given by $[(1, 1); 1, 1]$. 

**VII**: Type-VII$_{(a)}$: 

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 0 & 0 \\ 0 & 0 & a_{(1)} & b_{(1)} \\ 0 & 0 & -b_{(1)} & a_{(1)} \end{bmatrix}. \quad (A3.13)$$
Here, we take the real number $b_{(1)} \neq 0$. The Segre characteristic is furnished by $[2; 1, \bar{T}]$.

\[
[\Theta](J) = \begin{bmatrix}
  a_{(1)} & b_{(1)} & 0 & 0 \\
  -b_{(1)} & a_{(1)} & 0 & 0 \\
  0 & 0 & a_{(2)} & b_{(2)} \\
  0 & 0 & -b_{(2)} & a_{(2)}
\end{bmatrix}.
\] (A3.14)

Here, we have taken $b_{(1)} \neq 0$, $b_{(2)} \neq b_{(1)}$, and $b_{(2)} \neq 0$. The Segre characteristic is $[1, \bar{T}; 1, \bar{T}]$. Type-VIII(b): Here, we assume that real numbers $a_{(2)} = a_{(1)}$ and $b_{(2)} = b_{(1)} \neq 0$. The corresponding Segre characteristic is $[(1, \bar{T}; 1, \bar{T})]$.

Every type of $[\Theta](J)$ exhibited in the previous example (involving energy–momentum–stress tensor components $T_{(a)(b)}(x_0)$) violates energy conditions in (2.190)–(2.192). Therefore, in the usual conditions of the macroscopic universe, these types of energy–momentum–stress tensor components are usually deemed unphysical. However, in the quantum realm, it may be possible to violate energy conditions. An example of this is an effect known as the Casimir effect where quantum fields subject to certain boundary conditions may violate energy conditions (see, e.g., [236, 254] and references therein). Exotic solutions in general relativity usually possess energy condition violation, as may be seen in Appendix 6 and Sect. 5.3, as well as [68] where energy condition violation is shown to result inside a black hole in a certain class of gravitational collapse models.
Appendix 4
Conformally Flat Space–Times and “the Fifth Force”

Let us recall the criterion for conformal flatness of a Riemannian (or a pseudo-Riemannian) manifold discussed in Theorem 1.3.33. We reiterate the same theorem with different notations in the following theorem.

**Theorem A4.1.** Let \( D \subseteq \mathbb{R}^N \) (with \( N > 3 \)) be a domain corresponding to an open subset of an \( N \)-dimensional Riemannian (or a pseudo-Riemannian) manifold of differentiablility class \( C^3 \). Then, the domain \( D \subseteq \mathbb{R}^N \) is conformally flat if and only if the tensor field \( C_{jkl}^i(x) \frac{\partial}{\partial x^j} \otimes dx^j \otimes dx^k \otimes dx^l = O(\ldots)(x) \) in \( D \).

Proof of the above theorem is provided in [56, 90, 171].

Consider a conformally flat domain of a pseudo-Riemannian (or a Riemannian) manifold \( M_N \). It is endowed with a coordinate chart such that

\[
g_{ij}(x) = \exp [2\nu(x)] \cdot d_{ij}.
\]

We consider another chart, intersecting the preceding one, such that

\[
\widehat{g}_{ij}(\widehat{x}) = \exp [2\widehat{\mu}(\widehat{x})] \cdot \hat{d}_{ij}.
\]

Let us consider the following set of coordinate transformations furnished by

\[
\widehat{x}^i = \lambda \cdot x^i, \quad \lambda \neq 0, \quad (A4.3i)
\]

\[
\widehat{x}^i = x^i + c^i, \quad (A4.3ii)
\]

\[
\widehat{x}^i = l^i_j x^j, \quad d_{ij} \cdot l^i_k \cdot l^j_m = d_{jm}, \quad (A4.3iii)
\]

\[
\widehat{x}^i = \left[ x^i / (d_{jk} \cdot x^j x^k) \right], \quad (A4.3iv)
\]
Here, $\lambda, c^i, l^j_i$, and $b^i$ are constant-valued parameters. Moreover, $x_i := d_{ij} x^j$, and $x_i x^i$ is assumed to be nonzero.

The set of coordinate transformations presented above constitutes a group. It is called the conformal group $\mathcal{C}(p, n; \mathbb{R})$ (with $p + n = N$). This group involves $(1/2) \cdot (N + 1) \cdot (N + 2)$ independent parameters. The following theorem provides the importance of this group in regard to conformally flat manifolds.

**Theorem A4.2.** A conformally flat metric $g_{ij}(x) = \exp[2\nu(x)] \cdot d_{ij}$ goes over into another conformally flat metric $\tilde{g}_{ij}(x) = \exp[2\tilde{\nu}(x)] \cdot d_{ij}$ for $N \geq 3$ if and only if the coordinate transformation belongs to the conformal group $\mathcal{C}(p, n; \mathbb{R})$.

Partial proof of the above theorem is provided in [56].

We shall now state Willmore’s theorem [266] for a conformally flat domain of a pseudo-Riemannian (or a Riemannian) manifold.

**Theorem A4.3.** Let a domain $D \subset \mathbb{R}^N$ with $N > 3$, corresponding to a domain of a pseudo-Riemannian (or a Riemannian) manifold of differentiability class $C^3$, admit a tensor field $S_{ij}(x)$ of differentiability class $C^2$. Moreover, let the following tensor field equations hold:

\[
R_{ijk}(x) = g_{hk}(x) \cdot S_{ij}(x) + g_{ij}(x) \cdot S_{hk}(x) - g_{hj}(x) \cdot S_{ik}(x) - g_{ik}(x) \cdot S_{hj}(x). \tag{A4.4}
\]

(i) The conformal tensor $C_{ijk}(x) = O_{ijk}(x)$ in $D$. (ii) Moreover, $S_{ji}(x) \equiv S_{ij}(x)$ in $D$.

Proof of the above theorem is skipped here. However, the next theorem due to Das [51] incorporates the proof of the preceding theorem.

**Theorem A4.4.** Let the curvature tensor $R_{ijk}(x)$ of a pseudo-Riemannian (or a Riemannian) manifold of differentiability class $C^3$ and dimension $N > 3$ admit a twice-differentiable tensor field $T_{ij}(x)$ in the domain of consideration. Moreover, let the following tensor field equations hold:

\[
R_{ijk}(x) = \kappa \cdot (N - 2)^{-1} \cdot \left\{ g_{lj}(x) \cdot T_{ik}(x) + g_{ik}(x) \cdot T_{lj}(x) - g_{lk}(x) \cdot T_{ij}(x) - g_{ij}(x) \cdot T_{lk}(x) \right\} + 2(N - 1)^{-1} \cdot T(x) \cdot \left[ g_{ik}(x) \cdot g_{ij}(x) - g_{lj}(x) \cdot g_{ik}(x) \right],
\]

\[T(x) := g^{ij}(x) \cdot T_{ij}(x). \tag{A4.5}\]
(Here, \( \kappa \) is an arbitrary non-zero constant). Then,

(i) \( G_{ij}(x) = -\kappa \cdot T_{ij}(x) \),

(ii) \( T_{ji}(x) \equiv T_{ij}(x) \); \( \nabla_i T^{ij} = 0 \),

(iii) \( \nabla_k T_{ij} - \nabla_j T_{ik} + (N - 1)^{-1} \left[ g_{ik}(x) \cdot \nabla_j T - g_{ij}(x) \cdot \nabla_k T \right] = 0 \),

(iv) \( C^l_{ijk}(x) = 0 \).

(v) Moreover,
\[
R_{ijk}(x) := (N - 2)^{-1} \cdot (N - 3) \cdot \nabla_l C^l_{ijk} = 0.
\]

Proof. (i) By the single contraction in (A4.5), it follows that
\[
R_{ij}(x) = \kappa \cdot \left[ (N - 2)^{-1} \cdot T(x) \cdot g_{ij}(x) - T_{ij}(x) \right],
\]
\[
R(x) = 2\kappa \cdot (N - 2)^{-1} \cdot T(x),
\]
\[
G_{ij}(x) = -\kappa \cdot T_{ij}(x).
\]

(ii) By the algebraic symmetry of Einstein’s tensor and the differential identities \( \nabla_i G^{ij} \equiv 0 \), (A4.6ii) follow.

(iii) By the first contracted Bianchi’s identities (1.150i), (A4.6ii), and (A4.5), the proof of (A4.6iii) follows.

(iv) Substituting \( R_{ijk}(x) \) from (A4.5), using (A4.7i,ii), (1.169i) yields (A4.6iv).

(v) The covariant differentiation of \( C^l_{ijk}(x) \) in (1.169i), together with (A4.6iv), implies (A4.6v).

In case we would like to apply the preceding theorem to general relativity, we must choose \( p = 3 \), \( n = 1 \), and \( N = 4 \) for the pseudo-Riemannian space–time manifold. The theorem, that is suited to general relativity is stated and proved in the following:

**Theorem A4.5.** Let the curvature tensor \( R^{\cdot \cdot \cdot \cdot}(x) \) of the pseudo-Riemannian space–time manifold of differentiability class \( C^3 \) admit a twice differentiable tensor field \( T_{\cdot \cdot \cdot}(x) \) in the domain \( D \subset \mathbb{R}^4 \) of consideration. Moreover, let the following tensor field equation hold:
\[ R_{ijkl}(x) = (\kappa/2) \cdot \left\{ \left[ g_{lj}(x) \cdot T_{lk}(x) + g_{ik}(x) \cdot T_{lj}(x) ight. \\
- g_{lk}(x) \cdot T_{ij}(x) - g_{ij}(x) \cdot T_{lk}(x) \left. \right] + (2/3) \cdot T(x) \cdot \left[ g_{lk}(x) \cdot g_{ij}(x) - g_{lj}(x) \cdot g_{ik}(x) \right]\right\}. \quad \text{(A4.8)} \]

Then, the following implications for \( g_{ij} = \phi^2 \cdot d_{ij}, \Box \phi := d_{ij} \partial_i \partial_j \phi \), and \( \widetilde{T}_{(0)} := d_{ij} \widetilde{T}_{ij} \) hold true:

(i) \[ \partial_i \partial_j \phi - d_{ij} \cdot \Box \phi - 2\phi^{-1} \left[ \partial_i \phi \cdot \partial_j \phi - (1/4) d_{ij} d^{kl} \partial_k \phi \cdot \partial_l \phi \right] = -(\kappa/2) \cdot T_{ij}(x) \cdot \phi(x). \quad \text{(A4.9)} \]

(ii) \[ R(x) = 6\phi^{-3} \cdot \Box \phi. \quad \text{(A4.10)} \]

(iii) \[ R_{ij}(x) = 2\phi^{-1} \cdot \partial_i \partial_j \phi + d_{ij} \phi^{-1} \cdot \Box \phi - 4\phi^{-2} \cdot \partial_i \phi \cdot \partial_j \phi \]

\[ + \phi^{-2} d_{ij} d^{kl} \partial_k \phi \cdot \partial_l \phi = -\kappa \widetilde{T}_{ij} =: -\kappa \left[ T_{ij} - (1/2) d_{ij} d^{kl} T_{kl} \right]. \quad \text{(A4.11)} \]

(iv) \[ \Box \phi + (\kappa/6) \cdot \widetilde{T}_{(0)}(x) \cdot \phi(x) = 0. \quad \text{(A4.12)} \]

(v) Field equations (A4.11) are covariant under the conformal group \( C(3, 1; \mathbb{R}) \) involving 15 independent parameters.

**Proof.** (i) The proof follows by noting that \( g_{ij}(x) = [\phi(x)]^2 \cdot d_{ij} \) and inserting this metric into the usual expression for \( R_{ij}(x) \). Then, using (A4.7ii) and (1.149ii), the following equations emerge:

\[ G_{ij}(x) = 2\phi^{-1} \cdot \partial_i \partial_j \phi - 4\phi^{-2} \cdot \partial_i \phi \cdot \partial_j \phi \]

\[ - d_{ij} \cdot \left[ 2\phi^{-1} \cdot \Box \phi - \phi^{-2} \cdot d^{kl} \partial_k \phi \cdot \partial_l \phi \right] = -\kappa \cdot T_{ij}(x). \quad \text{(A4.13)} \]

Next, from Einstein’s field equations (A4.7ii), (A4.9) is derived.

(ii) Double contraction of (A4.13) leads the (A4.10).

(iii) By the equality \( R_{ij}(x) = G_{ij}(x) + (1/2) \cdot g_{ij}(x) \cdot R(x) \), and (A4.9), (A4.10) and definitions

\[ \widetilde{T}_{ij}(x) := T_{ij}(x) - (1/2) g_{ij}(x) \cdot T^k_k(x), \quad \text{(A4.14i)} \]

\[ \widetilde{T}_{(0)}(x) := d^{ij} \widetilde{T}_{ij}(x) = -T_{(0)}(x) =: -d^{ij} T_{ij}(x), \quad \text{(A4.14ii)} \]

(A4.11) is deduced.

(iv) Double contractions of equations (A4.11) (with \( d^{ij} \)) lead to (A4.12).

(v) This statement can be proved by using Theorem A4.2.
Remarks. (i) It is clear from (A4.5) and (A4.8) that $T_{ij}(x) \equiv 0$ implies that $R_{ijk}(x) \equiv 0$. In other words, mathematically, the support of $R_{ijk}(x)$ is the support of $T_{ij}(x)$ in conformally flat space–times. Therefore, the class of gravitational phenomena governed by field equation (A4.8) or (A4.9) is nontrivial only in the presence of material sources. Outside material sources, this class of gravitational forces just “switches off”! That is why we interpret gravitational force, arising from field equation (A4.9) as the “fifth force”.1 (See [101, 216].) This effect can equivalently be seen utilizing (1.169i) and field equation (2.163i). The trace of the latter implies that $R(x) = 0$ and $R_{ij}(x) = 0$ when $T_{ij}(x) = 0$ holds. Then, condition (A4.6iv) for conformally flat space–times for $N > 3$, when used in (1.169i), implies that $R_{ijkl}(x) = 0$ whenever $T_{ij}(x) = 0$.

(ii) Field equation (A4.9), governing the “fifth force”, does not admit general covariance. However, this equation does admit covariance under the 15-parameter conformal group $C(3, 1; \mathbb{R})$.

(iii) The system of field equations (A4.9) is highly overdetermined. (See [52].)

