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GREEN FUNCTIONS AND SCATTERING AMPLITUDES IN
LIOUVILLE STRING-FIELD THEORY I

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ABSTRACT

We develop a method for the calculation of the expectation value of products of conformally covariant operators in the two-dimensional Liouville theory. We apply it to the evaluation of the four-point functions of the corresponding open string model, and discuss the Regge behaviour and duality properties of the result.

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1. INTRODUCTION

In a previous publication ¹⁾, we constructed the quantum field operators for the two-dimensional Liouville theory in a box, $0 \leq \sigma \leq \pi$, in view of its application to the dual string model. In sect. 2 of ref. 1 we introduced two operators ψ_1 and ψ_2 which obey a closed algebra. Out of ψ_1 and ψ_2 we were able to construct a local Liouville field operator $e^{-\frac{1}{2}\varphi}$ with appropriate conformal properties. These operators act on the Hilbert space of harmonic oscillator excitations on top of a set of ground states, labelled by the zero mode $p_0 = \omega/2i\pi$ of a free field $P(\sigma)$. It was found in ref. 1 that, in accordance with the semi-classical analysis of ref. 2, the locality and positivity of the Liouville theory imposed a finite and discrete set of possible values of this zero mode together with a quantization of the boundary condition parameters at $\sigma = 0$ and $\sigma = \pi$.

The algebra obeyed by ψ_1 and ψ_2 was proven by showing first that its vacuum expectation value satisfies a simple, hypergeometric type equation. In sect. 2, we want to show that this same technique can be extended to compute the vacuum expectation value of an arbitrary product of conformally covariant operators. In sect. 3, we shall apply this to the calculation of a part of the new open string four-point function which replaces the old Veneziano formula. By keeping the simplest Liouville-Green function only, we show that we recover exactly the same amplitude as the one we constructed ten years ago ³⁾ entirely in terms of the orbital modes of the string. In general, this amplitude had Regge behaviour (or duality) problems. In the discussion of sect. 4, we argue that good Regge behaviour and duality can probably be recovered for, at least, the special values of the Liouville coupling constant (i.e. the space-time dimension), for which one can make more extensive use of the algebra of the operators ψ_1 and ψ_2 . The point is that one should use the complete vertex for ground state emission which is suggested by the work of Polyakov ⁴⁾, that is, $e^{-\varphi/2}$ raised to a certain power. The amplitude we earlier computed ³⁾ is only a part of the final answer. This explains its difficulties with duality and/or Regge behaviour.

2. A DIFFERENTIAL EQUATION FOR THE PRODUCT OF COVARIANT OPERATORS

Following the notations of ref. 1, an operator $\mathcal{A}(\sigma)$ in the Liouville theory is conformally covariant with dimension δ if it satisfies the following commutation relation with the Virasoro conformal generators L_n :

$$[L_n, \mathcal{A}(\sigma)] = -ie^{in\sigma} \left(\frac{d}{d\sigma} - in\delta \right) \mathcal{A}(\sigma). \quad (2.1)$$

In general, one could expect that there exist many independent operators with a given conformal dimension. However, this is not so. A conformally covariant operator with a given dimension δ is a linear superposition of elementary operators, all with the same dimension, each of them shifting the zero mode p_0 by a discrete different value from one of the ground states to another. In other words, the operator is determined by δ and its expectation value between pairs of ground states. This follows from eq. (2.1) and the fact ⁵⁾ that all excited states are obtained by repeated application of L_{-n} ($n > 0$) to the ground states. The ground states $|\omega\rangle$ are labelled by the value $\omega = 2ip_0$ of the zero Fourier mode p_0 of $P(\sigma)$.

