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FURTHER PROPERTIES OF CLASSICAL SOLUTIONS IN GRASSMANNIAN σ MODELS

A.M. Din ^{*)}

Institute of Theoretical Physics
University of Lausanne, Switzerland ^{**)}

and

W. J. Zakrzewski ^{†)}
CERN -- Geneva

ABSTRACT

We calculate the action and the topological charge of classical solutions in two-dimensional Euclidean Grassmannian σ models (for both generic and special solutions). We show that all the above non-instanton or non-anti-instanton solutions are unstable. We also find solutions of the Dirac equation describing fermions in the background of these Grassmannian fields, and we exhibit relations between these fermionic solutions and those of the associated Hilbert-Riemann problem.

^{*)} Research supported by the Swiss National Science Foundation.

^{**)} Postal address : University of Lausanne, BSP Dorigny, CH-1015 Lausanne, Switzerland.

^{†)} On leave of absence from Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom.

1. - INTRODUCTION

The Grassmann σ model $G(M,N)$ ¹⁾ may be defined in terms of a $N \times M$ matrix field $Z(x)$, with $1 \leq M \leq N$, where x belongs to the compactified two-dimensional Euclidean plane ^{*}). The field Z is chosen to fulfil the constraint

$$Z^+ Z = 1_M \quad (1.1)$$

and the Lagrangian is given by

$$\mathcal{L} = \text{Tr} (D_\mu Z)^+ D_\mu Z = 2 \text{Tr} [(D_+ Z)^+ D_+ Z + (D_- Z)^+ D_- Z]. \quad (1.2)$$

Here, and in what follows, the covariant derivatives $D_\pm$ are defined as

$$D_\pm \phi = \partial_\pm \phi - \phi Z^\pm \partial_\pm Z \quad (1.3)$$

in terms of the complex variables $x_\pm = x_1 \pm ix_2$. The Lagrangian is invariant under the non-Abelian $U(M)$ transformations $Z \rightarrow ZU$, and so the real degrees of freedom are the equivalence classes under these transformations.

In this paper, we analyse various properties of the large, possibly complete class of finite action classical solutions of this model, i.e., of the stationary points of the action functional $S = \int dx_+ dx_- \mathcal{L}$ subject to the requirement that S be finite. These solutions were derived in Refs. 3) and 4), where some of their properties were also analysed. In this paper, we calculate the action and the topological charge for these solutions, and show that, similarly to the $\mathbb{CP}^{N-1} = G(1,N)$ case, they have a simple form, which underlies their geometrical origin. We also show that, as in the $\mathbb{CP}^{N-1}$ case ⁵⁾, only the instanton or anti-instanton solutions are stable.

Next, we consider the problem of solving the Dirac equation describing fermions in the background of these Grassmannian fields. In analogy to the similar problem in the $\mathbb{CP}^{N-1}$ case ⁶⁾, we require the fermion fields to be orthogonal to the Grassmann fields - a constraint that would come in a natural way had we derived our Dirac equations from the supersymmetric form of (1.2) and neglected

^{*}) Here we return to the more usual notation; the notation of Ref. 2) and Ref. 3) differs from the present one by the transposition of all matrix quantities.

the bilinear fermionic term. We finish this paper by exhibiting the relation between the derived solutions of the Dirac equation and the solutions to the related Riemann-Hilbert problem⁷⁾.

2. - SOLUTIONS OF THE EQUATIONS OF MOTION

The solutions of the equations of motion,

$$D_+ D_- Z + Z (D_- Z)^+ D_+ Z = 0, \quad Z^+ Z = 1, \quad (2.1)$$

which have finite action, were analysed in Refs. 3) and 4). There it was shown that these solutions fall into several classes. Of these, the most important one (henceforth called generic) is characterized by the dependence on M analytic vectors with polynomial entries $f_1 \dots f_M$. The construction of solutions of the equations of motion Z from these vectors was described in Ref. 3); here we repeat it for completeness.

We choose integers K_i , $i = 1, \dots, M$ such that $K_1 \geq K_2 \geq K_3 \geq \dots \geq K_M \geq 1$ and $K_1 + K_2 + \dots + K_M = N$. Then, in general, the N vectors

$$\partial_+^{l_i} f_p; \quad l_i = 0, 1, \dots, K_p - 1; \quad p = 1, \dots, M \quad (2.2)$$

span $\mathbb{C}^N$. We choose a certain order of them [governed by the requirement to fulfil the condition (2.4) below], and denote them by g_β , $\beta = 1, \dots, N$. Next, we define the subspaces $H_\beta = \{g_1, \dots, g_\beta\}$ (with $H_0 = \emptyset$ and $H_\beta = \mathbb{C}^N$ for $\beta \geq N$) and perform a Gramm-Schmidt orthogonalization of g_i

$$\begin{aligned} \hat{e}_\beta &= g_\beta - g_\beta \downarrow H_{\beta-1} \quad ; \quad \beta = 1, \dots, N \\ e_\beta &= \hat{e}_\beta / |\hat{e}_\beta|. \end{aligned} \quad (2.3)$$

Then, as was shown in Ref. 3), provided that the order of the vectors g_β was chosen so that

$$\partial_+ H_\beta \subset H_{\beta+M} \quad (2.4)$$

the Grassmannian $Z^{(\beta)}$ $\beta = 1, \dots, N-M+1$, considered as a matrix whose columns are formed by the orthonormal vectors $e_\beta, e_{\beta+1}, \dots, e_{\beta+M-1}$, i.e.,

$$Z(\beta) = (e_\beta, e_{\beta+1}, \dots, e_{\beta+M-1}) \quad (2.5)$$

solves the equation of motion (2.1).