In spite of the fact that the system of field equations is overdetermined, many exact solutions of the system do exist. In fact, many of these exact solutions turn out to be extremely important for understanding of the cosmological universe. We shall furnish some of these exact solutions in the following examples.

Example A4.6. In this example, the following choice is made:

$$T_{ij}(x) := -(3K_0/\kappa) \cdot g_{ij}(x).$$

$$T(x) = -12 \cdot \kappa^{-1} \cdot K_0.$$  \hspace{1cm} (A4.15)

(Here, $K_0$ is a constant.)

Substituting (A4.15) into (A4.8), we obtain

$$R_{hijk}(x) = K_0 \left[ g_{hj}(x) \cdot g_{ki}(x) - g_{hk}(x) \cdot g_{ji}(x) \right].$$  \hspace{1cm} (A4.16)

Therefore, by (1.164i), the metric is that of a space–time of constant curvature. Such a space–time is also called the de Sitter universe (for $K_0 > 0$) and anti-de Sitter universe (for $K_0 < 0$). (See #1 of Exercise 6.1. Also see [126].)

By Theorem 1.3.30, we can transform this metric locally to the conformally flat form:

$$ds^2 = \left[ 1 + (K_0/4) \cdot (d_{kl} \cdot x^k x^l) \right]^{-2} \cdot d_{ij} \cdot dx^i dx^j.$$  \hspace{1cm} (A4.17)

By the equation $g_{ij}(x) = [\phi(x)]^2 \cdot d_{ij}$, we obtain in this example

$$\phi(x) = \left[ 1 + (K_0/4) \cdot (d_{ij} \cdot x^i x^j) \right]^{-1}.$$  \hspace{1cm} (A4.18)

Thus, an explicit expression for $\phi(x)$ is furnished. \hfill \square

1It should be stressed here that this is nomenclature only. This phenomenon is still a special class of gravitational effect, and not due to some new force.
The next example deals with F–L–R–W cosmological metrics of (6.7i) and (6.41). These are extremely important metrics in regard to the proper understanding of the cosmological universe.

Example A4.7. In this example, we choose the energy–momentum–stress tensor for a perfect fluid from Theorem 6.1.3 and (6.46). It is provided by

\[
T_{ij}(x) = \left[ \rho(x) + p(x) \right] \cdot U_i(x) \cdot U_j(x) + \rho(x) \cdot g_{ij}(x),
\]

(A4.19i)

\[
U^i(x) \cdot U_i(x) \equiv -1.
\]

(A4.19ii)

Recall the F–L–R–W metric from (6.41) as

\[
ds^2 = \left[ a(x^4) \right]^2 \cdot \left[ 1 + (k_0/4) \cdot \delta_{\alpha\beta} \cdot x^\alpha x^\beta \right]^{-2} \cdot \delta_{\mu\nu} \cdot dx^\mu dx^\nu - \left( dx^4 \right)^2,
\]

\[
D := \{ x : x \in D \subset \mathbb{R}^3; x^4 > 0 \},
\]

\[
k_0 \in \{ 0, 1, -1 \}.
\]

(A4.20)

The corresponding fluid velocity components, in the comoving frame (derivable from field equations \(E_{\alpha4}(\cdot) = 0\)) are given by

\[
U^\alpha(x) \equiv 0, \quad U^4(x) \equiv 1.
\]

(A4.21)

Einstein’s field equations from (6.47i–iii) are furnished by

\[
\mathcal{E}_1(\cdot) \equiv \mathcal{E}_2(\cdot) \equiv \mathcal{E}_3(\cdot) = \left[ \frac{2\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + k_0 \right] + \kappa \ p(x^4) = 0,
\]

(A4.22i)

\[
\mathcal{E}_4(\cdot) = 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + k_0 \right] - \kappa \ p(x^4) = 0,
\]

(A4.22ii)

\[
\mathcal{T}^4(\cdot) = \dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) \cdot (\rho + p) = 0,
\]

(A4.22iii)

\[
\ddot{a} := \frac{\text{d}a(x^4)}{\text{d}x^4}, \ etc.
\]

(A4.22iv)

In the case of \(k_0 = 0\), the metric from (A4.20) goes over into

\[
ds^2 = \left[ a(x^4) \right]^2 \cdot \left[ \delta_{\alpha\beta} \cdot dx^\alpha dx^\beta \right] - \left( dx^4 \right)^2.
\]

(A4.23)
(Consult Example 7.2.2.) Now we make a transformation,

$$\tilde{x}^\alpha = x^\alpha,$$

$$\tilde{x}^4 = \int \frac{dx^4}{a(x^4)},$$

$$\tilde{a}(\tilde{x}^4) := a(x^4) > 0. \tag{A4.24}$$

The metric in (A4.23) yields the conformally flat form:

$$ds^2 = [\tilde{a}(\tilde{x}^4)]^2 \cdot \left( \delta_{\alpha\beta} \cdot d\tilde{x}^\alpha d\tilde{x}^\beta - (d\tilde{x}^4)^2 \right),$$

$$=: [\tilde{\phi}(\tilde{x})]^2 \cdot d\tilde{\theta} \cdot d\tilde{\phi}. \tag{A4.25}$$

Therefore, the corresponding $\tilde{\phi}(\tilde{x})$ field is explicitly furnished by

$$\tilde{\phi}(\tilde{x}) = \tilde{a}(\tilde{x}^4). \tag{A4.26}$$

For the case of $k_0 = 1$, (6.7i), (A4.20), and the answer to # 2i of Exercise 6.1 yield

$$ds^2 = [\tilde{a}(\tilde{x}^4)]^2 \cdot \left( (d\tilde{\chi})^2 + \sin^2 \tilde{\chi} \cdot \left[ (d\tilde{\theta})^2 + \sin^2 \tilde{\theta} \cdot (d\tilde{\phi})^2 \right] - (d\tilde{x}^4)^2 \right). \tag{A4.27}$$

Now we make another coordinate transformation by

$$r^\# = \frac{2 \sin \tilde{\chi}}{\cos \tilde{\chi} + \cos \tilde{x}^4},$$

$$\tilde{x}^4 = \frac{2 \sin \tilde{x}^4}{\cos \tilde{\chi} + \cos \tilde{x}^4},$$

$$\left( \tilde{\theta}^\#, \tilde{\phi}^\# \right) = \left( \tilde{\theta}, \tilde{\phi} \right),$$

$$\cos \tilde{\chi} + \cos \tilde{x}^4 \neq 0;$$

$$a^\#(x^{4\#}) := \tilde{a}(\tilde{x}^4) > 0. \tag{A4.28}$$

The corresponding conformally flat metric from (A4.27) emerges as

$$ds^2 = \frac{1}{4} \cdot [a^\#(x^{4\#})]^2 \cdot \left[ \cos \chi^\# + \cos x^{4\#} \right]^2$$

$$\times \left\{ (d\chi)^2 + (r^\#)^2 \cdot \left[ (d\theta)^2 + \sin^2 \theta^\# \cdot (d\phi)^2 \right] - (dx^{4\#})^2 \right\},$$

$$=: [\phi^\#(x^\#)]^2 \cdot \left\{ (dr)^2 + (r^\#)^2 \cdot \left[ (d\theta)^2 + \sin^2 \theta^\# \cdot (d\phi)^2 \right] - (dx^{4\#})^2 \right\}. \tag{A4.29}$$
Therefore, the corresponding field $\phi^\#(x^\#)$ is analytically furnished by

$$
\phi^\#(x^\#) = \frac{1}{2} \cdot [\alpha^\#(x') \cdot [\cos \chi^\# + \cos x']].
$$

(A4.30)

Thus, in an important model of cosmology, an analytic expression for the $\phi^\#(x^\#)$ field is explicitly provided.
In this appendix we briefly review the linearized theory of gravitation and discuss a class of solutions known as gravitational waves. There are currently several major experiments in progress which hope to detect gravitational waves in the near future. If they are successful, a new arena, known as gravitational wave astronomy, will potentially yield insight into many interesting astrophysical phenomena. The subject of gravitational waves and the detection of gravitational waves is vast, and we can only touch upon the subject here. The interested reader is referred to [33, 103, 172, 230], and references therein.

Although wave solutions also exist in the full nonlinear theory, as mentioned in Chap. 7, we shall study here the more physically relevant scenario where the gravitational field is weak. In such a case the metric may be written as a perturbation about the flat space–time metric:

$$g_{ij}(x) = d_{ij} + \epsilon h_{ij}(x) + \mathcal{O}(\epsilon^2),$$

(A5.1)

with $\epsilon$, a small parameter such that terms of order $\epsilon^2$ or higher may be ignored.$^1$ (We will use equalities below instead of approximate equalities, and it is understood that this implies “equal to order $\epsilon$.”) The flat space–time metric is known as the background metric. Of course, one could also perturb a non-flat solution to the Einstein field equations (in fact, many interesting phenomena do indeed consider non-flat backgrounds); however, (A5.1) will suffice for our purpose.

In the linear approximation, the Einstein equations are computed utilizing metric (A5.1), and terms to order $\epsilon$ are retained. The conjugate metric tensor components $g^{ij}(x)$ from (A5.1) are furnished by

---

$^1$(1) Here, $h_{ij}(x)$ will be used to denote the perturbation of the metric. This notation should not be confused with the one of $h_{ij}(x)$ in Appendix 1, where it represented the variation of the metric tensor components.

(2) Weak gravitational fields, up to an arbitrary order $\mathcal{O}(\epsilon^n)$, have been investigated in [64].

\begin{align}
g^{ij} (x) = d^{ij} - \frac{\varepsilon}{2} \cdot \left( d^{ik} \cdot d^{jl} + d^{jk} \cdot d^{il} \right) \cdot h_{kl} (x). \quad \text{(A5.2)}
\end{align}

Therefore, we obtain by raising indices,

\begin{align}
h^{kl} (x) = g^{km} (x) \cdot g^{ln} (x) \cdot h_{mn} (x)
= \left\{ \frac{d^{km} \cdot d^{ln} - \varepsilon}{2} \cdot \left[ d^{ln} \cdot (d^{kp} \cdot d^{mq} + d^{mp} \cdot d^{kq}) + d^{km} \cdot (d^{jp} \cdot d^{nq} + d^{np} \cdot d^{lj}) \right] \cdot h_{pq} (x) \right\} \cdot h_{mn} (x), \quad \text{(A5.3i)}
\end{align}

\begin{align}
d_{ij} h^{ij} (x) =: h^i_i (x)
= \left\{ d^{ij} - \frac{\varepsilon}{2} \cdot \left( d^{ik} \cdot d^{jl} + d^{jk} \cdot d^{il} \right) \cdot h_{kl} (x) \right\} \cdot h_{ij} (x), \quad \text{(A5.3ii)}
\end{align}

\begin{align}
det \left[ g_{ij} (x) \right] = - \left[ 1 + \varepsilon h^k_k (x) \right]. \quad \text{(A5.3iii)}
\end{align}

Further, we shall require the Christoffel symbols, which are given as

\begin{align}
\left\{ \begin{array}{c}
i \\
i \\
j \\
k
\end{array} \right\} = \frac{\varepsilon}{2} d^{il} \left[ \partial_k h_{lj} (x) + \partial_j h_{lk} (x) - \partial_l h_{jk} (x) \right]. \quad \text{(A5.4)}
\end{align}

In addition, to form the Einstein tensor in linearized theory, we shall require the Riemann tensor and its contractions to order \( \varepsilon \) which may be computed from (A5.4) as:

\begin{align}
R^i_{jkl} (x) &= \partial_k \left\{ \begin{array}{c}
\partial_i \\
j \\
k
\end{array} \right\} - \partial_i \left\{ \begin{array}{c}
\partial_k \\
j \\
j
\end{array} \right\}
= \frac{\varepsilon}{2} \left[ \partial_k \partial_j h^i_l (x) + \partial_l \partial^j h_{jk} (x) - \partial_k \partial^j h_{jl} (x) - \partial_l \partial_j h^i_k (x) \right], \quad \text{(A5.5i)}

R_{ij} (x) &= \frac{\varepsilon}{2} \left[ \partial_i \partial_j h^k_k (x) + \partial_k \partial^k h_{ij} (x) - \partial^k \partial_i h_{jk} (x) - \partial_k \partial_j h^i_k (x) \right], \quad \text{(A5.5ii)}

R (x) &= \varepsilon \left[ \partial^k \partial_k h^i_i (x) - \partial^k \partial^i h^k_k (x) \right]. \quad \text{(A5.5iii)}
\end{align}

(Compare the above with (2.172).) From these, the Einstein tensor is furnished as

\begin{align}
G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R
= \frac{\varepsilon}{2} \left[ \partial_i \partial_j h^k_k (x) + \partial_k \partial^k h_{ij} (x) + d_{ij} \partial_k \partial^l h^k_l (x)ight. \\
\left. - d_{ij} \partial^k \partial_k h^l_l (x) - \partial^k \partial_i h_{jk} (x) - \partial_k \partial_j h^i_k (x) \right]. \quad \text{(A5.6)}
\end{align}
At this point it is worthwhile noting that if we introduce the *trace-reversed perturbation tensor*, defined by

\[
\overline{h}_{ij}(x) := h_{ij}(x) - \frac{1}{2} d_{ij} h^k_k(x),
\]

the expression (A5.6) simplifies greatly:

\[
G_{ij} = \frac{\varepsilon}{2} \left[ \partial_k \partial^k \overline{h}_{ij}(x) + d_{ij} \partial_k \partial^k \overline{h}_l^l(x) - \partial^k \partial_i \overline{h}_{jk}(x) - \partial_j \partial_j \overline{h}_{ki}(x) \right].
\]

One may simplify this expression even further by an appropriate choice of gauge (which is equivalent to a coordinate transformation in general relativity theory). From (A5.8) it can be seen that considerable simplification would occur if we could choose a coordinate system such that

\[
\partial_k \overline{h}_{kj}(x) = 0.
\]

The condition in (A5.9) is known as the *Lorentz gauge condition* (also known as the harmonic, deDonder, or Fock-deDonder gauge condition). Establishing such a condition is always possible under an infinitesimal (and differentiable) coordinate transformation of the form \( x^i \rightarrow x'^i = x^i + \varepsilon \xi^i(x) \). Under such a transformation, the metric perturbation becomes

\[
h_{ij}(x) \rightarrow h'_{ij}(x) = h_{ij}(x) - \partial_i \xi_j(x) - \partial_j \xi_i(x),
\]

\[
\overline{h}_{ij}(x) \rightarrow \overline{h}'_{ij}(x) = \overline{h}_{ij}(x) - \frac{1}{2} d_{ij} h^k_k(x) = \overline{h}_{ij}(x) - \partial_i \xi_j(x) - \partial_j \xi_i(x) + d_{ij} \partial_k \xi^k(x).
\]

(Note that \( \varepsilon \xi^i(x) \) is of order \( \varepsilon \), and therefore, its index is raised or lowered with the background metric.)

