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TOPOLOGY AND SADDLE POINTS IN FIELD THEORIES

P. Forgács *)

CERN — Geneva

and

Z. Horváth

Roland Eötvös University
H-1088 Budapest, Hungary

ABSTRACT

We point out that in a large class of models there are non-contractible loops in configuration space. This signals the possible existence of static, finite energy saddle points. We show that these solutions play a rôle in the tunnelling process.

*) On leave from the Central Research
Institute for Physics, Budapest.

In the last decade we witnessed the triumph of spontaneously broken gauge theories (SBGT) in the description of electroweak interactions ¹⁾. SBGT also provide us with an attractive framework for the unification of strong and electroweak forces ²⁾. A very important property of this class of theories is the fact that they are perturbatively renormalizable. Their other remarkable feature is the presence of "topological excitations" (vortices, monopoles), at least at the classical level. In other words, there are topologically inequivalent sectors in the space of all classical configurations.

As was first pointed out by Manton ³⁾, for the Weinberg-Salam theory even in the vacuum sector the perturbational approach becomes meaningless at large enough - of the order of $E \sim 1/\alpha M_W$ - energies due to the non-trivial topology of the space of field configurations in the vacuum sector.

Manton was inspired, in turn, by the very interesting work of Taubes ⁴⁾ who has rigorously proved the existence of a non-trivial, static, finite energy non-self-dual solution of a spontaneously broken $SU(2)$ theory (without potential), with zero topological charge.

In this paper, we pursue these ideas for other models (not only SBGTs). We point out that in most cases there is an intimate connection of the above-mentioned phenomenon with the tunnelling.

Let us start with the following energy functional, E , for static fields in D space dimensions :

$$E = \int \mathcal{E} d^D x, \quad \mathcal{E} = \frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \Phi, D^\mu \Phi) + V(\Phi) \quad (1)$$

where we denote by $(\cdot, \cdot)$ the scalar product in the representation space of the Higgs field, Φ . As is widely known, one can get necessary conditions for the existence of finite energy solutions in D dimensions, exploiting the scale transformations of E ⁵⁾. In $D=2,3$ one needs a Yang-Mills-Higgs (YMH) theory; in $D=4$, every finite energy YMH solution is gauge equivalent to a pure YM solution. In $D=1$, a pure scalar model is sufficient. For $D=2$, pure scalar models have non-trivial, static, finite energy solutions only in the case when they realize a symmetry in a non-linear way $[O(n) \text{ and } \mathbb{CP}^{n-1} \text{ models}]$.

First, we consider SBGTs. We denote by H the residual gauge group, and we set $\phi^\infty = \lim_{|x| \rightarrow \infty} \phi(x)$; for finite E configurations ϕ^∞ maps the S^{D-1} sphere at infinity in $\mathbb{R}^D$ into G/H . Thus $\phi^\infty: S^{D-1} \rightarrow G/H$, for

$\phi \in C$ where C is the space of static, finite E configurations (configuration space). In fact, due to the complications arising from gauge invariance, one should consider the space of gauge orbits, $\bar{C}$. The points of $\bar{C}$ are the classes of gauge equivalent fields. The simplest way to describe $\bar{C}$ is to choose a representative element from each class by fixing the gauge. For our purposes, it is sufficient to fix a gauge in such a way that the vacuum is a single point in $\bar{C}$.

The topology of $\bar{C}$ is characterized by the topological properties of the space of $\{\phi^\infty\} = \text{Maps}(S^{D-1}; G/H)$. For us the most important topological characteristics of $\{\phi^\infty\}$ are its homotopy groups. As is well known, if there are non-trivial mappings, that is if $\Pi_{D-1}(G/H)$ is non-trivial, then these models exhibit stable topological excitations (vortices, monopoles). This means that $\Pi_0(\{\phi^\infty\})$ - the number of connected components of $\bar{C}$ - is greater than one. In the last decade, there has been a tremendous progress in finding topologically stable finite E solutions.