Sometimes it is convenient to use the expression for the Grassmannian $Z(\beta)$, whose column vectors are not mutually orthogonal. In this case, having chosen β , we consider the M vectors $g_\beta, g_{\beta+1}, \dots, g_{\beta+M-1}$, and orthogonalize them with respect to $H_{\beta-1}$ (but not to each other), defining

$$\hat{g}_j = g_j - g_j \downarrow H_{\beta-1} \quad ; \quad j = \beta, \beta+1, \dots, \beta+M-1 \quad (2.6)$$

Then, having formed a matrix $\hat{Z}$, whose columns are given by $\hat{g}_j$, we find that the solution Z is given by

$$Z = \hat{Z} (M)^{-1/2} \quad (2.7)$$

where

$$M = \hat{Z}^+ \hat{Z} \quad (2.8)$$

The expression (2.7) is, of course, completely equivalent to (2.5), but happens to be more useful in some applications.

Let us add that there exist various choices of the vectors (2.2), and thus of their ordering which satisfies the condition (2.4). In particular, we can choose

$$\{g_\beta\} = \{f_1, f_2, \dots, f_M, \partial_+ f_1, \partial_+ f_2, \dots, \partial_+ f_M, \partial_+^2 f_1, \dots\} \quad (2.9)$$

but, of course, other choices are also possible, such as

$$\{g_\beta\} = \{f_1, \partial_+ f_1, \partial_+^2 f_1, f_2, f_3, \dots, f_M, \partial_+^3 f_1, \partial_+ f_2, \partial_+ f_3, \dots\} \quad (2.10a)$$

or even

$$\{ \partial_+ f_1, f_1, \partial_+^2 f_1, f_2, f_3, \dots, f_M, \partial_+^3 f_1, \partial_+ f_2, \dots \} \quad M \geq 2 \quad (2.10b)$$

It is the existence of this large class of choices, together with the existence of further special solutions (to be mentioned subsequently) which makes the problem of proving the completeness of the solution set (for arbitrary M) so difficult.

We shall shortly determine the value of the action corresponding to these solutions - we will see then why it was so important to require that all f_i have polynomial entries and how the value of the action is related to the degree of these polynomials. Meanwhile, let us say a few words about additional special solutions, pointing out why we may consider them as non-trivial embeddings of lower-order Grassmannians. The existence of such solutions was stated some time ago²⁾; their general structure was discussed at length in Refs. 3) and 4). The general structure of these special solutions is based on the fact that they are determined in terms of a smaller number of arbitrary holomorphic vectors (with polynomial components) $f_1 \dots f_{M'}$, where $M' < M$, in such a way that $\partial_+ H_\beta \subset H_{\beta+M'}$ for any β . A special solution is then given by any choice of M vectors from the ordered set $\{e_i\}$, such that the patches of the chosen adjacent vectors and also the patches of the "holes" between them (vectors not chosen) all have the length $\geq M'$.

Let us give examples of such solutions. First we consider $M' = 1$ (i.e., when there is only one vector f). Then $\partial_+ H_\beta \subset H_{\beta+1}$, and any choice of M distinct vectors from the sequence $\{e_\beta\}$ is a solution. In the case of two holomorphic vectors f_1, f_2 with, say, $\{g_\beta\} = \{f_1, f_2, \partial_+ f_1, \partial_+ f_2, \dots\}$, i.e., such that $\partial_+ H_\beta \subset H_{\beta+2}$, the sets $\{e_1, e_2, e_5, e_6\}$ or $\{e_3, e_4, e_7, e_8\}$ are solutions corresponding to $M = 4$. In checking the equations of motion, it is important to observe that in the collection of subsets, the condition that the length of each is $\geq M'$ guarantees that all subsets are completely decoupled from each other, and so all their properties are purely additive. The additivity of the action and the topological charge show that to derive a general result, it is sufficient to determine the expressions for the case of a sequence of, say, K adjacent vectors $\{e_\beta, e_{\beta+1}, \dots, e_{\beta+K-1}\}$ generated by the set of M' vectors where $M' \leq K$. Then, the expression for all the other cases can be stated in terms of the results determined for this example.