By taking the four-divergence of (A5.10ii) it can be seen that \( \overline{h}'_{ij}(x) \) respects the Lorentz gauge condition if

\[
\partial_k \partial^k \xi_i(x) = \partial^j \overline{h}_{ji}(x).
\]

Therefore, under this gauge condition, the Einstein tensor may be written as

\[
G_{ij} = \frac{\varepsilon}{2} \partial_k \partial^k \overline{h}'_{ij}(x),
\]

so that the linearized Einstein field equations read

\[
\varepsilon \partial_k \partial^k \overline{h}_{ij}(x) = -2\kappa \varepsilon T_{ij}(x).
\]

Note that since the background space-time is flat, \( T_{ij}(x) \) is of order \( \varepsilon \).
Appendix 5  Linearized Theory and Gravitational Waves

Fig. A5.1  An illustration of the quantities in (A5.13) in the three-dimensional spatial submanifold. The coordinates \( x_s \), known as the source points, span the entire source (shaded region). \( O \) represents an arbitrary origin of the coordinate system.

Remarks. (i) From physics, (A5.12) is the equation for a spin-two tensor field propagating on flat space–time with a source \( T_{ij}(x) \) [98]. Gravitons (hypothetical quanta mediating gravitational interactions) should therefore carry a spin of two in the corresponding quantum theory.

(ii) In the case where \( T_{ij}(x) = 0 \), (A5.12) is simply the wave equation for a second-rank tensor field \( h_{ij}(x) \) with speed equal to unity (the speed of light, or massless particles). Therefore, in the linearized theory, it can be argued that gravitational interactions propagate at the speed of light.

If \( T_{ij}(x) \) is known and of compact support, one may find a solution to (A5.12) utilizing the flat space–time retarded Green function for the wave operator:

\[
\tilde{h}_{ij}(\bar{x}, x^4) = \frac{\kappa}{2\pi} \int_B \frac{T_{ij}(\bar{x}, x^4 - |\bar{x} - \bar{x}_s|)}{|\bar{x} - \bar{x}_s|} \, dx_1^s dx_2^s dx_3^s,
\]

where the domain of integration, \( B \), is over the source (see Fig. A5.1).

For the remainder of this appendix, we concentrate on the case where \( T_{ij}(x) = 0 \) (i.e., away from sources). In this scenario, (A5.12) reduces to the wave equation:

\[
\partial_k \partial^k \tilde{h}_{ij}(x) = 0.
\]

The Lorentz gauge condition is actually a class of gauges, as \( \xi_i(x) \) is not completely fixed. Within the Lorentz gauge it is possible to add to \( \xi_i(x) \) a vector field, \( \zeta_i(x) \), such that the condition

\[
\partial_k \partial^k \zeta_i(x) = 0
\]

holds, as this will not spoil (A5.11). The Lorentz gauge condition, along with the conditions (A5.15), allows us to reduce the number of components of \( \tilde{h}_{ij}(x) \) (omitting primes from now on) from ten to two. At this stage, all coordinate freedom has been exhausted, and the system of equations contains two physical (i.e., not
arising from coordinate artifacts which do not affect the curvature) *degrees of freedom*.

For physical considerations, it is convenient to choose $\xi_4(x)$ such that $\overline{h}_k^k = 0$ and choose $\xi_{i\mu}(x)$ such that $\overline{h}_{4\mu}(x) = 0$. This choice constitutes what is known as the *traceless-transverse gauge*. Note that in this coordinate system there is no distinction between $h_{ij}(x)$ and $\overline{h}_{ij}(x)$. We therefore use the notation $^{TT}h_{ij}(x)$ to denote metric perturbations in the traceless-transverse gauge below.

A class of important solutions to (A5.14) is comprised of a superposition of plane waves. Let us consider a single wave

$$^{TT}h_{ij}(x) = \text{Re} \left[ A_{ij} \cdot e^{i k^i x^i} \right], \quad (A5.16)$$

with $A_{ij}$, the components of a symmetric, constant complex tensor (called the *polarization tensor*) and $k^i$, the components of a null vector (called the *wave vector*). The traceless condition implies that $A^i_{\ i} = 0$, and the condition $^{TT}h_{4\mu}(x) = 0$ implies $A_{4\mu} = 0$. Without loss of generality, let us consider the wave in (A5.16) traveling in the $x^3$ direction. The wave vector then possesses the form $[k^i] = [0, 0, \omega, \omega]$ with $\omega$ representing the *frequency of the wave*. The conditions (A5.9) and $A_{4\mu} = 0$ dictate that $A_{13} = 0$ and $A_{44} = 0$. Assembling all the information we have gathered on the tensor, we see that the two degrees of freedom mentioned previously are encoded in the two components $A_{11} = -A_{22}$ and $A_{12} = A_{21}$ (the first equality arising from the traceless condition and the second equality from the symmetric property of $h_{ij}(x)$).

To determine the physical effects of these gravitational waves, let us consider two nearby test particles, located at $x^2 = x^3 = 0$ in the space–time of a gravitational wave. Consider a vector connecting these two particles, whose components, at some initial time, $t_0$, are furnished by $[\delta^i]_{t_0} = [\epsilon, 0, 0, 0]$. To the linear order considered in this appendix, the geodesic deviation equations (1.191) yield, after some calculation:

$$\frac{\partial^2 \eta^1(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} \left[ ^{TT}h_{11}(x) \right], \quad (A5.17i)$$

$$\frac{\partial^2 \eta^2(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} \left[ ^{TT}h_{12}(x) \right]. \quad (A5.17ii)$$

(Note that since the Riemann tensor is already of order $\epsilon$, the components of the 4-velocity ($\xi^{i\ '}$ in the notation of (1.191)) are given by $\delta^i_4$.) On the other hand, if two
Fig. A5.2 The $+$ (top) and $\times$ (bottom) polarizations of gravitational waves. A loop of string is deformed as shown over time as a gravitational wave passes out of the page. Inset: a superposition of the two most extreme deformations of the string for the $+$ and $\times$ polarizations.

Particles are located at $x^1 = x^3 = 0$ and the initial separation vector components are given by $[\eta_j^i]_{t_0} = [0, \epsilon, 0, 0]$, then the separation vector satisfies the equations:

$$\frac{\partial^2 \eta^2(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} [\mathcal{T} \mathcal{T} h_{22}(x)], \quad (A5.18i)$$

$$\frac{\partial^2 \eta^1(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} [\mathcal{T} \mathcal{T} h_{12}(x)]. \quad (A5.18ii)$$

From the form of $\mathcal{T} \mathcal{T} h_{ij}(x)$ in (A5.16), it is obvious that the geodesic separation vector components, $\eta_j^i(x)$, are oscillatory. There exist two polarization states of gravitational waves. One state affects particle motion, as shown in Fig. A5.2a and is characterized by $\mathcal{T} \mathcal{T} h_{11}(x) = -\mathcal{T} \mathcal{T} h_{22}(x) \neq 0$ and $\mathcal{T} \mathcal{T} h_{12}(x) = 0 = \mathcal{T} \mathcal{T} h_{21}(x)$. This state is known as the $+$-state. The other state is characterized by $\mathcal{T} \mathcal{T} h_{12}(x) = \mathcal{T} \mathcal{T} h_{21}(x) \neq 0$ and $\mathcal{T} \mathcal{T} h_{22}(x) = 0 = -\mathcal{T} \mathcal{T} h_{11}(x)$ and is known as the $\times$-state. This motion is depicted in Fig. A5.2b. These two states correspond to two independent polarizations since $A_{11}$ (or equivalently $A_{22} = -A_{11}$) and $A_{12} (= A_{21})$ are independent of each other. The two polarizations are orthogonal to each other in the sense that $A^j_{\bar{i}} + A_{ij} | x = 0$ (the bar here representing complex conjugation).

There are currently a number of gravitational wave detectors in operation throughout the world, with several others in the advanced planning and design stages. The largest land-based detector, Laser Interferometer Gravitational Wave Observatory (LIGO), consists of two interferometers: one in Hanford, Washington, and one in Livingston, Louisiana, which have characteristic lengths of approximately 4 km. The LISA detector (Laser Interferometer Space Antenna), originally scheduled for launch in the next decade, would consist of three orbiting satellites, each separated by approximately 5 million kilometers. These mammoth devices possess the sensitivity to detect gravitational waves from various astrophysical sources (see Fig. A5.3) with amplitudes as small as $10^{-9}$ cm. (See, for example, [182].) The current status of the LISA project is under assessment.

We have treated here only the very simplest type of gravitational wave, as a more in-depth treatment is beyond the scope of this text. A more thorough review would
include solutions with sources \( T_{ij}(x) \neq 0 \), such as many of the phenomena in Fig. A5.3), which are astrophysically relevant, as well as estimations of the power emitted via gravitational wave emission for various astrophysical processes such as black hole collisions and binary neutron star orbits. (In closing this appendix we note that the 1993 Nobel Prize in Physics was awarded to Russell A. Hulse and Joseph H. Taylor, Jr., for their discovery of a binary pulsar system which is observed to be losing energy at a rate in agreement with gravitational wave calculations [137].)
Appendix 6
Exotic Solutions: Wormholes, Warp-Drives, and Time Machines

This appendix deals with several interesting classes of exotic solutions to the gravitational field equations. The purpose of studying such solutions is several-fold: On the one hand, they are interesting in their own right, as they illustrate just how rich the arena of solutions to the field equations can be. There are no analogs of these solutions in pregeneral relativistic theories of gravity. It is therefore of great interest in gravitational field research to study just what is possible within general relativity theory under extreme situations. Secondly, there is pedagogical value in studying such solutions. As we will show below, these solutions concretely illustrate the utility of many of the topics and techniques discussed in this text in a very simple and clear fashion. Finally, the types of solutions mentioned here actually have application in specialized physical topics within general relativity theory. Wormholes, for example, provide a simplistic picture of “space–time foam,” a potential model for the vacuum in theories of quantum gravity, depicted in Fig. A6.1. (Whether this topology is allowed to fluctuate is not clear at the moment, as some arguments against topology change are indifferent to whether one is considering classical or quantum effects [126].) An in-depth review of exotic solutions of the type discussed here may be found in [168].

A6.1 Wormholes

The first exotic solution we will discuss is the wormhole. This type of object has already been mentioned previously. (See Fig. 2.17c.) Qualitatively, the wormhole represents a handle (or shortcut) in space–time, which introduces a nontrivial topology. The most often studied topology is $R^2 \times S^2$. The wormhole handle may connect two otherwise disconnected universes (an interuniverse wormhole) or else connect two regions of the same universe (an intrauniverse wormhole). The two cases are depicted in Fig. A6.2. It should be noted that the near-throat region (the “narrowest” section of the handle) has an uncanny resemblance to the manifold depicted in Fig. 3.1 and, in fact, the part of the Schwarzschild manifold depicted in
Fig. A6.1 A possible picture for the space–time foam. Space-time that seems smooth on large scales (left) may actually be endowed with a sea of nontrivial topologies (represented by handles on the right) due to quantum gravity effects. (Note that, as discussed in the main text, this topology is not necessarily changing.) One of the simplest models for such a handle is the wormhole.

Fig. A6.2 A qualitative representation of an interuniverse wormhole (top) and an intra-universe wormhole (bottom). In the second scenario, the wormhole could provide a shortcut to otherwise distant parts of the universe.

Fig. 3.1 can actually represent one half of a type of wormhole known as an *Einstein-Rosen bridge*. The shading in the throat region here represents the presence of some matter field, which is the source of the gravitational field supporting the wormhole.

Let us now quantify the geometry represented in Fig. A6.2. We shall be limiting our attention to the case where the wormhole exhibits spherical symmetry and, for simplicity, we will assume the space–time is static. In such a case, we may take the line element to be that of (3.1) and utilize the field equations and conservation equations given by (3.44i–vii). Note that here, \( r_1 \) is equal to the coordinate radius of the throat. To avoid confusion, below we shall use coordinate indices \((r, \theta, \varphi, t)\) instead of \((1, 2, 3, 4)\) as was used in Chap. 3.
Since we have a specific geometry in mind, the $g$ or mixed methods of solving the field equations will be most useful here. To mathematically construct the wormhole, we consider Fig. A6.3 which illustrates a cross-section or profile curve of a two-dimensional section, $t = \text{constant}, \theta = \pi/2$, of the wormhole near the throat (see also Fig. 1.18). The wormhole itself is constructed via the creation of a surface of revolution when one rotates this profile curve about the $x^3$-axis. The surface of revolution may be parameterized as follows:

\[ x_\alpha = \xi^\alpha(r, \varphi), \quad (\xi^1(r, \varphi), \xi^2(r, \varphi), \xi^3(r, \varphi)) := (r \cos \varphi, r \sin \varphi, P(r)), \quad (A6.1) \]

where $r = \sqrt{(x^1)^2 + (x^2)^2}$ and $\varphi$ is shown in Fig. A6.3. Therefore, the induced metric on the surface of revolution is given by

\[ ds^2 = \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right]_{\Sigma} = \left\{ 1 + \left[ \partial_r P(r) \right]^2 \right\} dr^2 + r^2 d\varphi^2. \quad (A6.2) \]

Note that the corresponding four-dimensional space–time metric, (3.1), may now be written as

\[ ds^2 = \left\{ 1 + \left[ \partial_r P_{\pm}(r) \right]^2 \right\} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 - e^{\nu_{\pm}(r)} dt^2, \quad (A6.3) \]

with coordinate ranges

\[ r_1 < r < r_2, \quad 0 < \theta < \pi, \quad -\pi < \varphi < \pi, \quad -\infty < t < \infty. \]

For the case of a wormhole throat, the coordinate chart of (A6.3) is insufficient to describe the entire wormhole. Instead, two spherical charts are required, one for
the upper half of the wormhole (the “+” domain) and one for the lower half (the “−” domain). (See [74, 75, 107, 185, 254] for various charts which cover both the + and − domains.) Hence, we use the ± subscript in the above line element.

The constant $m_0$ in (3.45i) can be set by the requirement that $\frac{\partial P}{\partial r} \rightarrow \pm \infty$ as $r \rightarrow r_1$ or, equivalently, that (3.45i) vanish in the limit $r \rightarrow r_1$. This condition implies that the constant $2m_0 = r_1$.