The basic remark of this paper follows from eq. (2.4) of ref. 1 :

$$\begin{aligned} \left(\frac{d^2}{d\sigma^2} + i\hbar\eta^2 \frac{d}{d\sigma} - \frac{\hbar^2\eta^4}{4} \right) <\omega_2 - \hbar\eta | \psi_1(\sigma) \\ = 2\eta\hbar^2 <\omega_2 - \hbar\eta | \psi_1(\sigma) \sum_{n \geq 0} e^{-in\sigma} L_n, \end{aligned} \quad (2.2)$$

where $\hbar$ is the coupling constant of the Liouville theory and η is defined by

$$2\hbar\eta^2 - \eta + 1 = 0. \quad (2.3)$$

To evaluate the vacuum expectation value of a product of operators $\mathcal{A}_i(\sigma_i)$ with dimensions δ_i , one considers first

$$A = \langle \omega_2 | \psi_1(\sigma) \mathcal{A}_1(\sigma_1) \mathcal{A}_2(\sigma_2) \dots \mathcal{A}_n(\sigma_n) | \omega_1 \rangle. \quad (2.4)$$

One then applies the differential operator in σ of the left-hand side of eq. (2.2), and pushes the L_n to the right until one hits the vacuum $|\omega_1\rangle$; one thus obtains a partial differential equation in $n+1$ variables $\sigma, \sigma_1, \dots, \sigma_n$ for A

$$\begin{aligned} & \left(\frac{\partial^2}{\partial \sigma^2} + i \hbar \gamma^2 \frac{\partial}{\partial \sigma} - \frac{\hbar^2 \gamma^4}{4} \right) A + \sum_{i=1}^n \left[\frac{2 i \gamma^2 \hbar}{1 - e^{-i(\sigma - \sigma_i)}} \frac{\partial}{\partial \sigma_i} \right. \\ & \left. + 2 \frac{\delta_i \hbar \gamma^2 e^{-i(\sigma - \sigma_i)}}{[1 - e^{-i(\sigma - \sigma_i)}]^2} \right] A + \gamma^2 \frac{\omega_1^2}{4\pi^2} A = 0. \end{aligned} \quad (2.5)$$

This equation is the immediate generalization to n operators of eq. (2.7) of ref. 1, valid for one operator only.

Because of the analogy of (2.4) with a dual model $n+3$ -point function written in a multiperipheral configuration it is natural to introduce the variable

$$e^{-i(\sigma - \sigma_1)} = z_1, \quad e^{-i(\sigma_1 - \sigma_2)} = z_2, \quad \dots \quad e^{-i(\sigma_{n-1} - \sigma_n)} = z_n. \quad (2.6)$$

After solving eq. (2.5) for A , one will simply obtain the expectation value

$$\langle \omega_2 | \mathcal{A}_1(\sigma_1) \mathcal{A}_2(\sigma_2) \dots \mathcal{A}_n(\sigma_n) | \omega_1 \rangle \quad (2.7)$$

by letting z_1 go to zero in A , at fixed $\sigma_1, \sigma_2, \dots, \sigma_n$, keeping only the leading behaviour in z_1 . This is a classic factorization trick of the old dual model. We must thus concentrate on picking the correct solution for eq. (2.5).

This is done by an examination of the contribution of the ground states as intermediate states in the operator product (2.4).

As mentioned above, according to the analysis of ref. 1, there is only

a finite number of ground states $|\omega\rangle$, and, besides its dimension, a conformally covariant operator can be characterized by its expectation values between different ground states. Hence without loss of generality, we can restrict the analysis to the case where each $\mathcal{A}_i$ operator shifts p_0 by a single, well-defined value $2im_i\hbar\eta$:

$$p_0 \mathcal{A}_i = \mathcal{A}_i (p_0 - 2im_i\hbar\eta),$$

In the cases of physical interest, m_i must be an integer but the dimension δ_i is unrestricted. For the operators ψ_1 and ψ_2 , one has $m = 1/2$ and $m = -1/2$ respectively. By introducing arbitrary intermediate states in eq. (2.4) one then finds that A is of the form

$$A = e^{-i\frac{\hbar}{2}\eta^2\sigma} e^{-i\left(\frac{\omega_1}{2\pi} + 2\sum_i m_i\hbar\eta\right)\eta\hbar\sigma} f(\beta_1, \beta_2, \dots, \beta_n) \\ \times \prod_{i=1}^n e^{-2i\hbar m_i^2\eta^2\sigma_i} e^{-2i\left(\frac{\omega_1}{2\pi} + 2\sum_{j=1}^n m_j\hbar\eta\right)m_i\eta\hbar\sigma_i},$$