3. - ACTION AND THE TOPOLOGICAL CHARGE

The expressions for the action and the topological charge densities^{*)} are given by

$$s = 2 \text{Tr} \left[(D_+ Z)^\dagger D_+ Z + (D_- Z)^\dagger D_- Z \right] \quad (3.1)$$

and

$$q = 2 \text{Tr} \left[(D_+ Z)^\dagger D_+ Z - (D_- Z)^\dagger D_- Z \right]. \quad (3.2)$$

Let us first consider the case $M' = K = 2$. Then the two column vectors of $D_\pm Z$ are given by

$$D_\pm Z = \left\{ \partial_\pm e_\beta - e_\beta (e_\beta^\dagger \partial_\pm e_\beta) - e_{\beta+1} (e_{\beta+1}^\dagger \partial_\pm e_\beta), \partial_\pm e_{\beta+1} - e_\beta (e_\beta^\dagger \partial_\pm e_{\beta+1}) - e_{\beta+1} (e_{\beta+1}^\dagger \partial_\pm e_{\beta+1}) \right\} \quad (3.3)$$

The condition that $\partial_+ H_\beta \subset H_{\beta+2}$ translates into the statement that

$$\partial_+ e_l = \delta_l e_l + \beta_l e_{l+1} + \omega_l e_{l+2} \quad (3.4)$$

and also that

$$\partial_- e_l = \alpha_l e_l + \beta_l e_{l-1} + \gamma_l e_{l-2} \quad (3.5)$$

where the coefficients are related by

$$\alpha_l = -\delta_l^*, \quad \beta_l = -\beta_{l-1}^*, \quad \gamma_l = -\omega_{l-2}^*. \quad (3.6)$$

^{*)} Our topological charge is normalized to be given by $2\pi \times (\text{integer})$.

Thus

$$\begin{aligned} \text{Tr} (D_- Z)^\dagger D_- Z &= |\partial_- e_\rho|^2 - |e_\rho^\dagger \partial_- e_\rho|^2 - |e_{\rho+1}^\dagger \partial_- e_\rho|^2 + \\ &\quad + |\partial_- e_{\rho+1}|^2 - |e_\rho^\dagger \partial_- e_{\rho+1}|^2 - |e_{\rho+1}^\dagger \partial_- e_{\rho+1}|^2 = \\ &= |\beta_\rho|^2 + |\gamma_\rho|^2 + |\gamma_{\rho+1}|^2, \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \text{Tr} (D_+ Z)^\dagger D_+ Z &= |\partial_+ e_\rho|^2 - |e_\rho^\dagger \partial_+ e_\rho|^2 - |e_{\rho+1}^\dagger \partial_+ e_\rho|^2 + \\ &\quad + |\partial_+ e_{\rho+1}|^2 - |e_\rho^\dagger \partial_+ e_{\rho+1}|^2 - |e_{\rho+1}^\dagger \partial_+ e_{\rho+1}|^2 = \\ &= |\beta_{\rho+2}|^2 + |\gamma_{\rho+2}|^2 + |\gamma_{\rho+3}|^2. \end{aligned} \quad (3.8)$$

We see that

$$\text{Tr} (D_+ Z(\rho))^\dagger D_+ Z(\rho) = \text{Tr} (D_- Z(\rho+2))^\dagger D_- Z(\rho+2) \quad (3.9)$$

where, given that the $Z(\beta)$ Grassmannian is described by $(e_\beta, e_{\beta+1})$, the $Z(\beta+2)$ Grassmannian is described by $(e_{\beta+2}, e_{\beta+3})$. A quick look at the $K=3$ case, where

$$\partial_- e_\ell = \alpha_\ell e_\ell + \beta_\ell e_{\ell-1} + \gamma_\ell e_{\ell-2} + \chi_\ell e_{\ell-3} \quad (3.10)$$

shows that

$$\begin{aligned} \text{Tr} (D_+ Z)^\dagger D_+ Z &= |\beta_{\rho+3}|^2 + |\gamma_{\rho+3}|^2 + |\gamma_{\rho+4}|^2 + \\ &\quad + |\chi_{\rho+3}|^2 + |\chi_{\rho+4}|^2 + |\chi_{\rho+5}|^2 \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \text{Tr} (D_- Z)^\dagger D_- Z &= |\beta_\rho|^2 + |\gamma_\rho|^2 + |\gamma_{\rho+1}|^2 + |\chi_\rho|^2 + |\chi_{\rho+1}|^2 + \\ &\quad + |\chi_{\rho+2}|^2 \end{aligned} \quad (3.12)$$

and so

$$\text{Tr} \left(D_+ Z(\beta) \right)^\dagger D_+ Z(\beta) = \text{Tr} \left(D_- Z(\beta+3) \right)^\dagger D_- Z(\beta+3) . \quad (3.13)$$

In general,

$$\text{Tr} \left(D_+ Z(\beta) \right)^\dagger D_+ Z(\beta) = \text{Tr} \left(D_- Z(\beta+K) \right)^\dagger D_- Z(\beta+K) \quad 8) \quad (3.14)$$

which is a natural generalization of what was observed in the $\mathbb{CP}^{N-1}$ case.

Let us observe that if $M' < K$, then one can still express $\partial_\pm e_\ell$ in terms of $K+1$ vectors (but the coefficients of $K-M'$ of them vanish). The general result (3.14) then follows, even though in the expression for $\text{Tr}(D_\pm Z)^\dagger (D_\pm Z)$ as a sum of squares of the coefficients, some of these coefficients happen to vanish.