We are now in a position to analyze the geometry. For now we limit the analysis to the upper half of the profile curve (the “+” region) in Fig. A6.3 since the bottom half is easily obtained once we have the upper half (for example, via reflection symmetry through the $x^1 − x^2$ plane if one wants a wormhole symmetric on either side of the throat, although this is not required). It should also be noted that $r = 0$ is not part of the manifold, unlike in the case of stars studied in earlier chapters. Therefore, divergences as $r$ tends to zero are not an issue here. From the profile curve in Fig. A6.3, it may be seen that a function $P(r)$ is required with the following properties:

1. The derivative, $\frac{\partial P}{\partial r}$, must be “infinite” at the throat and finite elsewhere.
2. The derivative, $\frac{\partial P}{\partial r}$, must be positive at least near the throat (negative for the “−” domain).
3. The function $P_+(r)$ must possess a negative second derivative at least somewhere near the throat region (positive for the “−” domain).
4. Since there will be a cutoff at some $r = r_2$, where the solution is joined to vacuum (Schwarzschild or Kottler solution), no specification needs to be made as $r$ tends to infinity.

From now on, we often drop the explicit $r$ dependence in functions for brevity.

In terms of the function $P(r)$, the field equations read

\[
\frac{(\partial_r P)^2 - (\partial_r \gamma) r}{1 + (\partial_r P)^2} \frac{r}{r^2} = -\kappa T^r_r, \tag{A6.4i}
\]

\[
-\frac{1}{4r} \left[1 + (\partial_r P)^2\right]^{-2} \left\{ -4 (\partial_r P) \cdot (\partial_r \gamma) + 2 (\partial_r \gamma) + (\partial_r P)^2 \right.
\]

\[
+ r (\partial_r \gamma)^2 + 2 r (\partial_r \gamma) \cdot (\partial_r P)^2 + 2 r (\partial_r \gamma) + r (\partial_r \gamma)^2 \cdot (\partial_r P)^2
\]

\[
- 2 r (\partial_r P) \cdot (\partial_r \gamma) \cdot (\partial_r \gamma) \right\} = -\kappa T^\theta_\theta \equiv -\kappa T^\phi_\phi, \tag{A6.4ii}
\]

\[
\frac{(\partial_r P) \left[ (\partial_r P)^3 + 2 (\partial_r \gamma) \cdot (\partial_r P) + (\partial_r P) \right]}{r^2 \left[1 + (\partial_r P)^2\right]^2} = -\kappa T^r_t, \tag{A6.4iii}
\]

where it is understood that these equations must hold in both the + and − domain.

A natural question to ask at this stage is what are the properties a matter field must possess in order to support such a wormhole space–time? Analyzing (A6.4iii)
by performing a careful limit as \( r \to r_1 \), and taking into account the properties of \( P(r) \) listed above, we see that \( G'_r(r_1) = \kappa p_r(r_1) \leq \frac{1}{r_1^2} \), \( \rho(r) \) being the local energy density as measured by static observers. (We are assuming here that the stress-tensor components are nonsingular.) To make the arguments more physical, we shall employ a mixed method as opposed to the purely geometric \( g \)-method, and therefore we will treat \( T_{rr}(r) \) as a function that is to be prescribed.

The orthonormal Riemann tensor components are furnished by

\[
R_{(t)(t)(t)(t)}(r) = \frac{1}{2} \left( 1 - \frac{b_{\pm}}{r} \right) \left[ \partial_r \partial_r \gamma_{\pm} + \frac{1}{2} \left( \partial_r \gamma_{\pm} \right)^2 - \frac{1}{2} \partial_r b_{\pm} - \frac{b_{\pm}}{r} \right], \quad (A6.5i)
\]

\[
R_{(t)(t)(\theta)(\theta)}(r) = \frac{1}{2r} \partial_r \gamma_{\pm} \left( 1 - \frac{b_{\pm}}{r} \right) = R_{(t)(\theta)(\theta)(\theta)}(r), \quad (A6.5ii)
\]

\[
R_{(r)(r)(r)(r)}(r) = \frac{1}{2r^2} \left[ \partial_r b_{\pm} - \frac{b_{\pm}}{r} \right] = R_{(r)(r)(r)(r)}, \quad (A6.5iii)
\]

\[
R_{(\theta)(\theta)(\theta)(\theta)}(r) = \frac{b_{\pm}}{r^3}, \quad (A6.5iv)
\]

where we use the notation \( b_{\pm} := \frac{(\partial_r p_{\pm})^2}{1 + (\partial_r p_{\pm})^2} \) to simplify the expressions.

To continue the analysis, it is noted that to make the the orthonormal Riemann components finite in this scenario, \( \partial_r \gamma(r) \) must be finite. (See [74] for full details as to why this renders all components finite.) From the field equations, it can be shown that this quantity is given by

\[
\partial_r \gamma_{\pm} = r \left[ \kappa T'_{r\pm} + \frac{1}{r^2} \right] \left[ 1 + (\partial_r P_{\pm})^2 \right] - \frac{1}{r}, \quad (A6.6)
\]

from which it may be seen that a prescription of \( T'_r(r) \) compatible with \( \kappa T'_r(r_1) = -\frac{1}{r_1} \) must be applied. Therefore, a negative pressure (i.e., a tension) must be present at the wormhole throat in order to support it.

Finally, we consider the combination

\[
\kappa (\rho_{\pm} + p_{\pm}) = G'_r(r_1) - G'_r(r_1) = -\frac{e^{\gamma_{\pm}}}{r} \partial_r \left\{ e^{-\gamma_{\pm}} \left[ 1 + (\partial_r P_{\pm})^2 \right]^{-1} \right\} \quad (A6.7)
\]

in the immediate vicinity of the wormhole throat. This quantity is directly related to the energy conditions discussed in Chap. 2. From the discussions above, it can be seen that at the throat, nonnegativity of this combination can (barely) be met (i.e., \( \rho(r_1) + p_r(r_1) \leq 0 \)). However, as one moves away from the throat, note that the quantity in braces in (A6.7) must go from a value of zero at the throat.
(where $\partial_r P \to \infty$) to a positive value away from the throat. Therefore, around the throat region, the derivative of the expression in braces is positive and thus (A6.7) is negative in a neighborhood of the throat. That is, a static wormhole throat violates the weak/null and strong energy conditions! (See, e.g., [146].)

The reader interested in this topic is referred to Visser’s book on the subject of Lorentzian wormholes [254].

### A6.2  Warp-Drive Space–Times

We consider here briefly another interesting class of solutions to the field equations known as warp-drive space–times. The first such solution was put forward by Miguel Alcubierre in 1994 [4], and since then much analysis has been done on these types of solutions. We will present here the original Alcubierre warp-drive.

The basic idea is simple. One wishes to construct a space–time in which an object (popularly, a spaceship is chosen) can be propelled between two points, such that the travel time (as measured by both the proper time of the object and external observers) is much less than the distance separating the points divided by the speed of light. Ideally, one also wishes to demand that the interval of proper time of the traveling observers is the same as the proper time interval for static external observers (so that the two observers age by the same amount!). Before beginning the analysis, it should also be noted that no object will be traveling faster than the local speed of light.

It is useful to employ the ADM formalism here (see Appendix 1) utilizing the metric (A1.42). An ADM form of the metric is useful, as a space–time domain constructed from the ADM formalism will not contain any closed causal curves (to be discussed in the next section) unless one introduces a topological identification. The original warp-drive used the following quantities:

\[
N(x) = 1, \\
N^\alpha(x) = -s v(x^4) f(s r(x)) \delta^\alpha_1, \\
g_{\alpha\beta}(x) = \delta_{\alpha\beta},
\]

where we are employing a Cartesian-like coordinate system in the three-dimensional spatial hypersurface. The line element therefore is the following:

\[
ds^2 = [dx^1 - s v(x^4) f(s r(x)) dx^4]^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2.
\]

Here, $s v(x^4) := \frac{d_x x^1(x^4)}{dx^4}$ represents the coordinate velocity of the spaceship and $s r(x) := \left[ \left( x^1 - s v^1(x^4) \right)^2 + (x^2)^2 + (x^3)^2 \right]^{1/2}$. The coordinate position of the
Fig. A6.4 A “top-hat” function for the warp-drive space–time with one direction ($x^3$) suppressed. The center of the ship is located at the center of the top hat, corresponding to $\sigma r = 0$.

ship is $x^1 = \sigma x^1(x^4)$, $x^2 = 0$, $x^3 = 0$ (note that we are assuming spatial motion in the $x^1$ direction only) and therefore $\sigma r(x)$ represents the coordinate distance from the ship. For the system to function as desired, the function $f(\sigma r(x))$ should approximate a “top-hat” function. In the original warp-drive solution, the following function was chosen (the explicit $x$ dependence is henceforth dropped):

$$ f(\sigma r) := \frac{\tanh [\sigma (\sigma r + R)] - \tanh [\sigma (\sigma r - R)]}{2 \tanh (\sigma R)}, \quad (A6.10) $$

with $R > 0$ and $\sigma \gg 0$ but otherwise arbitrary constants. This function is displayed in Fig. A6.4 with the $x^3$ coordinate suppressed.

This space–time possesses several interesting properties. Note from (A6.9) that $x^4 = \text{constant hypersurfaces are always flat. Also, for distances } \sigma r \gg R^2, \text{the full four-dimensional space–time is flat, as is the space–time anywhere where the first two derivatives of } f(\sigma r) \text{ are zero. As well, it is not difficult to show that observers possessing 4-velocity components } u^i = [\sigma f(\sigma r), 0, 0, 1] \text{ are traveling along geodesics. As a concrete example, consider an observer located at } x^1_O = \sigma x^1(x^4), x^2_O = x^3_O = 0 \text{ and therefore traveling with the ship. It may immediately be seen that, for such an observer, the invariant infinitesimal line element } (A6.9) \text{ yields the condition } ds^2_O = -(dx^4)_O^2. \text{ (Note that } f(\sigma r) = 1 \text{ in the immediate vicinity of the ship.) Recalling that } ds^2 \text{ along a world line yields minus the infinitesimal proper time squared along that world line tells us that the lapse of proper time for an observer traveling with the ship is equal to the lapse of proper time for a static observer located at some distance } > R \text{ from the ship, for whom } ds^2 = -(dx^4)^2 \text{ also holds. Therefore, there is no relative time dilation between such observers. (We are assuming that the function } f(\sigma r) \text{ is very close to a true top-hat function, which is a good approximation as long as } \sigma \text{ is large.)}

Qualitatively, the top hat, often called the warp bubble, is “propelled” through the external space–time with a coordinate velocity $\sigma v$, and any object enclosed in the bubble moves along with it. This velocity can be very large, the important point being that objects inside the bubble always possess 4-velocities that are timelike. This emphasizes an important point in general relativity that the structure of the light cones is a local phenomenon. As long as 4-velocities remain within their local light cone, the velocity is timelike or subluminal (in the local sense).
To glean further properties of this space–time, let us next consider the extrinsic curvature components, $K^\alpha_{\beta}$, of the spatial slices. A small amount of calculation reveals that

$$K^\alpha_{\beta} = -s^v \frac{df(s)}{ds} \frac{\partial s}{\partial x^1} \delta^\alpha_1 \delta^\beta_1 + \text{off-diagonal terms.} \quad (A6.11)$$

The negative trace of this quantity yields an expression for the expansion of volume elements in this space:

$$-K^\alpha_{\alpha} = s^v \frac{df(s)}{ds} \left[ \frac{x^1 \chi^1(x^4)}{sr} \right]. \quad (A6.12)$$

This function is plotted (with the $x^3$ coordinate suppressed) in Fig. A6.5.

It should be stressed that the contraction of volume elements in front of the spaceship is simply a “by-product” phenomenon due to the type of metric chosen. The space ship does not travel at apparent superluminal speed by moving through the compressed space. The space ship is always located inside the warp bubble, where the space is approximately flat.

Finally, we briefly analyze the properties of the matter field required to support a warp bubble such as the one described here. The energy density, as measured by observers with 4-velocity components $u_i = [0, 0, 0, -1]$ (geodesic observers), is given by

$$T_{ij}(x)u^i(x)u^j(x) = -\frac{1}{\kappa} G_{ij}(x)u^i(x)u^j(x)$$

$$= -\frac{1}{4\kappa} (s^v)^2 \left[ \left( \frac{\partial f(s)}{\partial x^2} \right)^2 + \left( \frac{\partial f(s)}{\partial x^3} \right)^2 \right]. \quad (A6.13)$$
From this expression, it can be noted that a negative energy density is required in order to generate this exotic geometry. The distribution of this negative energy density is plotted in Fig. A6.6. Again, energy conditions are violated.

### A6.3 Time Machines

We conclude our survey of exotic solutions by considering some space-times which contain closed causal (null or timelike) or closed timelike curves (CTCs). We have already briefly touched upon such phenomena when discussing the Kerr space–time. We shall present a few more such solutions here.

CTCs are associated with space-times which possess some property which allows a continuous timelike curve to intersect with itself. That is, some observer may travel into his or her own past. (These definitions may be extended to include null curves as well.) Two qualitative examples of CTCs are displayed in Fig. A6.7, where “forward light cones” have been drawn to indicate the timelike nature of the curves. It should be noted that the space-times in the figures may possess an everywhere nonvanishing vector field which is timelike and continuous (and thus may be time-orientable). Also, locally, one can define a “past” and a “future” in the figures. However, globally this is not possible, as a forward-pointing timelike vector tangent to a timelike curve may be Fermi–Walker transported along the curve in the forward direction (see Fig. 2.13), only to wind up in the past of its point of origin.

Causality violation refers to the violation of the premise that cause precedes effect, as would be possible in a space–time with CTCs. We define the causality violating region of a space–time point \( p \), \( \mathcal{C}(p) \), as

\[
\mathcal{C}(p) := \mathcal{C}^{-}(p) \cap \mathcal{C}^{+}(p),
\]
where $\mathcal{C}^-(p)$ and $\mathcal{C}^+(p)$ represent the causal past and causal future of $p$, respectively (see Fig. 2.2). For causally well-behaved space-times, this intersection is empty for all $p$. The causality violating region of a space–time $M$ is

$$\mathcal{C}(M) = \bigcup_{p \in M} \mathcal{C}(p),$$

and a space–time whose $\mathcal{C}(M) = \emptyset$ is known as a causal space–time.

