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NOVEL TRIANGLE RELATION AND ABSENCE OF TACHYON
IN LIOUVILLE STRING FIELD THEORY

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ABSTRACT

We construct the quantum mechanical field operator of the two-dimensional Liouville theory in a finite box. This leads us to the discovery of a new type of triangle relation which does not reduce to the already known ones. We apply our result to the construction of the string model in an arbitrary number of space-time dimensions $\mathcal{D}$. We show that there are no tachyons in $-\infty < \mathcal{D} \leq 1$, and we indicate how to continue to $\mathcal{D} > 1$, which is a strong coupling region for the Liouville field theory.

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1. - INTRODUCTION

In a series of recent articles^{1),2)}, we have displayed successive progress in solving the quantum Liouville field theory in a box, relevant for open string dynamics. In the present paper we take more steps in this direction. In our construction, we shall come across mathematical structures which are of wider interest in the realm of two-dimensional integrable models. Therefore, the present paper is written for a more general audience. The reader who is not yet interested in Liouville field theory can follow up to Section 4, provided he has faith in the existence of the general operator correspondence which we established in Ref. 2). However, the Liouville field theory does provide a wonderful insight.

In Ref. 1), we have, among other things, separated the Liouville degrees of freedom into harmonic oscillator modes which one may take as a starting point. We have introduced a field $P(\sigma)$ such that

$$P(\sigma) = \sum_{m=-\infty}^{+\infty} p_m e^{-im\sigma} \quad (1.1)$$

$$[p_m, p_n] = m\hbar \delta_{m,-n}$$

σ is the space variable. $\hbar = 2\pi\alpha'$ is the effective Planck constant related to the Liouville coupling constant in a way which does not concern us yet. A basic feature of the theory is the Virasoro conformal group structure. The corresponding infinitesimal generators write^{1),2)}

$$L_n = \frac{1}{2\hbar} N \left(\sum_r p_r p_{n-r} - im p_n \right) \quad n \text{ integer}, \quad (1.2)$$

where the symbol $N(\dots)$ means normal ordering (all p_n , $n > 0$ to the right). The operators L_n satisfy¹⁾

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \left(\frac{m^3}{4\hbar} + \frac{1}{12}(m^3-n) \right) \quad (1.3)$$

The first term has the standard Lie algebra form. The second term is a c number central charge which plays a crucial role, as is well known.

In Ref. 2), we showed that there exists another equivalent set of harmonic oscillator modes $\tilde{p}_n$ expressible as power series in p_n , with the following basic properties

$$[\tilde{p}_m, \tilde{p}_m] = m\hbar \delta_{m,-m} \quad (1.4)$$

$$N\left(\sum_{\mathcal{N}} p_{\mathcal{N}} p_{m-\mathcal{N}} - im p_m\right) = \tilde{N}\left(\sum_{\mathcal{N}} \tilde{p}_{\mathcal{N}} \tilde{p}_{m-\mathcal{N}} - in \tilde{p}_m\right) \quad (1.5)$$

$$\tilde{p}_0 = -p_0 \quad (1.6)$$

$\tilde{N}$ means normal ordering with respect to $\tilde{p}_n$. There is a complete symmetry between p and $\tilde{p}$. In particular, (1.5) shows that the L_n 's can be expressed in the same way in terms of either set.

An operator $\mathcal{A}(\sigma)$ is conformally covariant with dimension δ if it satisfies

$$[L_m, \mathcal{A}(\sigma)] = -i e^{im\sigma} \left(\frac{d}{d\sigma} - im\delta \right) \mathcal{A}(\sigma) \quad (1.7)$$

for all n . It is easy to check that both operators

$$\psi_1 = K e^{p_0 \eta \sigma} e^{-i\eta^2 \hbar \sigma} N\left(\exp\left[i\eta \sum_{m \neq 0} \frac{p_m}{m} e^{-im\sigma}\right]\right) \quad (1.8)$$

$$\psi_2 = \tilde{K} e^{-p_0 \eta \sigma} e^{-i\eta^2 \hbar \sigma} \tilde{N}\left(\exp\left[i\eta \sum_{m \neq 0} \frac{\tilde{p}_m}{m} e^{-im\sigma}\right]\right) \quad (1.9)$$

satisfy (1.7) with dimension $\delta = \frac{1}{2}(\eta + \eta^2)$, provided K and $\tilde{K}$ are such that

$$p_0 K = K(p_0 - i\hbar\eta) \quad ; \quad [p_m, K] = 0, m \neq 0 \quad (1.10)$$

$$p_0 \tilde{K} = \tilde{K}(p_0 + i\hbar\eta) \quad ; \quad [\tilde{P}_m, \tilde{\pi}] = 0, m \neq 0 \quad (1.11)$$

η is a c number which we shall choose below. The checking of this is similar to standard dual model computations³⁾, once one has proved²⁾ that the L_n 's can be written in the two equivalent forms (1.5).

Our semiclassical study of ground states in Liouville theory¹⁾ suggests that the spectrum of p_0 is discrete and purely imaginary. We shall indeed see that this is an appropriate quantum space. Letting

$$\omega = 2\pi i p_0 \quad (1.12)$$

we introduce a basis of zero mode eigenstates $|\omega_\nu\rangle$, ν integer, such that

$$\omega |\omega_\nu\rangle = \omega_\nu |\omega_\nu\rangle ; \quad P_m |\omega_\nu\rangle = \tilde{P}_m |\omega_\nu\rangle = 0, m > 0 \quad (1.13)$$

$$\omega_\nu = \omega_0 + \eta \nu \hbar, \quad \langle \omega_\nu | \omega_{\nu'} \rangle = \delta_{\nu, \nu'}$$

ω_0 will be chosen later. The full Hilbert space is obtained by applying the creation operators p_{-n} (or, equivalently, $\tilde{p}_{-n}$), $n > 0$ on $|\omega_\nu\rangle$. In this space, K and $\tilde{K}$ are completely determined if we further normalize them to be such that

$$\begin{aligned} \langle \omega_{\nu'} | K | \omega_\nu \rangle &= \delta_{\nu', \nu+1} \\ \langle \omega_{\nu'} | \tilde{K} | \omega_\nu \rangle &= \delta_{\nu', \nu-1} \end{aligned} \quad (1.14)$$

One can make an analogy between ψ_1 (or ψ_2) and the dual vertex for tachyon emission³⁾ if one identifies K with $\exp(\eta q_0)$, q_0 being the "centre of mass position". This is only formal because here the spectrum of the "centre of mass momentum" p_0 is discrete and finite.