The general result (3.14) allows us to re-express the action density s for a general solution in terms of a topological charge density q of the related solutions. To see this, observe that, using (3.14), we can write (choosing first $M = 2$ for simplicity) and denoting

$$\begin{aligned} \text{Tr} \left(D_\pm Z(\beta) \right)^\dagger D_\pm Z(\beta) &= |D_\pm Z(\beta)|^2, \\ s &= |D_+ Z(\beta)|^2 + |D_- Z(\beta)|^2 = |D_+ Z(\beta)|^2 - |D_- Z(\beta)|^2 + \\ &+ 2 |D_+ Z(\beta-2)|^2 - 2 |D_- Z(\beta-2)|^2 + \\ &+ 2 |D_+ Z(\beta-4)|^2 - 2 |D_- Z(\beta-4)|^2 + \\ &\vdots \\ &+ 2 |D_+ Z(\beta+2)|^2 - 2 |D_- Z(\beta+2)|^2 + 2 |D_+ Z(\beta)|^2 \end{aligned} \quad (3.15)$$

where $p = 2$ if β is even and $p = 3$ if β is odd. However, a few lines of algebra show that

$$|D_- Z(3)|^2 = |D_+ Z(1)|^2 \quad (3.16)$$

even though the expression on the right-hand side involves a Grassmannian of lower order (here $M = 1$). Thus, using the definition of the topological charge density, and recalling that

$$|D_- Z(p)|^2 = 0 \quad (p=1, 2)$$

we see that we can write

$$S(\beta) = q(\beta) + 2q(\beta-2) + 2q(\beta-4) + \dots + 2q(p) \quad (3.17)$$

where $p = 2$ if β is even, or $p = 1$ if β is odd.

A similar discussion for the general case shows

$$S(\beta) = q(\beta) + 2q(\beta-M) + 2q(\beta-2M) + \dots + 2q(p) \quad (3.18)$$

where $p = \beta \pmod{M}$ or M if $\beta \pmod{M} = 0$. Hence, the problem of calculating the action density, and thus the action, has been reduced to the calculation of the topological charges. As

$$q = 2 \left[\text{Tr} \left((D_+ Z)^\dagger D_+ Z - (D_- Z)^\dagger D_- Z \right) \right] \quad (3.19)$$

we can rewrite q , using the cyclic permutation property of the trace as

$$q = 2 \text{Tr} \left(\partial_- Z^\dagger \partial_+ Z - \partial_+ Z^\dagger \partial_- Z \right). \quad (3.20)$$

Let us consider the case when

$$Z = (e_\beta, e_{\beta+1}, \dots, e_{\beta+M-1}).$$

We will show that

$$q = 2 \partial_+ \partial_- \text{Tr} \ln M \quad (3.21)$$

where M is given by (2.8). To see this, observe that as $\text{tr} \ln A = \ln \det A$, we can write

$$\begin{aligned} \text{Tr} \ln M &= \ln \det M = \ln \left[\prod_{i=0}^{M-1} (\hat{e}_{\rho+i}^+ \hat{e}_{\rho+i}) \right] = \\ &= \sum_{i=0}^{M-1} \ln (\hat{e}_{\rho+i}^+ \hat{e}_{\rho+i}), \end{aligned} \quad (3.22)$$

where we used the property that

$$\det M = \det (U^{-1} M U).$$

Next we calculate

$$\begin{aligned} \partial_+ \partial_- \ln (\hat{e}_{\rho+i}^+ \hat{e}_{\rho+i}) &= \\ &= \frac{|\hat{e}_{\rho+i}|^2 \partial_+ \partial_- |\hat{e}_{\rho+i}|^2 - \partial_+ |\hat{e}_{\rho+i}|^2 \partial_- |\hat{e}_{\rho+i}|^2}{|\hat{e}_{\rho+i}|^4} \end{aligned} \quad (3.23)$$

and compare this term with the $(\beta+i)^{\text{th}}$ term in (3.20). As

$$\partial_+ \hat{e}_{\rho+i} = \frac{\partial_+ \hat{e}_{\rho+i}}{|\hat{e}_{\rho+i}|} - \frac{1}{2} \hat{e}_{\rho+i} \frac{\partial_+ |\hat{e}_{\rho+i}|^2}{|\hat{e}_{\rho+i}|^3} \quad (3.24a)$$

and

$$\partial_- \hat{e}_{\rho+i}^+ = \frac{\partial_- \hat{e}_{\rho+i}^+}{|\hat{e}_{\rho+i}|} - \frac{1}{2} \hat{e}_{\rho+i}^+ \frac{\partial_- |\hat{e}_{\rho+i}|^2}{|\hat{e}_{\rho+i}|^3} \quad (3.24b)$$

we see that the $(\beta+i)^{\text{th}}$ term in (3.20) is given by

$$\begin{aligned} &\frac{\partial_- \hat{e}_{\rho+i}^+ \cdot \partial_+ \hat{e}_{\rho+i}}{|\hat{e}_{\rho+i}|^2} - \frac{1}{2} \frac{(\hat{e}_{\rho+i}^+ \cdot \partial_+ \hat{e}_{\rho+i})}{|\hat{e}_{\rho+i}|^4} \partial_- |\hat{e}_{\rho+i}|^2 - \\ &- \frac{1}{2} \frac{(\partial_- \hat{e}_{\rho+i}^+ \cdot \hat{e}_{\rho+i})}{|\hat{e}_{\rho+i}|^4} \partial_+ |\hat{e}_{\rho+i}|^2 - (\partial_- \leftrightarrow \partial_+) \end{aligned} \quad (3.25)$$