(2.8)

where f is a series of positive powers of its arguments. It satisfies a partial differential equation which can be easily obtained by substituting the form (2.8) for A by eq. (2.5) and which we write down only for $n = 2$:

$$\beta_1 \frac{\partial^2 f}{\partial \beta_1^2} + \left\{ 1 + 2\eta \left[2m_1\eta\hbar + 2m_2\eta\hbar + \frac{\omega_1}{2\pi} \right] + \frac{2\eta^2\hbar}{1-\beta_1} \right\} \frac{\partial f}{\partial \beta_1} \\ - 2\eta^2\hbar \frac{1-\beta_2}{(1-\beta_1)(1-\beta_1\beta_2)} \beta_2 \frac{\partial f}{\partial \beta_2} \\ - 2\eta^2\hbar f \left\{ \frac{2m_1\eta}{1-\beta_1} \left[\frac{\omega_1}{2\pi} + 2\hbar m_2\eta + \hbar m_1\eta \right] \right. \\ \left. + \frac{2m_2\eta\beta_2}{1-\beta_1\beta_2} \left[\frac{\omega_1}{2\pi} + \hbar m_2\eta \right] \right. \\ \left. + \frac{\delta_1}{(1-\beta_1)^2} + \frac{\delta_2\beta_2}{1-\beta_1\beta_2} \right\} = 0. \quad (2.9)$$

For $z_2 = 0$, eq. (2.9) can be reduced to a hypergeometric equation in z_1 ,¹⁾ whose regular solution at $z_1 = 0$ is, in Bateman's notation⁶⁾

$$f(z_1, z_2=0) = (1-z_1)^{-\gamma\beta_1} F(a, b; c; z_1), \quad (2.10)$$

with

$$\beta_1 = \frac{1}{2}\gamma(1 - \sqrt{1+8\epsilon\delta_1}), \quad (2.11)$$

$$a = -\gamma\epsilon(\beta_1 - 2m, \gamma),$$

$$b = \gamma\left(\frac{\omega_1}{\pi} - \epsilon(\beta_1 + 2\epsilon m, \gamma)\right),$$

$$c = \gamma\left(1 + 4\epsilon m, \gamma + \frac{\omega_1}{\pi}\right). \quad (2.12)$$

This function can be understood as the contribution to A of the ground state between the operators $\mathcal{A}_1$ and $\mathcal{A}_2$. The contributions of the excited states give successive terms in the expansion of f in power series in z_2 . Using, again, the fact that all excited states are obtained by application of the L_{-n} ($n > 0$) operators to the ground state, and working the L_{-n} to the left, one sees that the coefficient of z_2^n in $f(z_1, z_2)$ is a function of z_1 obtained by applying an n -th order differential operator to $f(z_1, z_2 = 0)$ of eq. (2.10). This remark plays the role of a boundary condition on f at $z_2 = 0$, and completely determines the solution of eq. (2.9) to all orders in z_1 .

In the case of n operators $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$, one can derive the generalization of eq. (2.9), which is a partial differential equation, second order in z_1 and first order in $z_2, z_3, \dots, z_n$. By the same argument as above for the $n = 2$ case, its solution is determined recursively on n by the fact that it should be a power series in z_n , which at $z_n = 0$ reduces to the solution of the equation with $n-1$ variables and whose coefficient of z_2^n is some n -th order differential operator in $z_1, z_2, \dots, z_{n-1}$, applied to this solution.