The basic result of the present paper is that η can be chosen in such a way that ψ_1 and ψ_2 satisfy a closed algebra. We will show in Section 2 that, if η is such that

$$2\hbar\eta^2 - \eta + 1 = 0 \quad (1.15)$$

then there exists a matrix $S_{il}^{jk}(\omega, \sigma - \sigma')$ such that

$$\psi_i(\sigma) \psi_j(\sigma') = \sum_{k,e} S_{ie}^{jk}(\omega, \sigma - \sigma') \psi_k(\sigma') \psi_e(\sigma) \quad (1.16)$$

This structure is remarkably similar to the algebra of the Zamolodchikov operators in two-dimensional integrable models⁴⁾. The difference is that here, S_{il}^{jk} depends upon ω and hence does not commute with ψ_1, ψ_2 . Instead, we have

$$S_{ie}^{jk}(\omega, \theta) \psi_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \psi_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} S_{ie}^{jk}(\omega \pm \eta \hbar, \theta) \quad (1.17)$$

The associativity of the products of three ψ 's leads to a new type of triangle relation⁵⁾

$$\begin{aligned} & S_{i\mu}^{dp}(\omega, \theta) S_{\mu e}^{k\nu}(\omega + \varepsilon_p, \theta') S_{p'm}^{\nu m}(\omega, \theta' - \theta) \\ &= S_{2\nu}^{kp}(\omega + \varepsilon_i, \theta' - \theta) S_{ip}^{\mu m}(\omega, \theta') S_{pe}^{\nu m}(\omega + \varepsilon_m, \theta) \end{aligned} \quad (1.18)$$

$$\varepsilon_2 = -\varepsilon_1 = \eta \hbar$$

This structure provides non-trivial generalizations of the standard triangle relations. In Section 2, we obtain the following explicit solution

$$\begin{aligned} S_{11}^{11}(\omega, \theta) &= S_{22}^{22}(\omega, \theta) = e^{-i \frac{\hbar \eta^2}{2} \varepsilon(\theta)} \\ S_{12}^{21}(\omega, \theta) &= \frac{\sinh(\hbar \eta^2)}{\sinh(\eta \omega)} e^{i \left(\frac{\hbar \eta^2}{2} - \eta \omega \right) \varepsilon(\theta)} \\ S_{21}^{12}(\omega, \theta) &= - \frac{\sinh(\hbar \eta^2)}{\sinh(\eta \omega)} e^{i \left(\frac{\hbar \eta^2}{2} + \eta \omega \right) \varepsilon(\theta)} \end{aligned} \quad (1.19)$$

$$S_{11}^{22}(\omega, \theta) = \frac{\Gamma(1 + \frac{\eta\omega}{\pi}) \Gamma(\frac{\eta\omega}{\pi})}{\Gamma(1 + \frac{\eta\omega}{\pi} + \frac{\hbar\eta^2}{\pi}) \Gamma(\frac{\eta\omega}{\pi} - \frac{\hbar\eta^2}{\pi})} e^{i \frac{\hbar\eta^2}{2} \varepsilon(\theta)}$$

$$S_{22}^{11}(\omega, \theta) = \frac{\Gamma(1 - \frac{\eta\omega}{\pi}) \Gamma(-\frac{\eta\omega}{\pi})}{\Gamma(1 - \frac{\eta\omega}{\pi} + \frac{\hbar\eta^2}{\pi}) \Gamma(-\frac{\eta\omega}{\pi} - \frac{\hbar\eta^2}{\pi})} e^{-i \frac{\hbar\eta^2}{2} \varepsilon(\theta)} \quad (1.19) \text{ cont.}$$

All other matrix elements are zero. $\varepsilon(\theta)$ is equal to the sign of θ for $-\pi < \theta < \pi$ and is further extended as a periodic function, with period 4π discontinuous at $\theta = 2\pi n$. In this S matrix theory, the "number of particles" N is conserved if ψ_1 and ψ_2 are given opposite values N_1 and N_2 of N . For definiteness, we shall take $N_1 = 1 = -N_2$. Of course, it is of interest to try and derive new, exactly solvable models from this new triangle relation by means of the usual techniques, but we shall leave this aside for the moment.

In Section 3, we turn back to the Liouville problem. With the open string boundary conditions we developed earlier¹⁾, the following relation holds at the classical level ($\hbar = 0$)

$$e^{-\varphi/2} = 2\sqrt{2} [b \psi_1(\sigma) \psi_1(2\pi - \sigma) + a \psi_1(\sigma) \psi_2(2\pi - \sigma) - d \psi_2(\sigma) \psi_1(2\pi - \sigma) - c \psi_2(\sigma) \psi_2(2\pi - \sigma)] \quad (1.20)$$

ϕ is the Liouville field; a, b, c, d are functions of ω determined by the boundary conditions at $\sigma = 0$ and $\sigma = \pi$, which are the values of the parameter σ at the ends of the string. Classically we checked¹⁾ that ϕ is local for $0 < \sigma < \pi$. In Section 3, we introduce the quantum version of (1.20) and determine a, b, c, d in such a way that

$$[e^{-\varphi/2(\sigma)}, e^{-\varphi/2(\sigma')}] = 0 \quad 0 < \sigma, \sigma' < \pi \quad (1.21)$$

Due to the algebra (1.16), this equation reduces to quadratic equations for a, b, c, d . We solve these equations, obtaining for a, b, c, d formulae which are surprisingly close to their classical analogues. Hence, our classical treatment¹⁾ of the open string has a direct quantum extension. From the viewpoint of the two-dimensional S matrix suggested by the analogy with the Zamolodchikov algebra, (1.21) indicates the existence of a composite state which does not interact with itself. Equation (1.2) may also be regarded as similar to the commutativity properties of the traces of transfer matrices.