But, as

$$\begin{aligned}\hat{e}_\gamma^+ \partial_- \hat{e}_\gamma &= 0 \\ \hat{e}_\gamma^+ \partial_+ \hat{e}_\gamma &= \partial_+ |\hat{e}_\gamma|^2\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}(\hat{e}_\gamma^+ \partial_+ \partial_- \hat{e}_\gamma) &= \partial_+ (\hat{e}_\gamma^+ \partial_- \hat{e}_\gamma) - \partial_+ \hat{e}_\gamma^+ \cdot \partial_- \hat{e}_\gamma = \\ &= -\partial_+ \hat{e}_\gamma^+ \cdot \partial_- \hat{e}_\gamma, \\ (\partial_+ \partial_- \hat{e}_\gamma^+ \cdot \hat{e}_\gamma) &= -\partial_+ \hat{e}_\gamma^+ \cdot \partial_- \hat{e}_\gamma\end{aligned}\tag{3.27}$$

and given that

$$\partial_+ \partial_- |\hat{e}_\gamma|^2 = \partial_+ (\partial_- \hat{e}_\gamma^+ \hat{e}_\gamma) = \partial_+ \partial_- \hat{e}_\gamma^+ \cdot \hat{e}_\gamma + \partial_- \hat{e}_\gamma^+ \partial_+ \hat{e}_\gamma\tag{3.28}$$

we see that

$$\begin{aligned}\partial_- \hat{e}_{\rho+i}^+ \partial_+ \hat{e}_{\rho+i} - \partial_+ \hat{e}_{\rho+i}^+ \partial_- \hat{e}_{\rho+i} &= \partial_+ \partial_- \hat{e}_{\rho+i}^+ \hat{e}_{\rho+i} + \\ + \partial_+ \partial_- |\hat{e}_{\rho+i}|^2 - \partial_+ \partial_- \hat{e}_{\rho+i}^+ \hat{e}_{\rho+i} &= \partial_+ \partial_- |\hat{e}_{\rho+i}|^2.\end{aligned}\tag{3.29}$$

This, together with the results (3.26), shows that (3.25) can be rewritten as

$$\begin{aligned}\frac{\partial_+ \partial_- |\hat{e}_{\rho+i}|^2}{|\hat{e}_{\rho+i}|^2} - \frac{1}{2} \frac{\partial_- |\hat{e}_{\rho+i}|^2 \partial_+ |\hat{e}_{\rho+i}|^2}{|\hat{e}_{\rho+i}|^4} - \frac{1}{2} \frac{\partial_+ |\hat{e}_{\rho+i}|^2 \partial_- |\hat{e}_{\rho+i}|^2}{|\hat{e}_{\rho+i}|^4} &= \\ &= \partial_+ \partial_- \ln |\hat{e}_{\rho+i}|^2\end{aligned}\tag{3.30}$$

thus proving (3.21) and (3.22).

We have shown, therefore, that

$$q = \sum_{i=p}^{p+M-1} q_i \quad (3.31)$$

where

$$q_i = 2 \partial_+ \partial_- \ln |\hat{e}_i|^2. \quad (3.32)$$

As expected, the topological charge density is completely additive. The additive form of the topological charge density allows us to rewrite the action density corresponding to the general case (3.18) as

$$S(p) = q_p + q_{p+1} + q_{p+2} + \dots + q_{p+M-1} + 2q_{p-1} + 2q_{p-2} + 2q_{p-3} + \dots + 2q_1 \quad (3.33)$$

Next, we calculate the integrated values of the topological charge and of the action. As

$$\partial_+ \partial_- \ln |p|^2 = \frac{1}{4} \partial_\mu \partial_\mu \ln |p|^2 \quad (3.34)$$

the use of the divergence theorem in two dimensions shows that

$$\int d^2x \partial_+ \partial_- \ln |p|^2 = \pi \alpha \quad (3.35)$$

where $|p| \rightarrow |x|^\alpha$ as $|x| \rightarrow \infty$, and we have assumed that $|p|$ has no singularities except those at ∞ .

To calculate the asymptotic behaviour of $|\hat{e}_i|^2$, it is convenient to use the wedge product formalism - originally used in the $\mathbb{CP}^{N-1}$ case⁸⁾. We define wedge products of the first i vectors of basis

$$h^{(i)} = g_1 \wedge g_2 \wedge g_3 \wedge \dots \wedge g_i \quad (3.36)$$

and then observe that

$$\hat{e}_p \sim h^{(p-1)+} \cdot h^{(p)} \quad (3.37)$$

where the dot product in (3.37) denotes the summation over all indices of $h^{(\beta-1)+}$ and all but one of $h^{(\beta)}$. As all g_i are analytic, so are $h^{(i)}$; if g_i have polynomial entries, so do $h^{(i)}$. To calculate the asymptotic behaviour, we require $|\hat{e}_\beta|^2$. It is easy to see that up to an overall constant