3. SCATTERING AMPLITUDES IN LIOUVILLE STRING-FIELD THEORY

Let us apply the results of the previous section to the calculation of dual

scattering amplitudes. In the old dual model, the intercept was restricted to be one by the requirement of conformal covariance, which provides the necessary ghost Killing mechanism. According to ref. 1, the Liouville string-field theory has only negative (or zero) intercept trajectories. Hence, the old dual vertex V_0 must be multiplied by an operator V_φ built out of the Liouville modes, which restores conformal invariance. Thus the dimension of V_φ is determined by the mass of the external particles, which we shall assume to be one of the ground states determined in ref. 1.

As we have seen above, a conformally covariant V_φ is not entirely determined by its dimension but one must also specify how it connects the various ground states. Polyakov's study⁴⁾ of the string functional integral with external sources suggests that V_φ is simply $e^{\lambda\varphi}$ where λ is chosen in such a way that $V_0 V_\varphi$ has dimension -1 . For $\hbar < 1/8$ we showed¹⁾ that there are several ground states. The corresponding values of λ are derived in the appendix. If $\hbar = n/2(n+1)^2$, n integer, 2λ is an integer and V_φ is formally obtained by raising $(e^{-\varphi/2})$ to a negative integer power. Hence, it can be written as a sum of terms $\mathcal{A}_i$ satisfying (2.8). We have built $e^{-\varphi/2}$ as a quadratic operator¹⁾, but we have not so far been able to derive useful formulae for its negative powers. On the other hand, we know¹⁾ that the number of ground states is finite, at least for $\hbar < 1/8$. Hence $(e^{\lambda\varphi})$ must involve only a finite number of operators $\mathcal{A}_i$ satisfying (2.8). All the terms of V_φ are presumably necessary for duality but we are not in a position to investigate this problem. In this paper, as a first sample calculation, we shall only consider the contribution of the simplest Liouville-Green function. Assume we are in a case where one ground state has $\omega = 0$. According to formula (4.11) of Ref. 1, the mass of the corresponding particle is $M_0^2 = -1 + 1/8\hbar$. The calculations of the Appendix show that the Liouville part of the vertex for emission of this particle has $\delta = -1/8\hbar$. It involves various operators $\mathcal{A}_i$ but, to make things simple, we only retain the part of V_φ which does not change ω [$m_i = 0$ in (2.8)].

The corresponding piece of the 4-point-Veneziano amplitude for the scattering of the ground state of the string with $\omega = 0$ is thus (fig. 1)

$$F(s, t) = \int_0^1 dz_2 z_2^{-\alpha_s-1} (1-z_2)^{2p_2 p_3} f(z_1=0, z_2). \quad (3.1)$$

where $f(z_1, z_2)$ is the solution of eq. (2.9) with the following values of the parameters

$$m_1 = m_2 = \omega_1 = 0, \quad (3.2)$$

$$\delta_1 = \delta_2 = -\frac{1}{8\epsilon} \quad (3.3)$$

and where

$$\alpha_s = -(p_1 + p_2)^2 + 1 - \frac{1}{8\epsilon},$$

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 1 - \frac{1}{8\epsilon}. \quad (3.4)$$

Duality properties of eq. (3.1) can be investigated by means of a simple change of variables in eq. (2.9) : by a straightforward calculation one finds that with the values (3.2) and (3.3) of the parameters, eq. (2.9) is invariant under the change

$$z_1 = -\frac{z'_1}{1-z'_1}, \quad z_2 = 1-z'_2,$$

$$f(z_1, z_2) = z'_2^{-\frac{1}{8\epsilon}} (1-z'_1)^{-\frac{1}{2}} (1-z'_2)^{\frac{1}{8\epsilon}} f(z'_1, z'_2).$$

(3.5)

Performing the same change of variables in eq. (3.1), one obtains

$$F(s, t) = \int_0^1 dz'_2 z'_2^{-\alpha_t-1} (1-z'_2)^{2p_1 p_2} f(z'_1=0, z'_2), \quad (3.6)$$

so that, formally, the fundamental duality equation

$$F(s, t) = F(t, s), \quad (3.7)$$

is satisfied.