In Section 4, we study in some detail how our results apply to the building of the dual model. Following Ref. 6), we argue that the Virasoro algebra of the conformal generators must, for arbitrary space-time dimension $\mathcal{D}$, have the same central charge as the old model has for leading intercept $\alpha_0 = 1$ and $\mathcal{D} = 26$. This implies

$$\frac{1}{\hbar} = \frac{25 - \mathcal{D}}{3} \quad (1.22)$$

$$\alpha_0 = 1 - \frac{1}{8\hbar} \quad (1.23)$$

Equation (1.22) differs from Polyakov's value⁷⁾, $(26 - \mathcal{D})/3$, by finite renormalization effects. On the other hand, all our discussion assumes that η is real. Otherwise, $e^{-\phi/2}$, for instance, would not be Hermitian. From (1.15) and (1.22), one sees that this implies

$$\hbar < \frac{1}{8} \quad ; \quad \mathcal{D} < 1 \quad (1.24)$$

At the end of Section 4, we comment on the meaning of this condition by analogy with the sine-Gordon theory at $\beta^2 = 4\pi$, and on the extension of the theory to physical dimensions ($\mathcal{D} > 1$).

2. - THE FUNDAMENTAL ALGEBRA

We shall first prove Eq. (1.16) for its vacuum expectation value. Conformal covariance will then be used to extend the equality to arbitrary matrix elements.

As a starting point, it is useful to write down a differential equation for the general vacuum expectation value $\langle \omega_f | \psi_\ell(\sigma) \mathcal{A}(\sigma') | \omega_i \rangle$; $\ell = 1, 2$, where $\mathcal{A}(\sigma)$ is a conformally covariant operator with dimension δ :

$$[L_n, \mathcal{A}(\sigma')] = -i e^{in\sigma'} \left(\frac{d}{d\sigma} - in\delta \right) \mathcal{A}(\sigma) \quad (2.1)$$

Straightforward calculations lead to

$$\begin{aligned} & \left(\frac{d^2}{d\sigma^2} + i\hbar\eta^2 \frac{d}{d\sigma} - \frac{\hbar^2\eta^4}{4} \right) \langle \omega_f | \psi_1(\sigma) \\ &= \langle \omega_f | \psi_1(\sigma) \sum_{m \geq 0} \bar{e}^{im\sigma} N \left(\eta^2 \sum_{\mathcal{R}} p_{\mathcal{R}} p_{m-\mathcal{R}} - im(\eta + 2\hbar\eta^3) p_m \right) \end{aligned} \quad (2.2)$$

We now specify η to be the solution of the equation

$$2\hbar\eta^2 - \eta + 1 = 0 \quad (2.3)$$

According to (2.1), this gives

$$\left(\frac{d^2}{d\sigma^2} + i\hbar\eta \frac{d}{d\sigma} - \frac{\hbar^2\eta^4}{4} \right) \langle \omega_f | \psi_1 = 2\eta^2\hbar \langle \omega_f | \psi_1 \left(\sum_{m \geq 0} \bar{e}^{im\sigma} L_m \right) \quad (2.4)$$

In view of (1.4) and (1.7), the same equation obviously holds with ψ_1 replaced by ψ_2 . Next, using the conformal transformation properties of $\mathcal{A}$ together with the relations

$$L_m | \omega_i \rangle = 0 \quad m > 0 \quad (2.5)$$

$$L_0 | \omega_i \rangle = - \frac{(\omega_i^2)}{4\pi\hbar} | \omega_i \rangle \quad (2.6)$$

we obtain

$$\begin{aligned} & \left\{ \frac{\partial^2}{\partial\sigma^2} + i\hbar\eta^2 \frac{\partial}{\partial\sigma} + \frac{2i\hbar\eta^2}{1 - e^{-i(\sigma-\sigma')}} \frac{\partial}{\partial\sigma} + \frac{2\hbar\eta^2 \delta e^{-i(\sigma-\sigma')}}{(1 - e^{-i(\sigma-\sigma')})^2} \right. \\ & \left. + \eta^2 \left(\frac{\omega_i}{2\pi} \right)^2 - \frac{\hbar^2\eta^4}{4} \right\} \langle \omega_f | \psi_e(\sigma) \mathcal{A}(\sigma') | \omega_i \rangle = 0 \end{aligned} \quad (2.7)$$

The dependence of ψ_ℓ and $\mathcal{A}$ upon σ follows from their commutation relation with L_0 :

$$\begin{aligned}\psi_\ell(\sigma) &= e^{iL_0\sigma} \psi_\ell(0) e^{-iL_0\sigma} \\ \mathcal{A}(\sigma) &= e^{iL_0\sigma} \mathcal{A}(0) e^{-iL_0\sigma}\end{aligned}\quad (2.8)$$

This shows that

$$\langle \omega_f | \psi_\ell(\sigma) \mathcal{A}(\sigma') | \omega_i \rangle = e^{i(\omega_i^2 - \omega_f^2)\sigma/4\pi\hbar} \tilde{F}(\sigma - \sigma') \quad (2.9)$$

where $\tilde{F}$ is a function of one variable. Letting $z = e^{i\sigma}$ and $z' = e^{i\sigma'}$, one derives a differential equation for $\tilde{F}$, which is further reduced to a hypergeometric differential equation by letting

$$\begin{aligned}\tilde{F}\left(\frac{z'}{z}\right) &= \left(\frac{z'}{z}\right)^\kappa \left(1 - \frac{z'}{z}\right)^\beta F\left(\frac{z'}{z}\right) \\ \kappa &= \frac{1}{2} \left[-\hbar\eta^2 \pm \eta \frac{\omega_f}{\pi} \right]\end{aligned}\quad (2.10)$$

$$\beta = \frac{\eta}{2} \left[1 - \sqrt{1 + 8\hbar\delta} \right] \quad (2.11)$$