$$|\hat{e}_\beta|^2 \equiv |h^{(\beta)}|^2 / |h^{(\beta-1)}|^2 \quad (3.38)$$

Then, defining

$$\alpha_\beta = \deg h^{(\beta)} \pmod{\text{(overall factors)}}; \quad \alpha_0 = 0,$$

we find that

$$Q_i = \int dx^2 q_i = 2\pi (\alpha_i - \alpha_{i-1}) \quad (3.39)$$

showing that the topological charge and the action of the general field configuration given in (2.5) are given by

$$\begin{aligned} Q &= 2\pi (\alpha_\beta - \alpha_{\beta-1} + \alpha_{\beta+1} - \alpha_\beta + \dots + \alpha_{\beta+n-1} - \alpha_{\beta+n-2}) = \\ &= 2\pi (\alpha_{\beta+n-1} - \alpha_{\beta-1}) \end{aligned} \quad (3.40)$$

and

$$\begin{aligned} S &= 2\pi (\alpha_{\beta+n-1} - \alpha_{\beta-1}) + 4\pi (\alpha_{\beta-1} - \alpha_{\beta-n-1}) + 4\pi (\alpha_{\beta-n-1} - \alpha_{\beta-2n-1}) + \\ &+ \dots + 4\pi (\alpha_3 - \alpha_2) + 4\pi (\alpha_2 - \alpha_1) + 4\pi \alpha_1 = \\ &= 2\pi (\alpha_{\beta+n-1} + \alpha_{\beta-1}). \end{aligned} \quad (3.41)$$

These are natural generalizations of the corresponding results obtained in the $\mathbb{CP}^{N-1}$ model⁸⁾.

All this discussion was concerned with the generic case. The special solutions with $M' \subset M$ can be treated in an analogous way, bearing in mind that in the final expression for Q and S , the dependence on M' is entirely contained in the definitions of α_i . What is, however, very important, is whether all the vectors which form Z are all adjacent or fall into groups of adjacent vectors - separated by "gaps" (adjacent with respect to their position in the ordered set $\{e_i\}$). If they are adjacent, the results (3.40) and (3.41) will

hold; if not, we consider each group of adjacent vectors as a separate unit for which the results (3.40) and (3.41) hold (with appropriately altered indices) and the total value of S and Q is given by a sum of such expressions corresponding to each subgroup. This, again, generalizes the result of Ref. 2).

4. - STABILITY

The non-instanton (and non-anti-instanton) solutions have $|Q| \neq S$, and so for them the usual topological arguments guaranteeing their stability do not apply - hence they may be unstable and so may not correspond to local minima of the action. This problem was considered for the $\mathbb{CP}^{N-1}$ case, where it was found that this was indeed so. Here we consider it for the more general Grassmannians. We find that, as in the $\mathbb{CP}^{N-1}$ case, all non-instanton (or non-anti-instanton) solutions are indeed unstable. To show this, we take a general solution Z for which $D_{\pm}Z \neq 0$, and consider a small fluctuation ϕ around Z . As a result of this fluctuation, the Z field is modified to

$$Z' = Z (1 - \phi^{\dagger} \phi)^{\frac{1}{2}} + \phi \quad (4.1)$$

where

$$\phi^{\dagger} \cdot Z = 0 \quad (4.2)$$

We can always describe the effect of the general fluctuation by (4.1); this is due to the fact that we are considering only classes of equivalence of Z fields [under $U(M)$ transformations], and any square matrix can be written as a product of a Hermitian and a unitary matrix - here, the Hermitian matrix is $(1 - \phi^{\dagger} \phi)^{1/2}$, and the unitary matrix is absorbed into Z .

The action for the modified field is

$$S' = 2 \int d^2x \text{Tr} [(D'_+ Z')^{\dagger} D'_+ Z' + (D'_- Z')^{\dagger} D'_- Z'] \quad (4.3)$$

where $D'_{\pm}$ are the usual covariant derivatives written in terms of Z' . Since the topological charge is invariant under the introduction of this fluctuation,

$$S' = 2 \int d^2x q(x) + 4 \int d^2x \text{Tr} [(D'_- Z')^{\dagger} D'_- Z']. \quad (4.4)$$

Calculating to second order in the small fluctuation ϕ , we have

$$\begin{aligned} \text{Tr} (\mathcal{D}'_+ \mathcal{Z}')^\dagger \mathcal{D}'_- \mathcal{Z}' &= \text{Tr} (\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z} + \text{Tr} (\mathcal{D}_- \phi)^\dagger \mathcal{D}_- \phi - \\ &- \text{Tr} \phi^\dagger \phi (\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z} - \text{Tr} [\mathcal{Z}^\dagger \mathcal{D}_- \phi + \phi^\dagger \mathcal{D}_- \mathcal{Z}] [\mathcal{Z}^\dagger \mathcal{D}_- \phi + \phi^\dagger \mathcal{D}_- \mathcal{Z}] \end{aligned} \quad (4.5)$$

giving

$$S' = S + 4 \int d^2 x V(\phi) \quad (4.6)$$

where S is the action of $\mathcal{Z}$ and

$$\begin{aligned} V(\phi) &= \text{Tr} (\mathcal{D}_- \phi)^\dagger \mathcal{D}_- \phi - \text{Tr} \phi^\dagger \phi (\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z} - \\ &- \text{Tr} [\mathcal{Z}^\dagger \mathcal{D}_- \phi + \phi^\dagger \mathcal{D}_- \mathcal{Z}] [\mathcal{Z}^\dagger \mathcal{D}_- \phi + \phi^\dagger \mathcal{D}_- \mathcal{Z}]. \end{aligned} \quad (4.7)$$