It is interesting to note that the source of causality violation in many space-times with closed timelike curves is the presence of large angular momentum. Several of the examples below shall reveal this phenomenon, and it will become clear that the coupling of a time coordinate with an angular coordinate (caused by the presence of rotation) can cause the angular coordinate to become timelike, and thus generate causality violating regions of the type in part b) of Fig. A6.7.

We consider first the anti-de Sitter (AdS) space–time (see Exercise 10 of Sect. 2.3). AdS space–time is a constant (negative) curvature solution to the field equations which is supported by a negative cosmological constant, $\Lambda$. The solution may be obtained from the induced metric on the hyper-hyperboloid defined by

$$-U^2 - V^2 + X^2 + Y^2 + Z^2 = -\frac{1}{2|\Lambda|}, \quad (A6.14)$$

embedded in the five-dimensional flat space admitting a line element of

$$ds^2_5 = -dU^2 - dV^2 + dX^2 + dY^2 + dZ^2. \quad (A6.15)$$

We show a schematic of this embedding (with 2 dimensions suppressed) in Fig. A6.8. (See Exercise #1iii of Sect. 6.1 for a discussion of the analogous diagram for de Sitter space–time.)
The topology of the AdS manifold is therefore $S^1 \times \mathbb{R}^3$. Referring to Fig. A6.8, it can be shown that, for example, curves represented by circles orbiting around the $X$-axis on the hyperbolic manifold are everywhere timelike. This space–time therefore exhibits closed timelike curves of the type displayed in Fig. A6.7a. Usually, when dealing with AdS space–time, one deals with the universal covering space of anti-de Sitter space–time, which involves unwrapping the $S^1$ part of the topology, thus yielding $\mathbb{R}^4$ topology.

Next, in our brief review of causality-violating solutions, we present the Gödel universe \cite{116}. The Gödel universe represents a rotating universe supported by a negative cosmological constant, $\Lambda$, and a dust matter source and possessing $\mathbb{R}^4$ topology. The rotation is somewhat peculiar in the sense that all points in this universe may be seen as equivalent. That is, there is no preferred point about which the Gödel universe is rotating. (See \cite{126} for details.)

In one popular coordinate system, the Gödel universe metric admits a line element of

$$
\text{d}s^2 = \frac{2}{|\Lambda|} \left[ \text{d} \varphi^2 + \text{d}z^2 + (\sinh^2 \varphi - \sinh^4 \varphi) \text{d} \varphi^2 + 2 \sqrt{2} \sinh^2 \varphi \, \text{d} \varphi \, \text{d}t - \text{d}t^2 \right],
$$

(A6.16)

with $\Lambda < 0$, $-\infty < t < \infty$, $0 < \varphi < \infty$, $-\infty < z < \infty$ and $-\pi < \varphi < \pi$.

One can see from (A6.16), like in the case of the Kerr geometry, that the $t$ and $\varphi$ coordinates are coupled. Furthermore, note that at $\varphi = \varphi_* := \ln(1 + \sqrt{2})$, the metric component $g_{\varphi \varphi}$ switches sign. This is one indication that the $\varphi$ coordinate changes character from a spacelike nature to a timelike nature. The $\varphi$ coordinate is truly an angular coordinate, that is, it is periodic. Therefore, for values of $\varphi > \ln(1 + \sqrt{2})$, this “angle” is timelike. Although not a geodesic, an observer in the Gödel space–time at $\varphi > \varphi_*$ could travel along a trajectory in the $\varphi$ direction and meet

\footnote{One has to be cautious when applying this loose criterion. A more rigorous discussion is based on the eigenvalue structure of the metric tensor as, for example, presented in various preceding chapters.}
The light-cone structure about an axis $\varrho = 0$ in the Gödel space–time. On the left, the light cones tip forward, and on the right, they tip backward. Note that at $\varrho = \ln(1 + \sqrt{2})$ the light cones are sufficiently tipped over that the $\varphi$ direction is null. At greater $\varrho$, the $\varphi$ direction is timelike, indicating the presence of a closed timelike curve.

The final solution we discuss is the Van Stockum space–time [251]. This solution to the field equations was one of the first to exhibit the behavior of CTCs. Physically, the solution describes an infinitely long cylinder of rotating dust with a vacuum exterior, the angular momentum preventing the gravitational collapse of the dust. The interior solution admits the following line element in corotating cylindrical coordinates:

$$ds^2 = e^{-\omega^2 \varphi^2} (d\varrho^2 + dz^2) + \varrho^2 (d\varphi)^2 - (dt - \omega \varrho^2 d\varphi)^2,$$

with $-\infty < t < \infty$, $0 < \varrho < \varrho_0$, $-\infty < z < \infty$ and $-\pi < \varphi < \pi$.

Here, $\omega$ represents the (constant) angular velocity of the dust cylinder. Again notice that the $\varphi$ and $t$ coordinates are coupled and that for $\omega \varrho > 1$ the periodic $\varphi$ coordinate becomes timelike. One could argue that by placing the boundary of the dust cylinder at some $\varrho = \varrho_0 < \frac{1}{\omega}$, the problem of CTCs would be cured, as the above interior metric is no longer valid at $\omega \varrho = 1$. However, it has been shown that if $\omega \varrho_0 > \frac{1}{2}$ CTCs still appear in the vacuum exterior [248]. But causality violation is avoided if $\omega \varrho_0 \leq \frac{1}{2}$, although there may exist other pathologies. As can be seen
here, strong rotation also causes the light cones to tip and a large enough value of \( \omega \) will cause CTCs to occur somewhere in the space–time. This is analogous to the situation shown in Fig. A6.7b.

Before closing this section, we mention that the wormholes described earlier in this appendix may also be used as a time machine. One process involves the time dilation effect when two mouths of a wormhole move with respect to each other. The interested reader is referred to [247].
Appendix 7
Gravitational Instantons

Let us go back to Tricomi’s p.d.e. of Example A2.5. It is given by

\[ \partial_1 \partial_1 w + x^1 \cdot \partial_2 \partial_2 w = 0. \]

The partial differential equation above originates in the theory of gas dynamics [43]. The p.d.e. is elliptic for the half-plane \( x^1 > 0 \), but hyperbolic for the other half-plane \( x^1 < 0 \). Thus, this equation motivates us to consider mathematically the following metric in order to introduce the subject of this appendix:

\[ g^+ (x^1, x^2) := \left( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \right) + x^1 \cdot \left( \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2} \right), \quad \text{(A7.1i)} \]

\[ g_- (x^1, x^2) = (dx^1 \otimes dx^1) + (x^1)^{-1} \cdot (dx^2 \otimes dx^2), \quad \text{(A7.1ii)} \]

\[ D_2 := \{(x^1, x^2) : x^1 > 0, x^2 \in \mathbb{R} \} \cup \{(x^1, x^2) : x^1 < 0, x^2 \in \mathbb{R} \}. \quad \text{(A7.1iii)} \]

The metric above is positive definite (or Riemannian) for \( x^1 > 0 \), but pseudo-Riemannian (of signature zero) for \( x^1 < 0 \). (A7.1iii) yields a union of apparently two disconnected two-dimensional differential manifolds.

Let us consider another toy model of a four-dimensional metric furnished by:

\[ g_\times (x) := (x^1)^4 \cdot \delta_{\alpha \beta} \cdot (dx^\alpha \otimes dx^\beta) - (x^1)^{-2} \cdot (x^4)^3 \cdot (dx^4 \otimes dx^4), \]

\[ ds^2 = (x^1)^4 \cdot \delta_{\alpha \beta} \cdot (dx^\alpha dx^\beta) - (x^1)^{-2} \cdot (x^4)^3 \cdot (dx^4)^2, \]

\[ D := \{x : x^1 > 0, x^2 \in \mathbb{R}, x^3 \in \mathbb{R}, x^4 > 0\} \cup \{x : x^1 > 0, x^2 \in \mathbb{R}, x^3 \in \mathbb{R}, x^4 < 0\}. \quad \text{(A7.2)} \]
In case $x^4 > 0$, the above metric is transformable to the Kasner metric of Example 4.2.1. However, in case $x^4 < 0$, the metric yields a Ricci-flat, positive definite metric. Two distinct sectors of the manifold have a common three-dimensional, positive definite hypersurface with the metric:

$$\tilde{g}_{\alpha\beta}(x) := (x^1)^4 \cdot \delta_{\alpha\beta} \cdot (dx^\alpha \otimes dx^\beta),$$

$$D := \{x : x^1 > 0, x^2 \in \mathbb{R}, x^3 \in \mathbb{R}\}.$$

The toy model in (A7.2) goes beyond the traditional definition of either a Riemannian or else a pseudo-Riemannian manifold. The common boundary $D$ can be called an instanton-horizon.

The static, vacuum, space–time metrics can generate positive definite, Ricci-flat metrics as in (A7.2).

At this point, we should mention that a gravitational instanton metric is not synonymous with a Riemannian metric. That is, the presence of a positive definite metric is necessary but not sufficient to guarantee that the solution is a gravitational instanton.\(^1\)

**Theorem A7.1.** Let the function $w(x)$ and the positive definite, three-dimensional metric $g_{\alpha\beta}(x)$ be of class $C^3$ in $D \subset \mathbb{R}^3$. Moreover, let

$$g_{\alpha\beta}(x) := e^{-2w(x)} \cdot \delta_{\alpha\beta} \cdot e^{2w(x)} \cdot (dx^4 \otimes dx^4)$$

satisfy vacuum field equations $R_{\alpha\beta}(x) = O_{\alpha\beta}(x)$ in $D := D \times \mathbb{R}$. Then, the four-dimensional, positive definite metric

$$\tilde{g}_{\alpha\beta}(x) := e^{-2w(x)} \cdot g_{\alpha\beta}(x) + e^{2w(x)} \cdot (dx^4 \otimes dx^4)$$

is Ricci flat in $D \times \mathbb{R}$.

**Proof.** For both of the metrics $g_{\alpha\beta}(x)$ and $\tilde{g}_{\alpha\beta}(x)$, conditions of Ricci flatness reduce to the same equations (4.41.i,ii).

However, such an easy transition to the positive definite status is not possible from stationary, vacuum metrics (which are not static). Let us explore this situation in more detail. Consider the “stationary,” positive definite, four-dimensional metric:

$$g_{\alpha\beta}(x) := e^{-2\omega(x)} \cdot g_{\alpha\beta}(x) + e^{2\omega(x)} \cdot \left[w_{\alpha}(x) \cdot dx^\alpha + dx^4\right] \otimes \left[w_{\beta}(x) \cdot dx^\beta + dx^4\right];$$

$$x = (x, x^4) \in D \times \mathbb{R}.$$  \((A7.3)\)

\(^1\)A gravitational instanton metric is a $C^3$ class of vacuum metrics in a “space–time” with a positive definite $+4$ signature which is geodesically complete. Often in cases of interest, they possess a self-dual or anti-self dual Riemann tensor (to be discussed shortly).
(Obviously, the metric above admits the Killing vector $\frac{\partial}{\partial x^4}$. Needless to say, the word timelike is undefined in this context.) Some of the Ricci-flat conditions imply the existence of potential $N(x)$ such that

$$\partial_\alpha N = (1/2) \cdot e^{4u} \cdot \eta_\alpha^\beta \gamma \cdot \left( \partial_\gamma w_\beta - \partial_\beta w_\gamma \right). \quad (A7.4)$$

The function $N(x)$ is called the NUT potential.

The remaining conditions of Ricci flatness reduce to the following p.d.e.s:

$$\tilde{R}_{\mu\nu}(x) + 2 \left( \partial_\mu u \cdot \partial_\nu u \right) - (1/2) \cdot e^{-4u} \cdot \partial_\mu N \cdot \partial_\nu N = 0, \quad (A7.5i)$$

$$\nabla^2 u - (1/2) \cdot e^{-4u} \cdot \tilde{g}^\mu^\nu \cdot \partial_\mu N \cdot \partial_\nu N = 0, \quad (A7.5ii)$$

$$\nabla^2 N - 4 \tilde{g}^\mu^\nu \cdot \partial_\mu u \cdot \partial_\nu N = 0. \quad (A7.5iii)$$

Now we shall derive some conclusions about potentials $u(x)$ and $N(x)$.

**Theorem A7.2.** Let field equations $(A7.5ii,iii)$ hold in a star-shaped domain $D \subset \mathbb{R}^3$.

(i) Moreover, let $N(x)$ be continuous up to and on the boundary $\partial D$. In case $N(x)$ attains the extremum at an interior point $x_0 \in D$, the metric (A7.3) is transformable to the “static form.”

(ii) Let the function $u(x)$, satisfying $(A7.5ii)$ attain the maximum at an interior point $x_0 \in D$. Then, the four-dimensional curvature tensor $R_{\ldots}(x) \equiv 0_{\ldots}(x)$ for all $x \in D \times \mathbb{R}$.

**Proof.** (i) Use Theorems 4.2.2 and 4.2.3 for the function $N(x)$.

(ii) Note that $\nabla^2 u = (1/2) \cdot e^{-4u} \cdot \tilde{g}^\mu^\nu \cdot \partial_\mu N \cdot \partial_\nu N \geq 0$. Use Hopf’s Theorem 4.2.2 for the function $u(x)$.