The last two relations suppress the poles in $(z'/z)^{-2}$ and $(1 - z'/z)^{-2}$ respectively and therefore reduce the differential equation to the standard hypergeometric form⁸⁾. One finally obtains

$$\begin{aligned}\langle \omega_f | \psi_\ell(z) \mathcal{A}(z') | \omega_i \rangle &= C(z')^{(\omega_i^2 - \omega_f^2)/4\pi\hbar} \left(\frac{z'}{z}\right)^\kappa \\ &\quad \left(1 - \frac{z'}{z}\right)^\beta F(a, b; c; \frac{z'}{z})\end{aligned}\quad (2.12)$$

$$a = \beta - \hbar\eta^2 \pm \eta \left(\frac{\omega_f - \omega_i}{2\pi} \right)$$

$$b = \beta - \hbar\eta^2 \pm \eta \left(\frac{\omega_f + \omega_i}{2\pi} \right)$$

$$c = 1 \pm \eta \frac{\omega_f}{\pi}$$

$F(a;b;c;z)$ is the standard hypergeometric function⁸⁾. The $\pm$ signs in (2.10) and (2.12) are correlated. The correct choice is easily made by inserting a complete set of eigenstates of L_0 between $\psi_\lambda(z)$ and $\mathcal{A}(z')$. Since the harmonic oscillators are equally spaced, one obtains a power series expansion in z'/z , up to a common non-analytic factor due to the vacuum energy, which has to match the term $(z'/z)^K$ in (2.12). One finds that the plus and minus signs correspond to ψ_1 and ψ_2 respectively. If the reader checks the above equations, he will find that there are two choices for β . Of course, they give the same answer due to the identity⁸⁾

$$F(a,b;c;y) = (1-y)^{c-a-b} F(c-a, c-b; c; y)$$

As usual, it is natural to continue in z and z' analytically; this is the standard Euclidean continuation.

Turning back to the case of present interest, we replace $\mathcal{A}$ by one of the two ψ operators and obtain

$$\begin{aligned} \langle \omega_f | \psi_e(z) \psi_e(z') | \omega_i \rangle &= \delta_{\nu_f, \nu_i + 2} z^{\frac{1}{2}(\hbar\eta^2 + \frac{\omega_f \eta}{\pi})} \\ &\quad (z')^{\frac{1}{2}(-\hbar\eta^2 + \frac{\eta \omega_i}{\pi})} \left(1 - \frac{z'}{z}\right)^{-\hbar\eta^2} \end{aligned} \quad (2.13)$$

$$\begin{aligned} \langle \omega_f | \psi_1(z) \psi_e(z') | \omega_i \rangle &= \delta_{\nu_f, \nu_i} \left(\frac{z'}{z}\right)^{\frac{1}{2}(-\hbar\eta^2 + \frac{\eta \omega_i}{\pi})} \\ &\quad \left(1 - \frac{z'}{z}\right)^{-\hbar\eta^2} F\left(-2\hbar\eta^2 + \frac{\eta \omega_i}{\pi}, -2\hbar\eta^2; 1 + \frac{\eta \omega_i}{\pi}; \frac{z'}{z}\right) \end{aligned} \quad (2.14)$$

$$\begin{aligned} \langle \omega_f | \psi_e(z) \psi_1(z') | \omega_i \rangle &= \delta_{\nu_f, \nu_i} \left(\frac{z'}{z}\right)^{\frac{1}{2}[-\hbar\eta^2 - \frac{\eta \omega_i}{\pi}]} \\ &\quad \left(1 - \frac{z'}{z}\right)^{-\hbar\eta^2} F\left(-2\hbar\eta^2 - \frac{\eta \omega_i}{\pi}, -2\hbar\eta^2; 1 - \frac{\eta \omega_i}{\pi}; \frac{z'}{z}\right) \end{aligned} \quad (2.15)$$

The first formula may, of course, be directly derived from (1.8) and (1.9). The normalization factors are easily checked from (1.9) by taking the limit $z'/z \rightarrow 0$ where the ground states dominate in the sum over intermediate states.

Finally the vacuum expectation value of (1.16) follows from the relation⁸⁾

$$F(a, b; c;) = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-x)^{-a} F(a, 1-c+a; 1-b+a; \frac{1}{x}) + \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-x)^{-b} F(b, 1-c+b; 1-a+b; \frac{1}{x}) \quad (2.16)$$

One has to take the limit $z \rightarrow e^{i\sigma}$, $z' \rightarrow e^{i\sigma'}$, remembering that (2.16) is valid for $|\arg(-x)| < \pi$. Equation (2.3) has real solutions only for $\kappa < \frac{1}{8}$; from now on, we assume that this is satisfied.