If we now choose

$$\phi = \varepsilon \mathcal{D}_+ \mathcal{Z} \quad (4.8)$$

where ε is a constant complex number, then as

$$\mathcal{D}_- \mathcal{D}_+ \mathcal{Z} = -\mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} \quad (4.9)$$

$$\begin{aligned} \text{we have } [\mathcal{Z}^\dagger \mathcal{D}_- \phi + \phi^\dagger \mathcal{D}_- \mathcal{Z}] [\mathcal{Z}^\dagger \mathcal{D}_- \phi + \phi^\dagger \mathcal{D}_- \mathcal{Z}] &= |\varepsilon|^2 (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} + \\ &+ |\varepsilon|^2 (\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z} - \varepsilon^2 (\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} - \varepsilon^{*2} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z} \end{aligned} \quad (4.10)$$

and so we find that

$$\begin{aligned} V &= -|\varepsilon|^2 \text{Tr} [(\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z}] - |\varepsilon|^2 \text{Tr} [(\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z}] + \\ &+ \varepsilon^2 \text{Tr} [(\mathcal{D}_- \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z}] + \varepsilon^{*2} \text{Tr} [(\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_+ \mathcal{Z} (\mathcal{D}_+ \mathcal{Z})^\dagger \mathcal{D}_- \mathcal{Z}]. \end{aligned} \quad (4.11)$$

This expression is definite negative, if neither $D_- Z$ nor $D_+ Z$ vanish, as the last two terms in (4.11) vanish. The disappearance of the last two terms can be seen as follows:

- 1) for the generic solutions $D_+ Z^\dagger D_- Z = 0$, and so the result follows trivially;
- 2) for the special solutions, the matrix $D_+ Z^\dagger D_- Z$ may be non-zero. If it does not vanish, its only non-zero elements lie in the upper right-hand corner. (Similarly, the non-zero elements of $D_- Z^\dagger D_+ Z$ lie in the lower left-hand corner.) However, it is easy to check that if an element of $D_- Z^\dagger D_+ Z$ is non-zero, the corresponding element of the Hermitian matrix $D_+ Z^\dagger D_+ Z$ vanishes. Thus, even though $(D_- Z)^\dagger D_+ Z (D_+ Z)^\dagger D_- Z \neq 0$, its trace vanishes.

As $S' < S$, we see that all non-instanton solutions are unstable, i.e., they correspond to the saddle points of the action.

5. - DIRAC BACKGROUND PROBLEM

In this section, we consider the supersymmetric Dirac equation describing fermions in the external fixed background of a generic Grassmannian solution discussed above.

The Dirac field equation is

$$\not{D}\psi - Z Z^\dagger \not{D}\psi = 0 \quad (5.1)$$

where the fermionic field ψ is, like Z , given by an $N \times M$ matrix. Moreover, the fermionic field is required to be orthogonal to Z :

$$Z^\dagger \psi = 0. \quad (5.2)$$

Decomposing ψ into eigenstates of γ_5 ⁶⁾

$$\psi = \begin{pmatrix} 1 \\ -i \end{pmatrix} \psi^+ + \begin{pmatrix} 1 \\ i \end{pmatrix} \psi^-, \quad (5.3)$$

one gets the equivalent equations

$$D_\pm \psi^\pm = Z \lambda_\pm \quad (5.4)$$

with the constraints

$$Z^\dagger \psi^\pm = 0, \quad (5.5)$$

where $\lambda_{\pm}$ are $M \times M$ matrix-valued functions of x_+ and x_- . We choose Z as given by (2.7) and (2.8), and observe that

$$\begin{aligned} D_+ M^{1/2} &= \partial_+ M^{1/2} - M^{1/2} M^{-1/2} \hat{Z}^+ \partial_+ (\hat{Z} M^{-1/2}) = \\ &= -\hat{Z}^+ \partial_+ (\hat{Z} M^{-1}) M^{1/2} = 0 \end{aligned} \quad (5.6)$$

as

$$\partial_+ (\hat{Z}^+ \hat{Z} M^{-1}) = 0 = \partial_+ \hat{Z}^+ \hat{Z} M^{-1} + \hat{Z}^+ \partial_+ (\hat{Z} M^{-1})$$

and

$$\partial_- \hat{Z} \in H_{i-1} \Rightarrow \partial_+ \hat{Z}^+ \hat{Z} = 0.$$

Similarly,

$$D_- M^{-1/2} = 0. \quad (5.7)$$

Thus, setting

$$\Psi^{\pm} = \phi_{\pm} M^{\pm \frac{1}{2}} \quad (5.8)$$

we find

$$\begin{aligned} D_{\pm} (\phi_{\pm} M^{\pm \frac{1}{2}}) &= \partial_{\pm} (\phi_{\pm} M^{\pm \frac{1}{2}}) - \phi_{\pm} M^{\pm \frac{1}{2}} M^{-\frac{1}{2}} \hat{Z}^+ \partial_{\pm} (\hat{Z} M^{-\frac{1}{2}}) = \\ &= \partial_{\pm} \phi_{\pm} M^{\pm \frac{1}{2}}. \end{aligned} \quad (5.9)$$