Now assume a functional relationship

$$e^{2u(x)} = f \left[ N(x) \right] > 0,$$

$$f' := \frac{d}{dN} \left[ f \right] \neq 0. \quad (A7.6)$$

Substituting $(A7.6)$ into $(A7.5ii,iii)$, subtracting $(A7.5iii)$ from $(A7.5ii)$, and assuming $\partial_\alpha N \neq 0$, we deduce that

$$\frac{f''}{f'} + \frac{f'}{f} - \frac{1}{ff'} = 0. \quad (A7.7)$$
(Compare and contrast the equation above with (4.156.)) The general solution of (A7.7) is furnished by

\[
\left\{ f[N] \right\}^2 = c_0 + 2c_1N + (N)^2,
\]

(A7.8)

\[ e^{2\alpha(x)} = \left[ N(x) + c_1 \right]^2 + \left( c_0 - c_1^2 \right) > 0. \]

Here, \( c_0 \) and \( c_1 \) are constants of integration. We now make special choices

\[ c_0 = c_1 = 0, \]

\[ e^{2\alpha(x)} = N(x) > 0. \]

(A7.9)

Field equations (A7.5i) yield

\[ \dddot{R}_{\mu\nu}(x) = 0. \]

(A7.10)

Thus, the three-dimensional metric \( \dddot{g}_{\mu\nu}(x) \) is (flat) Euclidean. Choosing a Cartesian chart for \( D \subset \mathbb{R}^3 \), (A7.5ii,iii) boil down to the Euclidean Laplace’s equation:

\[ \nabla^2 \left\{ N(x)^{-1} \right\} :\left[ \delta_{\mu\nu} \cdot \partial_{\mu} \partial_{\nu} \left\{ N(x)^{-1} \right\} \right] = 0. \]

(A7.11)

Equations (A7.4) reduce to

\[ \partial_{\alpha} w_{\beta} - \partial_{\alpha} w_{\beta} = -\epsilon_{\alpha\beta\gamma} \cdot \partial_{\gamma} \left\{ N(x)^{-1} \right\}, \]

or, \( \nabla \times \tilde{w}(x) = \nabla \cdot \left\{ N(x)^{-1} \right\}. \)

or, \( \text{curl} \tilde{w} = \text{grad} \left\{ N(x)^{-1} \right\}. \)

(A7.12)

The “stationary metric” reduces to

\[ ds^2 = [N(x)]^{-1} \cdot \delta_{\mu\nu} \cdot (dx^\mu dx^\nu) + [N(x)] \cdot \left[ w_{\alpha}(x) dx^\alpha + dx^4 \right]^2. \]

(A7.13)

The class of exact solutions given by (A7.13), (A7.11), and (A7.12) was discovered by Kloster, Som, and Das (K–S–D in short) in 1974 [149]. Hawking rediscovered the same metric in 1977 [125]. We shall call the metric in (A7.13) as the \( H–K–S–D \) metric.

Now recall the Hodge-dual mapping in (1.113). It implies for the curvature tensor

\[ ^*R_{ijkl}(x) = (1/2) \cdot \eta_{rskl}(x) \cdot R_{ij}^{rs}(x). \]

(A7.14)

In case

\[ ^*R_{ijkl}(x) \equiv R_{ijkl}(x), \]

(A7.15)
the curvature tensor $R_{ijkl}(x)$ is called \textit{self-dual}. On the other hand, if
\[
*R_{ijkl}(x) \equiv -(1/2) \cdot \eta_{rskl}(x) \cdot R_{ij}^{rs}(x) = -R_{ijkl}(x), \quad (A7.16)
\]
the curvature tensor is called \textit{anti-self-dual}. Both (A7.15) and (A7.16) can be combined into
\[
*R_{ijkl}(x) \equiv \varepsilon \cdot R_{ijkl}(x), \quad \varepsilon = \pm 1. \quad (A7.17)
\]
It can be proved that in case of the H–K–S–D metric of (A7.13), the condition in (A7.17) is identically satisfied by the use of (A7.9) [110].

\textit{Instantons are regular, finite-action solutions of the field equations in a four-dimensional, positive definite, flat (or curved) “space–time”.} For Yang-Mills type of gauge field theories, the construction of Atiyah, Drinfeld, Hitchin, and Manin [11] allows one to compute all regular, self-dual (or anti-self-dual) solutions. However, in the case of gravitation, the situation is more complicated. \textit{We define gravitational instantons as positive definite metrics of class $C^3$, which are Ricci flat and geodesically complete.} There exist gravitational instantons which are neither self-dual, nor anti-self-dual. A class of these are provided by the “static metrics” of Theorem A7.1. Fortunately, H–K–S–D metrics supply infinitely many self-dual (or anti-self-dual) gravitational instantons which are useful for some path-integral approaches to quantum gravity.

\textit{Example A7.3.} Consider the multicenter H–K–S–D gravitational instantons furnished by the function
\[
[N(x)]^{-1} = a + \sum_{A=1}^{N} \frac{2m_A}{\|x - x_A\|} > 0. \quad (A7.18)
\]
Here, $a \geq 0$ and $m_A > 0$ are constants. (Compare with the pseudo-Riemannian Example 4.2.8.) If $m_A = m$ for all $A \in \{1, \ldots, N\}$ and if $x^4$ is a periodic variable with the fundamental period in $0 \leq x^4 \leq (\kappa m / N)$, then singularities at $x = x_A$ are removable. Such solutions qualify as gravitational instantons. In case $a = 1$ and $N = 1$, the metric is called the self-dual\textit{Taub-NUT metric} [126]. In case $a = 0$ and $N = 2$, we obtain the \textit{Eguchi-Hansen gravitational instantons} [85]. □
Appendix 8
Computational Symbolic Algebra Calculations

In this appendix, we present some calculations relevant to general relativity utilizing popular symbolic algebra computer programs. Calculations in general relativity are notoriously lengthy and complicated, and therefore, computational symbolic algebra programs are very useful in this field. Two popular symbolic algebra programs are Maple™ and Mathematica™, both of which we concentrate on here. For the sake of brevity, we do not discuss special packages, available for both programs, which are designed to supplement the ability of these programs to do general relativistic calculations. (For example, we do not discuss the use of the excellent add-on package available for both Maple™ and Mathematica™ called GRTensorII, which is available at http://grtensor.phy.queensu.ca/).

A8.1 Sample Maple™ Work Sheet

This work sheet deals with quantities related to the vacuum Schwarzschild metric:

> #MAPLE worksheet involving the vacuum Schwarzschild metric.

> restart: #The restart command clears all set variables.

> with(tensor): #This loads Maple’s tensor package.

> coord:=[r, theta, phi, t]; #The coord command defines~the four coordinates.

        coord := [r, \theta, \phi, t]
> g_compts := array(symmetric, sparse, 1..4, 1..4); #create a 4x4 sparse array, to be used for metric components. The symmetric command means that if you have non-diagonal components, you only have to enter one of the symmetric pair, the other will automatically be assumed.

\[
g_{\text{compts}} := \text{array(symmetric, sparse, 1..4, 1..4, [])}
\]

> g_compts[1,1] := 1/(1-2*m/r); #This is \(g_{11}\).

\[
g_{\text{compts}_{1,1}} := \frac{1}{1 - \frac{2m}{r}}
\]

> g_compts[2,2] := r^2; #This is \(g_{22}\).

\[
g_{\text{compts}_{2,2}} := r^2
\]

> g_compts[3,3] := r^2*(sin(theta))^2; #This is \(g_{33}\).

\[
g_{\text{compts}_{3,3}} := r^2 \sin^2(\theta)
\]

> g_compts[4,4] := -1/g_compts[1,1]; #This is \(g_{44}\).

\[
g_{\text{compts}_{4,4}} := -1 + \frac{2m}{r}
\]

> g := create([-1,-1], eval(g_compts)); #This command creates the metric. The [-1,-1] indicates covariant components on both indices.

\[
g := \text{table(index_char = [-1, -1],}
\begin{align*}
\frac{1}{1 - \frac{2m}{r}} & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 \\
0 & 0 & r^2 \sin^2(\theta) & 0 \\
0 & 0 & 0 & -1 + \frac{2m}{r}
\end{align*}
\)

> ginv := invert(g, 'detg'); #Creates the inverse metric.

\[
ginv := \text{table(index_char = [1, 1],}
\begin{align*}
-\frac{r + 2m}{r} & 0 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & 0 & \frac{1}{r^2 \sin^2(\theta)} & 0 \\
0 & 0 & 0 & -\frac{r}{r + 2m}
\end{align*}
\)
\( \text{D1g} := \text{d1metric}(g, \text{coord}) \): # Creates first partial derivatives of the metric. Note the colon (instead of semi-colon). This suppresses the output, which can be lengthy.

\( \text{D2g} := \text{d2metric} \text{(D1g, coord)} \): # Creates the second partial derivatives of the metric.

\( \text{Cf1} := \text{Christoffel1} \text{(D1g)} \): # Creates Christoffel symbols of the first kind.

\( \text{Cf2} := \text{Christoffel2} \text{(ginv, Cf1)} \): # Creates Christoffel symbols of the second kind.

displayGR(Christoffel2, Cf2); # Display only the non-zero components of the Christoffel symbols of the second kind. The \{a, bc\} indicates that the first index is ‘‘contravariant’’ and the second and third indices are ‘‘covariant’’.

\( \text{The Christoffel Symbols of the Second Kind} \)

\( \text{non-zero components:} \)

\( \{1, 11\} = \frac{m}{(-r + 2m) r} \)

\( \{1, 22\} = -r + 2m \)

\( \{1, 33\} = (-r + 2m) \sin^2(\theta) \)

\( \{1, 44\} = \frac{(-r + 2m) m}{r^3} \)

\( \{2, 12\} = \frac{1}{r} \)

\( \{2, 33\} = -\sin(\theta) \cos(\theta) \)

\( \{3, 13\} = \frac{1}{r} \)

\( \{3, 23\} = \frac{\cos(\theta)}{\sin(\theta)} \)

\( \{4, 14\} = -\frac{m}{(-r + 2m) r} \)

\( \text{RMN} := \text{Riemann} \text{(ginv, D2g, Cf1)} \): # Creates the Riemann curvature tensor.

\( \text{displayGR(Riemann, RMN)} \); # Display only the non-zero components of the Riemann tensor.

\( \text{The Riemann Tensor} \)

\( \text{non-zero components:} \)
\[ R_{1212} = \frac{m}{-r + 2m} \]
\[ R_{1313} = \frac{m \sin(\theta)^2}{-r + 2m} \]
\[ R_{1414} = -\frac{2m}{r^3} \]
\[ R_{2323} = 2m^2 \sin(\theta)^2 \]
\[ R_{2424} = \frac{-(r + 2m)m}{r^2} \]
\[ R_{3434} = -\frac{(r + 2m)m \sin(\theta)^2}{r^2} \]

character: \([-1, -1, -1, -1]\)

> \texttt{RICCI:=Ricci(ginv,RMN); #Create the Ricci tensor.}  
Maple contracts the first and fourth index of the Riemann tensor in order to create the Ricci tensor, compatible with the conventions in this book. It is zero here since we are dealing with a vacuum solution.

\[
\text{RICCI} := \text{table}([\text{index\_char} = [-1, -1], \text{compts} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}])
\]

> \texttt{RS:=Ricciscalar(ginv,RICCI); #The Ricci scalar.}  
\[
\text{RS} := \text{table}([\text{index\_char} = [], \text{compts} = 0])
\]

> \texttt{upRMN:=raise(ginv, RMN, 1,2,3,4): #Calculates the contravariant Riemann tensor by raising the first, second, third and fourth index using the inverse metric.}  
\[
\text{Kret} := \text{prod(RMN,upRMN,[1,1],[2,2],[3,3],[4,4])};  
\text{#Creates the Kretschmann scalar by multiplying the covariant and contravariant Riemann tensor and contracting first index with first, second with second, third with third and fourth with fourth.}  
\[
\text{Kret} := \text{table}([\text{index\_char} = [], \text{compts} = \frac{48m^2}{r^6}])
\]

> \texttt{Estn:=Einstein(g,RICCI,RS): #Calculates the covariant Einstein tensor.}
Appendix 8  Computational Symbolic Algebra Calculations

> mixeinst := raise(ginv, Estn, 1): # Raises the first index of the covariant Einstein tensor, creating the mixed-character Einstein tensor.

> displayGR(Einstein, mixeinst); # Displays only the non-zero components of the mixed-character Einstein tensor. There are no non-zero components. Caution must be applied when using this command to display this tensor’s components; only components where the row-index is less than or equal to the column-index will be displayed.

The Einstein Tensor

non-zero components :
None
character : [1, -1]

A8.2  Sample Mathematica™ Notebook

This notebook deals with quantities related to the vacuum Schwarzschild metric. It is based on a notebook which originally supplemented the text *Gravity: An introduction to Einstein’s general relativity* by Hartle [124]. Mathematica notebook involving the vacuum Schwarzschild metric.

This next command clears various variables that might be set. Note that in Mathematica, you need to press shift-enter after you are finished typing a cell.

Clear[coord, metric, ginv, riemann, ricci, rs, einstein, christoffel, r, theta, phi, t]

This next command sets the coordinates to be used.

coord = \{r, theta, phi, t\}

Out[2]=\{r, theta, phi, t\}

This next command creates the metric.

metric = \{\{1/(1 - 2 * m/r), 0, 0, 0\}, \{0, r^2, 0, 0\}, \{0, 0, r^2 * (Sin[theta])^2, 0\}, \{0, 0, 0, -(1 - 2 * m/r)\}\}
This next command will display the metric in matrix form.

```
metric//MatrixForm
```

```
Out[4]//MatrixForm=
\[
\begin{pmatrix}
\frac{1}{1-\frac{2m}{r}} & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 \\
0 & 0 & r^2\text{Sin}[\theta]^2 & 0 \\
0 & 0 & 0 & -1 + \frac{2m}{r}
\end{pmatrix}
\]
```

This next command will create the inverse metric.

```
ginv1 = Inverse[metric]
```

```
Out[5]=
\[
\begin{pmatrix}
\frac{2mr\text{Sin}[\theta]^2}{1-\frac{2m}{r}} & r^2\text{Sin}[\theta]^2 & 0 & 0 \\
0 & \frac{2mr\text{Sin}[\theta]^2}{1-\frac{2m}{r}} & r^2\text{Sin}[\theta]^2 & 0 \\
0 & 0 & \frac{2mr\text{Sin}[\theta]^2}{1-\frac{2m}{r}} & r^2\text{Sin}[\theta]^2 \\
0 & 0 & 0 & \frac{2mr\text{Sin}[\theta]^2}{1-\frac{2m}{r}}
\end{pmatrix}
\]
```

The last output was messy. Let’s simplify it a bit with the Simplify command.

```
ginv = Simplify[ginv1]
```

```
Out[6]=
\[
\begin{pmatrix}
1 - \frac{2m}{r} & 0 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & 0 & \frac{\text{Csc}[\theta]^2}{r^2} & 0 \\
0 & 0 & 0 & \frac{r}{2m-r}
\end{pmatrix}
\]
```

We can also display the inverse metric in matrix form.

```
ginv//MatrixForm
```

```
Out[7]=
\[
\begin{pmatrix}
1 - \frac{2m}{r} & 0 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & 0 & \frac{\text{Csc}[\theta]^2}{r^2} & 0 \\
0 & 0 & 0 & \frac{r}{2m-r}
\end{pmatrix}
\]
```