In order to generalize the above result to arbitrary matrix elements, it is useful to examine more closely the Hilbert space, where ψ_1 and ψ_2 live. We start from the standard Fock space generated by the operators p_{-n} acting on the vacuum. Taking, as is usual in dual models, p_{-n} as the Hermitian conjugate of p_n is in conflict with the antihermiticity of p_0 expected from the semi-classical analysis of Ref. 1). Hence, another hermiticity condition must be found. Separating for a while the zero mode $\omega = 2\pi i p_0$ from the other operators p_n , $n \neq 0$, represented by $\{p_n\}$, we note⁹⁾ that the following set of ket vectors forms a complete set:

$$|\omega, \{m\}\rangle = \prod_{k=1}^{\infty} [L_{-k}(\omega, \{p\})]^{m_k} |\omega\rangle \quad (2.17)$$

A convenient Hermitian conjugate is defined by the set of bra vectors:

$$\langle \omega, \{m\}| = \langle \omega| \prod_{k=1}^{\infty} [L_k(\omega, \{p\})]^{m_k} \quad (2.18)$$

We shall denote by a dagger the corresponding Hermitian conjugates. By means of Eq. (1.5), that is,

$$L_m(\omega, \{p\}) = L_m(-\omega, \{\tilde{p}\}) \quad (2.19)$$

one can verify the following equations:

$$(\omega, \{m'\} | \omega, \{m\})^* = (\omega, \{m\} | \omega, \{m'\}) \quad (2.20)$$

$$p_n^+ = \tilde{p}_{-n} \quad (2.21)$$

$$K^+ = \tilde{K} \quad (2.22)$$

Equation (2.21) is, of course, necessary for consistency. Equation (2.22), together with (2.20), shows that $L_n^+ = L_{-n}$, as expected. We adopt this Hermitian structure from now on. With round brackets, the matrix elements of the operators which satisfy closed commutation relations with the L_n are algebraically related to their vacuum expectation values (Wigner-Eckart theorem).

Going back to the algebra of ψ_1, ψ_2 we note that, due to their conformal covariance, one can derive any matrix element of $\psi_\ell(\sigma)\psi_\ell(\sigma')$ from a differential operator in σ, σ' acting on the corresponding vacuum expectation value. This differential operator is symmetric in σ and σ' since ψ_1 and ψ_2 have the same dimension $\frac{1}{2}(\eta + \eta^2)$. Moreover, $S_{i\ell}^{jk}$ is a constant for $\sigma = \sigma'$. Hence, the algebra (1.6) and (1.18) is preserved by the differential operator. It therefore holds for any matrix element between states of the form (2.17) and (2.18) QED.

As a further check, one can verify the triangle relation (1.18) and the inverse relation

$$S_{im}^{\partial m}(\omega, \theta) S_{me}^{mk}(\omega, -\theta) = \delta_{ik} \delta_{je} \quad (2.23)$$

This latter equation has the usual form. Both relations follow from elementary trigonometric identities and from the equation

$$S_{11}^{22}(\omega, \theta) S_{22}^{11}(\omega, \theta) = \frac{\sin(\eta\omega + \hbar\eta^2) \sin(\eta\omega - \hbar\eta^2)}{\sin^2(\eta\omega)} e^{i\hbar\eta^2 \xi(\theta)} \quad (2.24)$$

Finally, we note that, as a consequence of (2.21) and (2.22), we have

$$\psi_1^+ = \psi_2 \quad (2.25)$$

The Hermitian conjugation we have defined exchanges particles and antiparticles in the associated S matrix which is easily seen to be invariant. Hence, it is very natural from this viewpoint.

3. - THE LOCAL QUADRATIC OPERATOR

In this section, we build a quantum mechanical field operator as a quadratic form in ψ_1 and ψ_2 , starting from the direct extension of the classical formula (1.20):

$$e^{-\varphi/2} = 2\sqrt{2} \left[b(\omega) \psi_1(\sigma) \psi_1(2\pi - \sigma) + a(\omega) \psi_1(\sigma) \psi_2(2\pi - \sigma) - d(\omega) \psi_2(\sigma) \psi_1(2\pi - \sigma) - c(\omega) \psi_2(\sigma) \psi_2(2\pi - \sigma) \right] \quad (3.1)$$

The ordering of a, b, c, d with ψ_1 and ψ_2 is now relevant, but can be changed by simple shifts of ω .

In checking the locality condition

$$\left[e^{-\varphi(\sigma)/2}, e^{-\varphi(\sigma')/2} \right] = 0, \quad (3.2)$$

we conveniently separate the possible values of the total particle number N .

For $N = \pm 4$, (3.2) reduces to a trivial identity.

For $N = \pm 2$, there arise two linear relations in a and d which become simpler if we let

$$\begin{aligned} a(\omega) &= T\left(\frac{\eta\omega}{\pi} - \frac{\hbar\eta^2}{\pi}\right) T\left(-\frac{\eta\omega}{\pi}\right) \frac{e^{-i\eta\omega/2}}{\pi} \tilde{a}(\omega) \\ d(\omega) &= T\left(-\frac{\eta\omega}{\pi} - \frac{\hbar\eta^2}{\pi}\right) T\left(\frac{\eta\omega}{\pi}\right) \frac{e^{i\eta\omega/2}}{\pi} \tilde{d}(\omega) \end{aligned} \quad (3.3)$$

One obtains

$$\begin{aligned} \psi_{\nu+1} &= T_{\nu} \psi_{\nu} & \psi_{\nu} &= \begin{pmatrix} \tilde{a}(\eta\omega + 2\nu\eta^2\hbar) \\ \tilde{d}(\eta\omega - 2\nu\eta^2\hbar) \end{pmatrix} \\ T_{\nu} &= \frac{1}{\sin(\eta\omega + 2\nu\hbar\eta^2)} \begin{pmatrix} \sin(\eta\omega + (2\nu+1)\hbar\eta^2) & \sin(\hbar\eta^2) \\ \sin(\eta\omega + (2\nu-1)\hbar\eta^2) & -\sin(\hbar\eta^2) \end{pmatrix} \end{aligned} \quad (3.4)$$

In (3.4), the values of ω jump by $2\hbar\eta$. This is a general feature in the verification of (3.2), since $e^{-\phi/2}$ is quadratic in ψ_1, ψ_2 . Instead of (1.13), we can thus retain only the zero modes:

$$\omega y \alpha^{\pm 0} m = \alpha^{\pm 0} m \quad \{ \langle \alpha^{\pm 0} m | \} \quad (3.5)$$

After diagonalizing T_{ν} , one obtains the general solution

$$\begin{aligned} \tilde{a}(\omega) &= e^{i\hbar\eta^2/4} \left(\rho_1 e^{i(\eta\omega - \hbar\eta^2)/2} + \rho_2 e^{-i(\eta\omega - \hbar\eta^2)/2} \right) \\ \tilde{d}(\omega) &= - e^{i\hbar\eta^2/4} \left(\rho_1 e^{-i(\eta\omega + \hbar\eta^2)/2} + \rho_2 e^{i(\eta\omega + \hbar\eta^2)/2} \right) \end{aligned} \quad (3.6)$$

where ρ_1 and ρ_2 are two arbitrary constants.