One can also investigate the behaviour of the integrand of eq. (3.1) under non-cyclic permutation of the external lines of fig. 1. This is done by the change

of variables

$$z_1 = z_1'', z_2 = \frac{1}{z_2''} \quad (3.8)$$

One finds that this change of variables, together with $f(z_1, z_2) = f(z_1'', z_2'')$ leaves eq. (2.9) invariant, again provided that eqs. (3.2) and (3.3) are satisfied. The same change of variables in eq. (3.1) shows that the integrand is formally invariant under non-cyclic permutations of the external lines

$$\begin{aligned} & d z_2 z_2^{-\alpha_3-1} (1-z_2)^{2p_2 p_3} f(z_1=0, z_2) \\ &= d z_2'' z_2''^{-\alpha_3-1} (1-z_2'')^{2p_2 p_3} f(z_1''=0, z_2''), \end{aligned} \quad (3.9)$$

exactly as in the old dual model for $\alpha_0 = 1$.

The above analysis can be summarized by the fact that $f(z_1=0, z_2)$ can be written as

$$f(z_1=0, z_2) = (1-z_2)^{-\frac{1}{8\epsilon}} (1-z_2+z_2^2)^{\frac{1}{16\epsilon}} g(y), \quad (3.10)$$

with

$$y = \frac{27 z_2^2 (1-z_2)^2}{4 (1-z_2+z_2^2)^2}, \quad (3.11)$$

$g(y)$ being a formal power series in y , so that

$$F(s, t) = \int_0^1 d z_2 z_2^{-\alpha_3-1} (1-z_2)^{-\alpha_4-1} (1-z_2+z_2^2)^{\frac{1}{16\epsilon}} g(y) \quad (3.12)$$

is a superficially perfectly satisfactory result. We have computed the function $g(y)$ to first order in y , from the z_2 power series solution of eq. (2.9) (a rather tedious calculation), and we find :

$$g(y) = 1 - \frac{128 \epsilon^2 + 244 \epsilon + 119}{2048 \epsilon (1+\epsilon)^2} y + O(y^2). \quad (3.13)$$

It is interesting to note that $g(y)$ does not exhibit any pathology, at least at this order, when $\hbar$ crosses the value $1/8$, when η becomes complex. The explicit calculation to order z_2^2 of $f(z_1, z_2)$ shows that f is complex as soon as $z_1 \neq 0$, for $\hbar > 1/8$. This is an example of our analogy in ref. 1 : at $\hbar = 1/8$, the ground state of the Liouville theory disappears, and the Liouville field itself ceases to exist, but the corresponding scattering amplitudes show no pathology. In ref. 1, we argued that this was analogous to what happens in the sine-Gordon theory at $\beta^2 = 4\pi$, where the "elementary" particle of the sine-Gordon Lagrangian unbinds into a soliton-antisoliton pair, but its S-matrix is regular.

At this point, one may remember our attempts³⁾, ten years ago, to build ghost-free dual models at arbitrary intercept and space-time dimension out of the orbital modes of the old dual model. These attempts were based on a modification of the orbital vertex by a function of the orbital L_{-n} operators, to be determined so that conformal covariance is restored. We had found a level-by-level recursive scheme to determine the new vertex, together with a new four-point function $F_1(s, t)$ which could be written in exactly the same form as eq. (3.12) :

$$F_1(s, t) = \int_0^1 dz_2 z_2^{-\alpha_s-1} (1-z_2)^{-\alpha_t-1} (1-z_2+z_2^2)^{\frac{1}{2}(1-\alpha_0)} g_1(y), \quad (3.14)$$

and in ref. 3 we had determined g_1 to order y^2 included. It is thus interesting to compute g_1 for the particular case of the present paper