Hence, our equations (5.4) and (5.5) can be rewritten as

$$\partial_{\pm} \phi_{\pm} = Z \mu_{\pm} \quad (5.10)$$

and

$$Z^+ \phi_{\pm} = 0 \quad (5.11)$$

where $\mu_{\pm}$ are some $M \times M$ matrix-valued functions of x_+ and x_- . To solve these equations, one uses some basic properties of the projectors P_{β} on the subspaces $H_{\beta} = \{g_1, \dots, g_{\beta}\}$. Let us denote by P the projector on the subspace defined by $Z = (e_{\beta}, e_{\beta+1}, \dots, e_{\beta+M-1})$ i.e., $P = P_{\beta+M-1} - P_{\beta-1}$. Then it follows³⁾ that $\partial_+ P_{\beta-1} = -P \partial_+ P$. We can now choose

$$\phi_+ = P_{\beta-1} h^+(x_-) \quad (5.12)$$

where h^+ is an $N \times M$ matrix dependent only on x_- . The orthogonality constraint (5.11) is now automatically satisfied and one finds

$$\partial_+ \phi_+ = \partial_+ P_{\beta-1} h^+ = -P \partial_+ P h^+. \quad (5.13)$$

However, since $P = ZZ^+$, we see that the Eq. (5.10) is fulfilled with

$$\mu_+ = -Z^+ \partial_+ P h^+. \quad (5.14)$$

In a similar way, we find that

$$\phi_- = (1 - P_{\beta+M-1}) h^-(x_+) \quad (5.15)$$

solves Eq. (5.10).

It can now be shown that there exist polynomial solutions for h^- and h^+ matrices, such that the resultant solutions of the Dirac equation are normalizable on the sphere (in analogy to the situation in the $\mathbb{CP}^{N-1}$ case)⁹⁾. One can count the parameters involved and show that the Atiyah-Singer theorem is satisfied.

The solutions

$$\psi^+ = P_{\beta-1} h^+ M^{1/2} \quad (5.16)$$

and

$$\psi^- = (1 - P_{\beta+M-1}) h^- M^{-1/2}, \quad (5.17)$$

are solutions of the Dirac background problem, i.e., are given by linear equations in which the Grassmannian field comes in as a parameter. A similar situation arises in the Hilbert-Riemann problem. There, one solves the linear equations

$$\begin{aligned}\partial_+ \Psi &= \frac{2}{1+\lambda} [\partial_+ \mathbb{P}, \mathbb{P}] \Psi \\ \partial_- \Psi &= \frac{2}{1-\lambda} [\partial_- \mathbb{P}, \mathbb{P}] \Psi\end{aligned}\tag{5.18}$$

this time for the $N \times N$ matrix-valued functions Ψ . Here, $\mathbb{P}$ is the projector operator formulation of the Grassmannian Z , i.e., is given by

$$\mathbb{P} = \mathbb{P}_{\beta+n-1} - \mathbb{P}_{\beta-1}\tag{5.19}$$

and the integrability conditions for the system (5.18) are equivalent to the equations of motions for the Grassmannian Z , which, in the projector operator formulation are given by

$$[\mathbb{P}, \partial_+ \partial_- \mathbb{P}] = 0.\tag{5.20}$$

Is there any relation between these two linear problems?

The solutions to the Hilbert-Riemann problem were discussed in Ref. 3). There, it was found that Ψ , which solves (5.18) using the notation of the present paper, is given by

$$\begin{aligned}\Psi &= \mathbb{1} + \frac{4\lambda}{(\lambda-1)^2} \mathbb{P}_{\beta-1} + \frac{2}{\lambda-1} (\mathbb{P}_{\beta+n-1} - \mathbb{P}_{\beta-1}) = \\ &= \left[\mathbb{1} + \frac{2(\lambda+1)}{(\lambda-1)^2} \mathbb{P}_{\beta-1} + \frac{2}{\lambda-1} \mathbb{P}_{\beta+n-1} \right].\end{aligned}\tag{5.21}$$

Hence, dropping the irrelevant overall factor $\lambda+1/\lambda-1$, we observe that

$$\Psi^\pm = \Psi H^\pm\tag{5.22}$$

where

$$H^\pm = \frac{(\lambda-1)^2}{(\lambda+1)^2} \mathbb{P}_{\beta-1} h^\pm M^{1/2}\tag{5.23}$$

and

$$H^- = (1 - P_{\beta+M-1}) h^- M^{-1/2}. \quad (5.24)$$

Deriving these expressions, we have used the property that

$$P_{\beta+M-1} P_{\beta-1} = P_{\beta-1} \quad (5.25)$$

which also shows that

$$(H^+)^+ H^- = 0 \quad (5.26)$$

as is to be expected.

Similar expressions can be derived for the solutions of the Dirac equation and the Hilbert-Riemann problem corresponding to the special solutions.

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