Finally, for $N = 0$, one derives a finite difference equation for $b(\omega)c(\omega - 2\hbar\eta)$, which can be solved by analogy with the classical expression for bc ¹⁾. Letting

$$b(\omega) c(\omega - 2\hbar\eta) = \frac{\Gamma(\eta\frac{\omega}{\pi} - 2\frac{\hbar\eta^2}{\pi}) \Gamma(\eta\frac{\omega}{\pi} - \frac{\hbar\eta^2}{\pi})}{\Gamma(1 + \eta\frac{\omega}{\pi}) \Gamma(1 + \eta\frac{\omega}{\pi} - \frac{\hbar\eta^2}{\pi})} \tilde{b}(\omega) \tilde{c}(\omega - 2\hbar\eta) \quad (3.7)$$

one obtains, using (3.6),

$$\begin{aligned} \tilde{b}(\omega) \tilde{c}(\omega - 2\hbar\eta) &= e^{i\hbar\eta^2/2} \lambda \frac{\sin(\eta\omega - \hbar\eta^2)}{\sin(\eta\omega)} \\ &\quad - \frac{\tilde{a}(\omega) \tilde{d}(\omega - 2\hbar\eta^2)}{\sin(\eta\omega) \sin(\eta\omega - \hbar\eta^2)} \end{aligned} \quad (3.8)$$

Our equations are homogeneous in a, b, c, d , so we can choose $\lambda = 1$ by an appropriate shift of ϕ .

Next, we remark that $e^{-\phi/2}$ should be Hermitian. From (1.16) and (2.25), we deduce

$$\begin{aligned} p_1 &= p_1^* & p_2 &= p_2^* \\ b(\omega) &= c^*(\omega - 2\hbar\eta) e^{-i\hbar/2} \end{aligned} \quad (3.9)$$

From (3.6), (3.8) and (3.9), one sees that b and c are determined up to a phase factor by the relation

$$|b(\omega)|^2 = |c(\omega - 2\hbar\eta)|^2 = \frac{(1-p_1^2)(1-p_2^2) - [\omega(\eta\omega - \hbar\eta^2) + p_1 p_2] e^{\phi}}{\sin(\eta\omega) \sin(\eta\omega - \hbar\eta^2)} \quad (3.10)$$

In order to develop our next point, we have to look in more detail at our Hilbert space with the special metric introduced in Section 2. The problem is that this metric is not always positive definite. Consider, for example, the simplest excited state $L_{-1}|\omega\rangle$; its norm follows from (1.3)

$$\langle \omega | L_1 L_{-1} | \omega \rangle = \frac{1}{4\hbar} \left(1 - \left(\frac{\omega}{\pi} \right)^2 \right) \quad (3.11)$$

Hence, it becomes negative for $|\omega| > \pi$. Next, following an argument we already developed¹⁾, we look for possible additional ghost states.

If $|\omega| < \pi$, the replacement

$$L_0 \rightarrow L_0 + \frac{1}{8\pi}$$

gives a positive definite L_0 operator and changes (1.3) into

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \left(\frac{1}{4\pi} + \frac{1}{12} \right) (m^3 - n^3) \quad (3.12)$$

This is of the standard form much studied in dual models³⁾, where this algebra is realized in a Hilbert space of harmonic oscillators in a Euclidean space-time with $1 + \frac{3}{4}$ dimensions. A complete set of states is obtained by repeated application of the L_n 's on $|\omega\rangle$ ⁹⁾. Their norms can be computed from (3.12) only. Therefore, all states have a positive norm for $|\omega| < \pi$.

We then try to see whether we can restrict $e^{-\phi/2}$ to being an operator in a space built on top of a finite family of vacua $|\omega_v\rangle$, with $0 \leq v \leq M$ and such that

$$|\omega_0| < \pi \quad |\omega_M| < \pi \quad (3.13)$$

This will be the case if

$$\langle \omega_M + 2\pi\eta | e^{-\phi/2} | \omega_M \rangle = \langle \omega_0 - 2\pi\eta | e^{-\phi/2} | \omega_0 \rangle = 0 \quad (3.14)$$

For this, one realizes from (3.1) and (3.10) that the function

$$G(\omega) \equiv (1 - p_1^2)(1 - p_2^2) - [\cos(\eta\omega - \pi\eta^2) + p_1 p_2]^2 \quad (3.15)$$

should vanish at $\omega = \omega_0$ and $\omega = \omega_M + 2h\eta$, and should be positive in between. This is possible only if $|\rho_1| < 1$, $|\rho_2| < 1$, and we let

$$\rho_1 = \sin \Omega_1 \quad \rho_2 = \sin \Omega_2 \quad (3.16)$$

obtaining

$$G(\omega) = \frac{[\cos(\eta\omega - h\eta^2) - \cos(\Omega_1 + \Omega_2)] \times [\cos(\eta\omega - h\eta^2) + \cos(\Omega_1 - \Omega_2)]}{2} \quad (3.17)$$