We next calculate the Christoffel symbols of the second kind. The triplets such as \(\{i,1,4\}\) in the braces correspond to the index \(i\) and the range of the index \(1\) to \(4\). The operator \(D[\text{metric}[[s,j]],\text{coord}[[k]]]\) calculates the partial derivative of the \(s-j\) metric component with respect to the coordinate \(k\).

```
christoffel = Simplify[Table[(1/2) * Sum[(ginv[[i, h]]) * (D[metric[[h, j]], coord [[k]]] + D[metric[[h, k]], coord[[j]]] - D[metric[[j, k]], coord[[h]]]), \{h, 1, 4\}, \{i, 1, 4\}, \{j, 1, 4\}, \{k, 1, 4\}]]
```

This next command will display the metric in matrix form.

```
matrix//MatrixForm
```

```
Out[7]//MatrixForm=
\[
\begin{pmatrix}
\frac{1}{1-\frac{2m}{r}} & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 \\
0 & 0 & r^2\text{Sin}[\theta]^2 & 0 \\
0 & 0 & 0 & -1 + \frac{2m}{r}
\end{pmatrix}
\]
Next the Riemann tensor (first index contravariant, second through fourth index covariant) is created. **Note that this differs from the one calculated in the Maple\textsuperscript{TM} worksheet which was completely covariant in character.**

\begin{align*}
\text{riemann} &:= \text{riemann} = \text{Simplify}[\text{Table}[D[\text{christoffel}[[i, j, l]], \text{coord}[[k]]] - D[\text{christoffel}[[i, j, k]], \text{coord}[[l]]] + \text{Sum}[\text{christoffel}[[s, j, l]]\text{christoffel}[[i, k, s]] - \text{christoffel}[[s, j, k]]\text{christoffel}[[i, l, s]], \{s, 1, 4\}, \{i, 1, 4\}, \{j, 1, 4\}, \{k, 1, 4\}, \{l, 1, 4\}]])
\end{align*}

This next command will display only the non-zero Riemann components. **Note that these differ from the ones calculated in the Maple\textsuperscript{TM} worksheet which was completely covariant in character.** The output from the first command is supressed.

\begin{align*}
\text{riemannarray} &:= \text{Table}[\text{If}[\text{UnsameQ}[\text{riemann}[[i, j, k, l]], 0], \{\text{ToString}[R[i, j, k, l]], \text{riemann}[[i, j, k, l]]\}, \{i, 1, 4\}, \{j, 1, 4\}, \{k, 1, 4\}, \{l, 1, k - 1\}]])
\end{align*}

\begin{align*}
\text{TableForm}[\text{Partition}[\text{DeleteCases}[\text{Flatten}[\text{riemannarray}], \text{Null}], 2]]
\end{align*}

\begin{align*}
\text{Out}[11]/\text{TableForm} &= \\
R[1, 2, 2, 1] &\rightarrow \frac{m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[1, 3, 3, 1] &\rightarrow \frac{2m(-2m+r)}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[1, 4, 4, 1] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[2, 1, 2, 1] &\rightarrow \frac{m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[2, 3, 3, 2] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[2, 4, 4, 2] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[3, 1, 3, 1] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[3, 2, 3, 2] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[3, 4, 4, 3] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[4, 1, 4, 1] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[4, 2, 4, 2] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m} \\
R[4, 3, 4, 3] &\rightarrow \frac{2m}{r} \\quad \frac{m\text{Sin}[^\theta]^2}{2m}
\end{align*}

This next command creates the Ricci tensor. Note the summation over the index i.

\begin{align*}
\text{ricci} &:= \text{Simplify}[\text{Table}[\text{Sum}[\text{riemann}[[i, j, l, i]], \{i, 1, 4\}, \{j, 1, 4\}, \{l, 1, 4\}]]
\end{align*}
Next we display only the non-zero components of the Ricci tensor. There are no non-zero components.

\[
\text{ricciarray} := \text{Table}[\text{If}[\text{UnsameQ}[\text{ricci}[[i, j]], 0], \{\text{ToString}[R[j, l]], \text{ricci}[[j, l]]\}, \{j, 1, 4\}, \{l, 1, 4\}]]
\]

\[
\text{TableForm[Partition[DeleteCases[Flatten[ricciarray], Null], 2]]}
\]

\[
\text{Out[14]//TableForm} = \{
\}
\]

We next calculate the Ricci curvature scalar.

\[
\text{RS} = \text{Sum}[\text{ginv}[[i, j]] \text{ricci}[[i, j]], \{i, 1, 4\}, \{j, 1, 4\}]
\]

\[
\text{Out[15]} = 0
\]

We next calculate the Einstein tensor, by explicitly creating it with the Ricci tensor, the Ricci scalar and the metric.

\[
\text{einstein} := \text{einstein} = \text{ricci} - (1/2) \text{RS} \ast \text{metric}
\]

Display only the non-zero components of the Einstein tensor. There are no non-zero components, indicating a vacuum solution.

\[
\text{einsteinarray} := \text{Table}[\text{If}[\text{UnsameQ}[\text{einstein}[[i, l]], 0], \{\text{ToString}[G[j, l]], \text{einstein}[[j, l]]\}, \{j, 1, 4\}, \{l, 1, 4\}]
\]

\[
\text{TableForm[Partition[DeleteCases[Flatten[einsteinarray], Null], 2]]}
\]

\[
\text{Out[18]//TableForm} = \{
\}
References

References
References

666 References

195. Noll, W., Notes on tensor analysis (prepared by Wang, C.C.), Mathematics Department, Johns Hopkins University, 1963.
219. Robertson, H.P., Rev. Mod. Phys. 5 62, 1933.
Symbols
1-form, 9
4-acceleration, 181
4-force, 115
4-momentum, 116
    total, 121
4-velocity, 113

A
acceleration
    4-acceleration, 181
Newtonian, 74
action function or functional
    (see also Lagrangian), 570, 574
ADM action, 582
affine parameter, 77, 79
alternating operation, antisymmetrization, 27
angle field, 42
anisotropic fluid, 212–214, 265
    collapse, 408–415
anti-de Sitter space–time, 192, 621, 642–643
anti-self-dual, 651
antisymmetric oriented tensor, 48
antisymmetric tensor, 27
antisymmetrization, 27
arc length parameter, 82
arc separation function, 85
arc separation parameter, 77
Arnowitt-Deser-Misner action integral, 582
atlas, 3
    complete, 3
    maximal, 3

B
Betti number, 565
Bianchi type-I models, 433

Bianchi’s differential identities, 60
    complex-valued, 512
    consequences of in Newman–Penrose
    formalism, 527–529
    first contracted, 63
    second contracted, 63
bicharacteristic curves, 596
big crunch, 426
big-bang cosmological model, 423, 434
Birkhoff’s theorem, 260
bivector space, 470
black hole, 351–418
Bondi-Metzner-Sachs group, 231
boost, 109
Born-Infeld (or tachyonic) scalar field,
    451–455
Boyer–Lindquist coordinate chart, 322, 384
Brinkman–Robinson–Trautman metric, 495
Buchdahl inequality, 252
bugle, 69

canonical energy-momentum-stress tensor, 119
canonical or normal forms, 492
Cartesian chart, 43, 68
Casimir effect, 615
Cauchy horizon, 388, 402
Cauchy problem, 203
Cauchy-Kowalewski theorem, 203
causal cone, 107
causal space–time, 642
causality violation, 641
characteristic hypersurface, 600
characteristic matrix, 601
characteristic ordinary differential equations, 588
characteristic polynomial equation, 174
characteristic surface, 595
charge-current, 124
charged dust, 125, 133, 220, 305
chart, 1
Christoffel symbol, 57
  first kind, 57
  second kind, 57
closed form, 33
Codazzi-Mainardi equations, 100
commutator or Lie bracket, 39
comoving coordinate chart, 193
complete atlas, 3
complex conjugate coordinates, 598
complex electromagnetic potential, 342
complex gravo-electromagnetic potential, 342
complex null tetrad field, 468
complex potential, 319, 333
components
  coordinate, 20
  covariant, 10
  orthonormal, 46
  physical, 120
conformal group, 618
conformal mapping, 70
conformal tensor, 71
conformally flat domain, 71
conformally flat space–time, 146, 617–624
conformastat metric, 297
conformastationary metric, 344
conjugate holomorphic function, 598
conjugate points, 81
conjugate, contravariant, or inverse metric tensor, 43
constant curvature, space, 68, 73
continuity equation, 122, 125, 183, 225
continuum mechanics, 117
contraction operation, 21
contravariant index, 50
contravariant order, 17
coordinate chart or system
  Boyer–Lindquist, 322, 384
  Cartesian, 43, 68
  comoving, 193
  Doran, 400
doubly-null, 130
  Eddington–Finkelstein, 358, 370
  Gaussian normal, 67, 197
  geodesic normal, 67, 102, 197
Kerr-Schild, 400
Kruskal–Szekeres, 366–367
local, 1
local Minkowskian, 151
Minkowskian, 105
normal or hypersurface orthogonal, 67
orthogonal, 61
Painlevé–Gullstrand, 357
pseudo-Cartesian, 43, 68
Regge-Wheeler tortoise, 358
Riemann normal, 67
Synges doubly-null, 359
Weyl’s, 278
Weyl-Lewis–Papapetrou (WLP), 317
coordinate components, 20
coordinate conditions, 164
cosmological constant, 419
cosmological principle, 162, 226, 420
degrees (system of p.d.e.s), 587
delta
  Kronecker, 8
derivative
  covariant, 50, 63
dark energy, 446
de Sitter space–time, 416, 436, 621
deceleration function, 433
deceleration parameter, 433
deDonder gauge, 627
defeormable solid, 214–215, 265
degree (system of p.d.e.s), 587
delta
  Kronecker, 8
derivative
  covariant, 50, 63
covariant directional, 51
directional, 7, 51
exterior, 31
Fermi, 145
gauge covariant, 538
Lie, 35–38
variational, 571, 575
derivative mapping, 22
determinate system, 164
differentiable manifold, 3
differential conservation of energy-momentum, 119
differential form, 30
directional derivative, 7
domain
curved, 64
flat, 64
dominant energy condition, 178
Doran coordinate chart, 400
dot product or inner-product, 40
double Hodge-dual, 85
doubly-null coordinates, 130
dual vector space, 9
dubiosity, 82
dust, 122, 125, 133, 182, 183, 210, 220, 305,
420, 422, 424, 426, 433, 438, 455
collapse, 370–374

eccentricity, 236
Eddington–Finkelstein coordinates, 358, 370
Eguchi-Hansen gravitational instantons, 651
Einstein space, 69
Einstein static universe, 419
Einstein summation convention, 6
Einstein tensor, 63
Einstein’s field equations, 162
complex-valued form, 511
Einstein–Maxwell equations, 253
Einstein–Maxwell–Klein–Gordon equations, 537
orthonormal form, 541
Einstein-Hilbert Lagrangian density, 166
Einstein-Maxwell equations, 218
Einstein-Rosen bridge, 634
electrical charge density, 125
electro-vac universe, 253
electromagnetic duality-rotation, 226
electromagnetic field tensor, 33, 50
electromagnetic four-potential, 34, 218, 226,
486, 538
electromagneto-vac domain, 218
electrostatic potential, 307
Index

form (cont.)

differential, 30
exact, 33
Frenet-Serret formulas, 83
generalized, 82
frequency of a wave, 486
Friedmann–Lemaitre–Robertson–Walker
metrics (or F–L–R–W), 421–456

G

Gödel universe, 643–644
gauge covariant derivatives, 538
gauge transformation, 34
Gauss’ equations, 100
Gauss’ theorem, generalized, 65
Gauss-Bonnet term, 584
Gaussian curvature, 91
Gaussian normal coordinate chart, 67, 197
general covariance, 130
general relativity, 166
generalized D’Alembertian, 216
generalized Frenet-Serret formulas, 82
generalized Kronecker tensor, 29
generalized wave operator, 216
goingesic, 76
non-null, 77
null, 77
goingesimal deviation equations, 80
goingesimal equations, 77
for dust, 183
for particle in $T$-domain, 356
for particle in Schwarzschild space-time, 234
for the Euclidean plane $E_2$, 78
goingesimal normal coordinate chart, 67, 102, 197
goingesically complete manifold, 168
geometrized units, 162
geons, 567
Global Positioning System (G.P.S.), 323
gravitational constant, $G$, 66, 160
gravitational field equations, 162
gravitational instantons, 337, 647–651
gravitational mass, 136
gravitational redshift, 242
gravitational wave astronomy, 625
gravitational waves, 166, 625–631
gravitons, 628
goingavo-electromagnetic potential, 342
group
Bondi-Metzner-Sachs, 231
conformal, 618
Lie, 44
Lorentz, 44, 108

Möbius, 342
Poincaré, 108
symplectic, 171
group of motion, 73

H

H–K–S–D metric (or Hawking-Kloster-Som-Das), 650
Hamilton-Jacobi equation, 591
Hamiltonian
for particle in a static gravitational
potential, 313
for particle in an electromagnetic field, 158
relativistic, 156
relativistic equations of motion, 158
relativistic mechanics, 156
super, 157
harmonic coordinate conditions, 172
harmonic function, 296, 599
harmonic gauge, 172, 627
Hausdorff manifold, 1
Hawking-Kloster-Som-Das metric (or
H–K–S–D), 650
heat flow vector, 225
Heaviside-Lorentz units, 122
Hessian, 592
Higgs fields, 456
Hilbert-Palatini approach for deriving field
equations, 577
Hodge star operation, 49
Hodge-dual operation, 479
holomorphic function, 598
holonomic function, 157
homeomorphism, 1
homogeneity, 427
homogeneous p.d.e., 587
homogeneous state of the complex wave
function, 547
Hopf’s theorem, 296
Hubble function, 433
Hubble parameter, 433
hybrid tensor field, 96–98
hyperbolic p.d.e., 594
hypersurface, 354
null, 130
hypersurface orthogonal coordinate, 67

I

I-S-L-D jump conditions, 165
identity tensor, 45
immersion, 93
incoherent dust, 182
index
  contravariant, 50
  covariant, 50
index lowering, 47
index raising, 47
inertial mass, 136
inertial observer, 112
inflationary era, 431
initial value problem, 203
inner-product or dot product, 40
instanton, 337, 647–651
instanton-horizon, 648
integrability conditions, 98
integral conservation, 120
integral curve, 35
intrinsic curvature
  extended, 580
intrinsic metric, 90