$$\alpha_0 = 1 - \frac{1}{8\hbar} = \frac{2-1}{24}. \quad (3.15)$$

To order y , the general formula for g_1 is, after correction of a misprint in ref. 3,

$$g_1(y) = 1 - \frac{(1-\alpha_0)(-11\alpha_0^2 - \frac{3}{2}\alpha_0\mathcal{D} + 117\alpha_0 + \frac{13}{2}\mathcal{D} - 158)}{54[-8\alpha_0^2 + (37-\mathcal{D})\alpha_0 - 42 + \frac{3}{2}\mathcal{D}]}. \quad (3.16)$$

Upon substitution of (3.15) into (3.16), one recovers (3.13) exactly !

4. DISCUSSION

The final results of sect. 3 represent the first results of an exact calculation of open string scattering amplitudes following Polyakov's approach. They call for several comments.

It is rather remarkable that, from a completely different starting point, we seem to recover the same amplitude as the one we constructed ten years ago. This implies that one should be able to eliminate the Liouville modes and effectively rewrite the vertex operator V_φ entirely in terms of the orbital modes. We have not attempted this ; however, we can note the following fact : the set of physical states in the Liouville string-field theory is defined through the condition (4.2) of ref. 1 :

$$\mathcal{L}_n | \text{physical state} \rangle = 0 \quad , \quad n \geq 0 . \quad (4.1)$$

This set, when $\omega = 0$, and $\alpha_0 = 1 - \frac{1}{8\alpha} = \frac{D-1}{24}$, contains a set of zero norm states, so that effectively the set of positive norm physical states is identical to a set of $D-1$ harmonic excitation states ; these states have exactly the same multiplicity as the states defined by the conditions :

$$L_n(\text{orbital}) | \text{physical state} \rangle = 0 \quad , \quad n > 0 ,$$

$$[L_0(\text{orbital}) - \alpha_0] | \text{physical state} \rangle = 0 , \quad (4.2)$$

of ref. 3 .

The procedure of ref. 3 gave rise to amplitudes which in general had Regge behaviour of duality problems. These problems arose from the fact that the function $g_1(y)$ in eq. (3.14) does not in general converge for all z_2 between 0 and 1. In the case of the Liouville string-field theory, we may expect the same phenomenon : eq. (3.10) would hold for z_2 (and y) small enough, and break down at some value of z_2 , above which the determination of $g(y)$ to be used would change. The invariance of the differential equation (2.9) under the changes of

variables (3.5) and (3.8) does not mean that a given solution is invariant. It rather means only that the whole set of solutions is transformed into itself. Indeed, such a behaviour is well known in the theory of hypergeometric functions⁵⁾, and the following toy model with one variable shows all this.

Consider the hypergeometric equation

$$z(1-z) \frac{d^2 f}{dz^2} - 2(1-2z) \frac{df}{dz} - 4f = 0, \quad (4.3)$$

where a regular solution at $z = 0$ is for example

$$f(z) = (1+z)(1-z)^3. \quad (4.4)$$

Equation (4.3) is invariant under the following changes :

$$z' = 1-z, \quad f(z) = f'(z'), \quad (4.5)$$

and

$$z'' = \frac{1}{z}, \quad f(z'') = z''^{-4} f''(z''), \quad (4.6)$$

i.e., $f'(z')$ and $f''(z'')$ satisfy eq. (4.3) with their respective variables.

It follows that one can write, formally :

$$f(z) = (1-z+z^2)^2 g(y), \quad (4.7)$$

$$y = \frac{27z^2(1-z)^2}{4(1-z+z^2)^3}, \quad (4.8)$$

but the solution (4.4) does not seem to be of this type. Actually, it is, provided one writes

$$1-2z = \sqrt{1-4z(1-z)}, \quad (4.9)$$

and one solves eq. (4.8) as a cubic equation for $z(1-z)$ in terms of y .