The discussion is now very easy. We assume, for example, that $\rho_1 > 0$, $\rho_2 > 0$, that is $0 \leq \Omega_1 \leq \frac{\pi}{2}$. One obtains the solution

$$\omega_M = -\omega_0 = \frac{\Omega_1 + \Omega_2}{\eta} - h\eta \quad (3.18)$$

which is valid if

$$\Omega_1 + \Omega_2 = h\eta^2(M+1) \quad (3.19)$$

M integer,

The corresponding ρ_1, ρ_2 are indeed positive if

$$M+1 < \frac{\pi}{h\eta^2} \quad (3.20)$$

This condition automatically ensures that $|\omega_0| < \pi$, $|\omega_1| < \pi$. It can be fulfilled only if $h\eta^2 < \pi$, that is, if

$$\eta < 2 \quad (3.21)$$

This is satisfied only by one of the two possible choices for η in Eq. (2.3):

$$\eta = \frac{1}{4h} (1 - \sqrt{1 - 8h}) \quad (3.22)$$

In the classical limit to which we now turn, one lets $\hbar \rightarrow 0$, $\nu \sim O(1/\hbar)$, at fixed Ω_1, ω . Equation (2.3) has two solutions $\eta_{\pm}$ with $\eta_+ \eta_- = 1/2\hbar$. η_- tends to one, while η_+ blows up. Condition (3.21) has selected the regular solution. Our formulae for a, b, c, d change very little if one sets $\hbar = 0$, $\eta = 1$. One recovers our classical results¹⁾ straightforwardly generalized to $\rho_1 \neq \rho_2$. Classically, ρ_1 and ρ_2 are the parameters of our classical boundary condition

$$\begin{aligned} \partial_{\sigma}(e^{-\psi/2}) &= -\rho_1 \sqrt{2} \Big|_{\sigma=0} \\ \partial_{\sigma}(e^{-\psi/2}) &= \rho_2 \sqrt{2} \Big|_{\sigma=\pi} \end{aligned} \quad (3.23)$$

It is rather remarkable that, classically, locality gives, for a, b, c, d , the same values as the boundary conditions (3.23). Quantum mechanically, we have not tried to impose similar boundary conditions on $e^{-\phi/2}$ because of the singularities in the operator products of Eq. (3.1) for $\sigma \rightarrow 0$ and $\sigma \rightarrow \pi$. It is also remarkable that locality forces a quantization of the boundary parameters ρ_1 and ρ_2 , as can be seen in Eq. (3.19).

The classical limit sheds some light on the troublesome poles in ω which appear in (3.3), (3.7) and (3.10). In our quantum solution, the values of ω are

$$\omega_{\nu} = (2\nu - M) \hbar \eta \quad (3.24)$$

This shows that we have poles at $\omega = 0$ for even M and at $\omega = i\hbar\eta$ for odd M . These poles are the quantum equivalents of an unusual feature of the classical theory. There, $\omega = 0$ corresponds to the perturbative vacuum ϕ_0 which is at the minimum of the potential $V[\phi] = e^{\phi}$ and hence at $\phi_0 = -\infty$. Classically, we have $e^{-\phi_0/2} = +\infty$ and $e^{\phi_0/2} = 0$. Quantum mechanically, we expect a similar situation, namely, $e^{\phi/2}$ will have a zero instead of a pole. We can thus conjecture that

$$\begin{aligned} e^{\psi/2} |0\rangle &= 0 & M &= 2p \\ e^{\psi/2} |\hbar\eta\rangle &= 0 & M &= 2p+1 \end{aligned} \quad (3.25)$$

Classically, the ground state solutions at ω and $-\omega$ are the same, since these two choices are related by a Möbius transformation¹⁾. At the quantum level, we have the same situation: throughout our formulae, one can see that if $\omega \leftrightarrow -\omega$, $\psi_1 \leftrightarrow \psi_2$, $b \leftrightarrow c$, $a \leftrightarrow d$, then $e^{-\phi/2}$ is unchanged. The states $\omega < 0$ are mere replicas of the states $\omega > 0$. This replication had already been noticed by the authors of Ref. 10) in the closed string case. Assuming that (3.25) holds, we disregard them and the final set of ground states is

$$\begin{aligned} \omega_m &= 2m\hbar\eta \quad 0 \leq m \leq p \\ \text{if } \Omega_1 + \Omega_2 &= (2p+1)\hbar\eta^2 \end{aligned} \quad (3.26)$$

$$\begin{aligned} \omega_m &= (2m-1)\hbar\eta \quad 1 \leq m \leq p \\ \text{if } \Omega_1 + \Omega_2 &= 2p\hbar\eta^2 \end{aligned} \quad (3.27)$$

The corresponding numbers of states are

$$\mathcal{N} = \frac{\Omega_1 + \Omega_2}{2\hbar\eta^2} + \frac{1}{2} \quad (3.28)$$

and

$$\mathcal{N} = \frac{\Omega_1 + \Omega_2}{2\hbar\eta^2} \quad (3.29)$$

respectively. The effective Planck constant $\hbar$ is related to the Liouville coupling constant γ^2 of our earlier works^{1),2)} through the relation

$$\hbar = \frac{\gamma^2}{8} \quad (3.30)$$

With this equality, one sees that our exact results for ω_m agree with our semiclassical quantization¹⁾ of the ground state classical solutions if we replace $\hbar$ by $\hbar\eta^2$, or, equivalently, $1/\gamma^2$ by $1/\gamma^2(1 + \gamma^2/4)^2$. This replacement has already been found in the central charge of the Virasoro algebra upon canonical quantization of the Liouville theory¹⁰⁾. In view of the applications to string dynamics, to be discussed in the next section, we prefer to keep the simpler central charge of Eq. (1.3), and take a modified Planck constant $\hbar\eta^2$ in the quantization of the ground states.

4. - STRING CONNECTION

In the two previous sections, we have discovered rather interesting mathematical structures in the quantization of the Liouville theory in a box. We now wish to discuss their application to string dynamics.