invariant eigenvalue problem, 91
invariant eigenvalues, 174
irrotational motions, 181
isomorphism, 483
on tensor space, 47
isotropic sectional curvature, 427
isotropy, 427
isotropy equation of static spherical symmetry, 250

J
Jacobian, 3
Jacobian mapping, 22
Jordan decomposition theorem, 607
junction conditions
  I-S-L-D, 165
  Synge’s, 164

K
Kaluza-Klein theory, 456, 464
Kasner metric, 207, 294, 298, 311
Kerr metric, 322–326, 384–403
  higher dimensional generalization, 329
Kerr–Newman solution, 401
Kerr–Newman solution, 323
Kerr-Schild coordinate chart, 400
Killing tensor, 399
Killing vector, 72
Klein–Gordon field
  (see also scalar field, massive), 537–567
Kottler solution, 243
Kretschmann invariant, 86, 385
Kronecker delta, 8
Kronecker tensor, generalized, 29

Kruskal–Szekeres coordinates, 366–367
Kruskal–Szekeres metrics, 366–367

L
Lagrange multiplier, 157
Lagrange’s solution method, 588
Lagrangian, 570
  alternative Lagrangians, 159
  augmented gravitational, 579
  classical mechanics, 572
  Einstein–Maxwell–Klein–Gordon, 541
  Einstein-Hilbert, 577
  Einstein-Maxwell, 218
  first-order gravitational, 581
  for particle in $T$ domain, 356
  for particle in curved space-time, 154, 160
  for particle in electromagnetic field, 158
  for particle in Schwarzchild space-time, 233, 234
  for scalar field, 575
  for static field equations, 292
  for stationary field equations, 339, 340
  for tachyonic scalar field, 453
  relation to super-Hamiltonian, 157
  square-root particle, 453
Lagrangian density, 166
  augmented gravitational, 579
  Einstein–Maxwell, 346
  Einstein-Hilbert, 166, 577
  most general curvature yielding
    second-order equations, 584
Lambert W-Function, 361
Lane-Emden equation, 551
Laplacian operator, 84
Legendre transformation, 156
Lemaître metric, 269
lemma of Poincaré, 32
length or norm, 42
Levi-Civita permutation symbol, 29
Levi-Civita tensor, 48
Lichnerowicz-Syngue lemma, 204
Lie bracket or commutator, 39
Lie derivative, 35–38
Lie group, 44
light cone, 107
linear equation, 587
linear p.d.e., 587
linearized theory of gravitation, 625–631
Liouville equation, 597
Lipschitz condition, 35
local coordinate system, 1
local Minkowskian coordinates, 151
London equation, 547
Lorentz gauge condition, 172, 216, 627
Lorentz group, 44, 108
Lorentz matrix, 132
Lorentz metric, 43, 64
Lorentz signature, 42
Lorentz transformation, 44, 108
orthochronus, 189
Lorentz–Heaviside units, 122

M
magnetic potential, 303
manifold
constant curvature, 68, 73
differentiable, 3
Einstein, 69
embedded, 93
flat, 67
goingesically complete, 168
Hausdorff, 1
orientable, 3
paracompact, 1
pseudo-Riemannian, 64
Ricci flat, 70, 162
Riemannian, 64
Riemannian or pseudo-Riemannian, 56, 64
topological, 3
Maple(TM), 653
mass
gravitational, 136
inertial, 136
mass density, 117
mass-shell, 116
Mathematica(TM), 653
matrix
characteristic, 601
Lorentz, 132
nilpotent, 606
rank of, 17
unit, 8
matter-dominated era, 430
maximal atlas, 3
maximum-minimum principle, 296
Maxwell’s equations, 33, 122
Maxwell–Lorentz equations, 124
mean curvature, 91
Meissner effect, 547
metric
anti-de Sitter, 192, 621, 642
Bianchi type-I, 433
Brinkman–Robinson–Trautman, 495
components, 43
conformastat, 297
conformastationary, 344
conjugate, contravariant, or inverse, 43
Curzon-Chazy, 288
de Sitter, 416, 436, 621
Einstein static universe, 419
Finsler, 156
Friedmann–Lemaître–Robertson–Walker
(or F–L–R–W), 421–456
Gödel, 643
Hawking-Kloster-Som-Das (or H–K–S–D),
650
Kasner, 207, 294, 298, 311
Kerr, 322–326, 384–403
higher dimensional generalization, 329
Kerr–Newman, 401
Kerr-Newman, 323
Kottler, 243
Kruskal–Szekeres, 366–367
Lemaître, 269
Lorentz, 43, 64
positive-definite, 42, 64
pseudo-Schwarzschild, 299
Reissner-Nordström-Jeffery, 255, 381
Robinson–Trautman, 527
Schwarzschild, 231–242, 299, 336,
351–369
higher dimensional generalization, 244
Schwarzschild’s interior, 253
signature, 42
Taub-NUT, 651
tensor, 41
Tolman-Bondi-Lemaître, 265
Vaidya, 273
Van Stockum, 644
warp-drive, 638
Weyl-Majumdar-Papapetrou-Das
(or W-M-P-D), 307
wormhole, 635
metric equations, 309
metric tensor, 41
Minkowskian chart, 105
Minkowskian chronometry, 112
Monge surface, 102
monogenic forces, 154
most general solution, 585
multilinearity conditions, 17
Möbius group, 342

N
Newman–Penrose equations, 517, 523–525
Newtonian acceleration, 74
Newtonian approximation, 236
Newtonian gravitational constant, G, 66, 160
Newtonian mass, 66
nilpotent matrix, 606
nilpotent operator, 33
no hair theorem, 398
non-degeneracy conditions, 12
non-linear eigenvalue problem for a theoretical fine-structure constant, 559
non-null differentiable curve, 81
non-null geodesic, 77	norm or length, 42
normal coordinate, 67
normal forms, 595
normal or canonical forms, 492
null cone, 107
null curve, 111
null electromagnetic field, 224, 485
null energy condition, 178
null hypersurface, 130
null vectors, 106
numerical tensor, 29, 110
NUT potential, 649

O
Oppenheimer-Volkoff equation, 250
order (system of p.d.e.s), 587
order of a tensor, 16, 17
orientable manifold, 3
oriented relative tensor field, 25
oriented tensor field, 49
oriented volume, 64
orthochronous Lorentz transformation, 189
orthogonal coordinate chart, 61
orthogonal vector fields, 41
orthonormal basis, 41
orthonormal components, 46
outer ergosurface, 388
outer product, 18
overdetermined system, 164

P
Painlevé–Gullstrand coordinate chart, 357
Palatini approach for deriving field equations, 577
Papapetrou-Ehlers class, 336
parabola, semi-cubical, 16
parabolic p.d.e., 594
paracompact manifold, 1
parallelly propagated, 74
parameterized curve, 11
parametric surface, 88
parametrized hypersurface, 93
Penrose-Carter compactification, 394
perfect fluid, 177, 210–212
perihelion, 236
perihelion shift, 237
permutation symbol, 29
Petrov-types-I, D, II, N and III, 493
physical components, 120
Planck length, 457
Planck mass, 567
Planck’s constant, 538
Poincaré group, 108
Poincaré’s lemma, 32
Poisson’s equation, 161
polarization states of gravitational waves, 630
polarization tensor, 629
polytropic equation of state, 416
positive-definite metric, 42, 64
potential
complex, 319, 333
complex electromagnetic, 342
complex gravo-electromagnetic, 342
electromagnetic four, 34, 218, 226, 486, 538
electrostatic, 307
equation, 309, 311
function, 296
gravo-electromagnetic, 342
magnetic, 303
NUT, 649
scalar field, 446, 449
twist, 333, 340, 342
potential equation, 309, 311
potential function, 296
pressure, 177
principal curvature, 145
principal pressures, 179
principal tensions, 179
principle of equivalence, 137
Proca equation, 547
profile curve, 91
projection tensor, 180
proper time, 112
pseudo-angle, 43
pseudo-Cartesian chart, 43, 68
pseudo-Riemannian manifold, 64
pseudo-Schwarzschild metric, 299
Index

Q
quantum field theory in curved space–times, 538
quantum gravity, 583, 633, 651
quasi-linear p.d.e., 587
quintessence scalar field, 446–450

R
radiation era, 431
Rainich problem, 542
raising and lowering of indices, 47
rank
matrix, 17
tensor, 16, 17
Raychaudhuri-Landau equation, 181–182
redshift, 242
Regge-Wheeler tortoise coordinate, 358
region, 395
regularity conditions, 12
Reissner–Nordström–JefferY metric, 381
Reissner-Nordström-Jeffery metric, 255
relative tensor field, 25, 63
relativistic canonical equations of motion, 158
relativistic Hamiltonian equations of motion, 158
relativistic Hamiltonian mechanics, 156
relativistic Lorentz equations of motion, 126
relativistic total angular momentum, 121
reparameterization of a curve, 13, 14
reparametrization, 94
Ricci flat space, 70, 162
Ricci identities, 61
Ricci rotation coefficients, 58
complex-valued, 511
Ricci tensor, 62
Riemann curvature tensor, 54
Riemann normal coordinate chart, 67
Riemann-Christoffel curvature tensor, 59, 62
algebraic identities, 59
differential identities, 60
Riemannian manifold, 64
Riemannian or pseudo-Riemannian manifold, 56, 64
rigid motion, 39
rigid motions, 181
Robinson–Trautman metric, 527
Routh’s theorem, 234
ruled surface, 85, 96

S
scalar field, 19, 440–456
Born-Infeld or tachyonic, 451–455
massive, 443–446
massless, 440–443, 456–464, 575
quintessence, 446–450
scalar field potential, 446, 449
Schwarzschild metric, 231–242, 299, 336, 351–369
higher dimensional generalization, 244
Schwarzschild radius, 232, 351
Schwarzschild’s interior solution, 253
second contracted Bianchi’s identities, 63
second fundamental form, 91
Segre characteristic, 609–611
of energy-momentum-stress tensors, 175–177
of Ricci tensor, 498
self-dual, 651
self-duality, 480
semi-cubical parabola, 16
semi-linear p.d.e., 587
semiclassical gravity, 538
separation of a vector field, 41
shear tensor, 181
shock waves, 586
signature of the metric, 42
singularity theorems, 417–418, 434–435
solid particle, 221
space of constant curvature, 68, 73
space–time foam, 633, 634
spacelike curve, 111
spacelike vectors, 106
spin coefficients, 517
spinor algebra, 525
spinor analysis, 525
spinor fields, 110
stationary limit surface, 388
Stokes’ theorem, generalized, 34
stress density, 117
stress tensor field, 118
string theory, 453, 456, 464, 537
strong energy condition, 178
summation convention, 6
surface
Monge, 102
parametric, 88
ruled, 85, 96
surface of revolution, 91
Sylvester’s law, 600
symmetric tensor, 27
symmetrization operation, 26
symmetrized curvature tensor, 153, 170
symplectic group, 171
Synge’s doubly-null coordinates, 359
Synge’s junction conditions, 164
Synge’s world-function, 86
Index

**T**
- T-domain, 269
- tachyon condensation, 453
- tachyonic (or Born-Infeld) scalar field, 451–455
- tangent vector, 6, 7
- tangent vector space, 7
- Taub-NUT metric, 651
- tensor
  - antisymmetric, 27
  - canonical energy-momentum-stress, 119
  - conformal, 71
  - conjugate, contravariant, or inverse metric, 43
  - Cotton-Schouten-York, 297
- curvature
  - Riemann, 54
  - extended extrinsic, 580
  - extrinsic, 88–104, 165
  - Ricci, 62
  - Riemann-Christoffel, 59, 62
  - symmetrized, 153, 170
- density, 25
- Einstein, 63
- electromagnetic field, 33, 50
- energy-momentum-stress, 118
- expansion, 181
- extended extrinsic curvature, 580
- extrinsic curvature, 88–104, 165
- field, 19
- generalized Kronecker, 29
- hybrid field, 96–98
- identity, 45
- Killing, 399
- Levi-Civita, 48
- metric, 41
- numerical, 29, 110
- order of, 16, 17
- oriented, 25, 49, 63
- oriented antisymmetric, 48
- polarization, 629
- projection, 180
- rank of, 16, 17
- relative, 25, 63
- Ricci, 62
- Riemann, 54
- Riemann-Christoffel, 59, 62
- shear, 181
- stress, 118
- symmetric, 27
- symmetrized curvature, 153, 170
- torsion, 54
- totally antisymmetric, 27, 48
- trace-reversed perturbation, 627
- vorticity, 181
- Weyl conformal, 71
- Weyl’s projective, 71
- tensor density field, 25
- tensor field, 19
- tensor product, 18
- tetrad, 106, 465
  - complex null, 468
- Theorema Egregium, 101
- tidal forces, 141
- time machine, 641–645
- time orientable space-times, 641
- timelike curve, 111
- timelike vectors, 106
- Tolman-Bondi-Lemaître metric, 265
- Tomimatsu–Sato solution, 329
- topological manifold, 3
- topology, 1
- torsion tensor, 54
- total 4-momentum, 121
- total charge, 133
- total conservation of energy-momentum, 120
- total mass, 133
- total mass function, 261, 262
- totally antisymmetric oriented tensor, 48
- totally antisymmetric tensor, 27
- totally differentiable function, 10
- totally geodesic hypersurface, 313
- trace-reversed perturbation tensor, 627
- traceless-transverse gauge, 629
- twist potential, 333, 340, 342

**U**
- underdetermined system, 164
- unit matrix, 8
- unit vector field, 41
- upper triangular form, 606

**V**
- vacuum field equations, 162
- Vaidya metric, 273
- Van Stockum space–time, 644–645
- variational derivative, 571, 575
- variationally permissible boundary conditions, 571, 575
- vector
  - covariant, 9, 10
  - heat flow, 225
  - Killing, 72
  - norm or length, 42
  - null, 106
  - separation, 41
vector (cont.)
  spacelike, 106
  tangent, 6, 7
  timelike, 106
  unit, 41
vector space
  cotangent, 9
  dual, 9
  tangent, 7
volume
  oriented, 64
vorticity tensor, 181

W
warp-drive, 638–641
wave number, 486
weak energy condition, 178
weak solution, 586
wedge product, 28

weight of relative tensor field, 25
Weingarten’s equations, 98
Weyl conformal tensor, 71
Weyl’s coordinate chart, 278
Weyl’s projective tensor, 71
Weyl-Lewis-Papapetrou (WLP) charts, 317
Weyl-Majumdar-Papapetrou-Das
  (or W-M-P-D) metric, 307
white hole, 366
Willmore’s theorem, 618
winding number, 14
world line, 112
world tube, 117
world-function (Synge’s), 86
wormhole, 169, 255, 633–638

Y
Yukawa equation, 549