But eq. (4.9) has a different meaning for $z < 1/2$ and $z > 1/2$! This kind of pathology, which could be called spontaneous breaking of duality, is disastrous for Regge behaviour, as discussed in ref. 3. The disaster comes from the non-analyticity of $g(y)$ as a function of z at $z = 1/2$, which is apparent in eq. (4.9). Alternatively, one may keep Regge behaviour, but, in general, crossing symmetry is lost if one does not perform the changes of variables (3.5) and (3.8). This

means that the poles of the amplitude in the t channel, arising from the behaviour of the function $f(z_1 = 0, z_2)$ at $z_2 = 1$, are not in general located at the same position as those in the s channel.

This leads us back to the duality problem. The poles which occur in the t channel should coincide with those of the s channel. Allowing for this avoids the non-convergence of the series $g(y)$ of eq. (3.10), but one may expect that, in general, it will force the existence of a well-defined set of ground states in the s channel to match the poles of the t channel. One may even be forced to only special values of the coupling constant $\hbar$, i.e. the space-time dimension D *). In a forthcoming paper, we shall show that for $\hbar = 2n - 1/8n^2$, n integer, there exists a closed algebra for the integer negative powers of ψ_1 and ψ_2 , which enables us to construct a crossing symmetric set of amplitudes, and that the set of ground states is the maximal one found in ref. 1 for these values of $\hbar$.

For $\hbar < 1/8$ we already anticipated that we were going to run into trouble with duality. Indeed $V_\varphi = e^{-\lambda\varphi}$ has terms which shift ω , and therefore the amplitude studied in sect. 3 is only one of several terms which get mixed by duality transformation. Hence, the Liouville theory while giving back our results of ten years ago ³⁾, tells us precisely why we ran into trouble and what the remedy is.

APPENDIX

The orbital part of the vertex for ground state emission has a dimension equal to the mass squared M^2 of the emitted particle. On the other hand, $e^{\lambda\varphi}$ has the same dimension as $(\psi_1^{-4\lambda})$, that is $\frac{1}{2}(-4\lambda\eta + (4\lambda\eta)^2)$. Altogether $V_0 e^{\lambda\varphi}$ has dimension -1 if

$$2(M^2 + 1) - 4\lambda\eta + 16\lambda^2\eta^2 = 0. \quad (A.1)$$

*) Duality and Regge behaviour had already been recovered in Ref. 3 for special values of the intercept (odd negative integer), together with special values of the dimension. This corresponded to cases where the function $g_1(y)$ of Eq. (3.14) was a polynomial. Unfortunately, these models had ghosts.

From Ref. 1, we know that there is a set of states of mass M_m , m integer, such that

$$M_m^2 + 1 = \frac{1}{8\pi\eta} \left(1 - \frac{\omega_m^2}{\pi^2} \right), \quad (\text{A.2})$$

and (A.1) gives

$$4\lambda = \frac{1}{2\pi\eta} \left(1 \pm \frac{\omega_m}{\pi} \right). \quad (\text{A.3})$$

There are two cases, according to ref. 1 :

$$\text{a) } \omega_m = 2m\pi\eta, \quad 0 \leq m \leq p, \quad 4\lambda = \frac{\eta}{\eta-1} \pm 2m, \quad (\text{A.4})$$

$$\text{b) } \omega_m = (2m-1)\pi\eta, \quad 1 \leq m \leq p, \quad 4\lambda = \frac{\eta}{\eta-1} \pm (2m-1). \quad (\text{A.5})$$

In either case, 4λ is an integer if $\eta/\eta-1$ is an integer, that is if $\eta = n/2(n+1)^2$ with n integer. In the general case where $\eta/\eta-1$ is not integer, the various vertices give shifts of ω which are not of the form (2.8) but which differ one from the other by integers. This case is more difficult since $e^{\lambda\varphi}$ is not built in the same space as the original $e^{-\varphi/2}$. The dual model may even not exist.

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Fig. 1 : The four-point Veneziano amplitude.