The usual orbital string modes X_μ have their own Virasoro generators $L_n^{(x)}$, which commute with the Liouville L_n 's. The generators of conformal transformations are the sum of the two:

$$\mathcal{L}_m = L_m + L_m^{(x)} \quad (4.1)$$

Let us define $\mathcal{L}_0$ to be such that the physical string spectrum is determined through the condition³⁾

$$\mathcal{L}_m |\text{phys}\rangle = 0 \quad m \geq 0 \quad (4.2)$$

With this convention, one has

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n) \mathcal{L}_{m+n} + \frac{1}{12} (m^3 C_3 - m C_1) \delta_{m,-n} \quad (4.3)$$

$$C_3 = \mathcal{D} + 1 + \frac{3}{h} \quad , \quad C_1 = \mathcal{D} + 1 - 24\alpha_0 \quad (4.4)$$

$\mathcal{D}$ is the number of space-time dimensions, and $(-\alpha_0)$ is the vacuum energy of the modes X_μ .

The values of C_3 and C_1 are essential for the consistency of the whole theory. In the old days, without Liouville dynamics, one had

$$C_3 = \mathcal{D} \quad , \quad C_1 = \mathcal{D} - 24\alpha_0 \quad (4.5)$$

The corresponding Veneziano model is consistent only if $\alpha_0 = 1$, $\mathcal{D} = 26$, that is, if

$$C_3 = 26 \quad , \quad C_1 = 2 \quad (4.6)$$

More modern arguments based on B.R.S. techniques indicate that these last values are indeed the only possible ones⁶⁾. With the Liouville dynamics, we use formulae (4.4) instead of (4.5), obtaining

$$\frac{1}{h} = \frac{25-28}{3} \quad (4.7)$$

$$\alpha_0 = 1 - \frac{1}{8h} \quad (4.8)$$

Equation (4.7) differs from Polyakov's famous relation⁷⁾

$$\frac{1}{h_p} = \frac{26-28}{3} \quad (4.9)$$

This last value is, however, obtained by functionally integrating over X_μ for fixed ϕ , that is, for classical Liouville fields. The discrepancy between (4.7) and (4.9) is a finite renormalization effect of the quantum Liouville field theory.

The construction of the Liouville field $e^{-\phi/2}$ which we performed in Section 3 is valid only for $\star < \frac{1}{8}$, i.e., for $D < 1$. Let us first discuss this case. The point $\star = \frac{1}{8}$ is also precisely the one where, according to (4.8), α_0 changes sign. This brings in the question of tachyons, to which we now turn. For $\star < \frac{1}{8}$, condition (4.2) applied to the states which have neither X_μ nor ϕ excitations gives the mass squared of the lightest particles in each family

$$M_m^2 = -\left(\alpha_0 + \frac{\omega_m^2}{4\pi h}\right) \quad (4.10)$$

If the right-hand side is negative, we have tachyons. We use Eqs. (2.3), (3.26), (3.27) and (4.8) in order to check it. Letting $\xi = (\Omega_1 + \Omega_2)/\pi$, $0 \leq \xi \leq 1$, we obtain

$$M_m^2 = \frac{1}{4(\eta-1)} \left[(2-\eta)^2 - 4m^2(\eta-1)^2 \right] \quad (4.11)$$

$$0 \leq 2m \leq \frac{1+\xi-\eta}{\eta-1} \quad \text{if} \quad \eta-1 = \frac{\xi}{2p+1}$$

$$M_m^2 = \frac{1}{4(\eta-1)} \left[(2-\eta)^2 - (2m-1)^2(\eta-1)^2 \right]$$

$$0 \leq 2m-1 \leq \frac{1+\xi-\eta}{\eta-1} \quad \text{if} \quad \eta-1 = \frac{\xi}{2p} \quad (4.12)$$

We recall that $1 < \eta < 2$. A miracle has happened! The range of m which we determined purely from Liouville dynamics is precisely such that $M_m^2 \geq 0$. For $\xi = 1$, the highest value $m = p$ precisely gives $M_p^2 = 0$. Hence, the theory has no tachyon for $\kappa < \frac{1}{8}$. For $\xi = 1$, the condition of positivity of our Hilbert space is realized if, and only if, the lightest particle is massless. As κ approaches $\frac{1}{8}$, the range of values of m decreases. At $\kappa = \frac{1}{8}$ only $m = 0$, $\xi = 1$ is possible. All this has a remarkable similarity to well-known features of the sine-Gordon model, with coupling constant β^2 . For $\beta^2 < 4\pi$, there is a finite number of soliton-antisoliton bound states which reduces to one as β^2 approaches 4π . If a condition similar to (3.26) or (3.27) with $\xi = 1$ holds, one of them has zero binding energy. At $\beta^2 = 4\pi$, the sine-Gordon fundamental particle which is one of the soliton-antisoliton bound states, is precisely at threshold. For $\beta^2 > 4\pi$, only solitons are left in the spectrum. In the Zamolodchikov construction⁴⁾, the field of the fundamental particle ceases to exist, although, strangely enough, the S matrix element¹¹⁾ for the scattering of two of them is not so trivially pathological. On the other hand, from the exact solution, one knows that the soliton field and scattering amplitudes extend above $\beta^2 = 4\pi$ without any difficulty¹¹⁾.

We can thus expect a similar phenomenon in our case: for $\kappa < \frac{1}{8}$, any of the ground states of Section 3 can be used as an external state in a dual amplitude, and there are no tachyons. As κ increases, fewer and fewer ground states are allowed, and when $\mathcal{D}$ enters the physically interesting region ($\mathcal{D} > 1$, i.e., $\kappa > \frac{1}{8}$), none remains. However, this does not mean that the theory does not exist. Rather, one must find the correct Liouville excited states, analogous to the sine-Gordon soliton, which still permit the construction of a tachyon-free dual model. Once this is done, all consistency problems of the old dual model will be solved simultaneously.

We shall discuss these problems in a forthcoming publication; we shall also discuss a generalization of the differential equation of Section 2 to a product of an arbitrary number of covariant operators $\mathcal{A}_\ell(\sigma_\ell)$. This generalized

differential equation will later be used to derive closed formulae for the scattering amplitudes of the new dual model.

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