As it was shown by Taubes⁴⁾, the higher homotopy groups of $\bar{C}$ are also important from both the mathematical and the physical point of view. Analyzing $\Pi_n(\bar{C})$ Taubes deduced the existence of a saddle point of (1) in the vacuum sector for an $SU(2)$ theory broken down to $U(1)$. At the heart of Taubes' approach lies the Ljusternik-Snirelman theory (LST) for infinite dimensional manifolds⁶⁾.

In this paper, we argue that such saddle point solutions may exist in a very large class of physically interesting models. We will restrict ourselves to the vacuum sector [if $\Pi_0(\bar{C})$ is non-trivial], but this is not essential for the validity of our arguments.

Before we enter into the mathematics, let us give a very elementary but illustrative example. We take the following scalar theory in $D=1$:

$$\mathcal{L} = \frac{1}{2} \partial_0 \phi \partial_0 \phi^* - \left[\frac{1}{2} \partial_1 \phi \partial_1 \phi^* + \frac{\lambda}{2} (\phi \phi^* - 1)^2 \right]$$

Since $\Pi_0(S^1) \simeq 0$ there are no topologically stable, static, finite E solutions. Here $\phi^\infty: \{[-\infty; +\infty]\} \rightarrow S^1$. Now, let us consider a family of maps, $\phi^\infty(\mu)$, depending on μ continuously, $0 \leq \mu \leq 2\pi$ beginning and ending at the same vacuum state, say, $\phi_{\text{vac}} = 1$. With such a family, we associate a loop in configuration space $\phi(\mu, x)$. The asymptotic configurations on the loop are given by $\phi^\infty(\mu): \{[-\infty; +\infty]\} \rightarrow S^1$, $\phi^\infty(0) = \phi^\infty(2\pi) = 1$. The crucial point is to realize that the family of maps $\phi^\infty(\mu)$ is topologically equivalent to a single map $\psi: S^1 \rightarrow S^1$.

More mathematically, this means that $\phi^\omega(\mu)$ is the generator of $\pi_1(S^1)$. The simplest map, $\phi^\omega(\mu)$, whose degree is non-zero, is given by $\phi^\omega(\mu) = e^{i\mu}$. The fact that $\phi^\omega(\mu)$ is a non-trivial $S^1 \rightarrow S^1$ map guarantees, that our loop in C is non-contractible. We take the following non-contractible loop (NCL) :

$$\phi(\mu, x) = f(x) e^{i\mu} + 1 - f(x), \quad 0 \leq \mu \leq 2\pi \quad (2)$$

Finite energy implies : $f \xrightarrow{x \rightarrow \pm\infty} 0$ and where f is a suitably regular function on $\mathbb{R}^1$.

One can easily prove that the only finite energy solution in this model is $\phi_0 = \tanh \sqrt{\lambda} x$, [up to translations and global $U(1)$ rotations]. It is trivial to construct a NCL passing through the saddle point ϕ_0 ; just take $f = e^{-\sqrt{\lambda} x} [e^{\sqrt{\lambda} x} + e^{-\sqrt{\lambda} x}]^{-1}$. The energy (mass) of the solution is : $E[\phi_0] = \frac{4}{3}\sqrt{\lambda}$. We know that this solution is unstable, but it is amusing to calculate the instability modes : for $\phi = (\phi_0 + \chi_1 e^{i\omega t}, \chi_2 e^{i\omega t})$ we get $\chi_1 = 0$ and $\chi_2 = a(\cosh \sqrt{\lambda} x)^{-1}$ (a is the normalizing constant) and $\omega^2 = -\lambda$. This is naturally in complete agreement with the homotopy arguments. The instability of the solution can also be understood if one thinks of it as an embedding of a kink of the real (one component) ϕ^4 theory. Then the instability is due to the presence of extra degrees of freedom.

Next, we outline very briefly the mathematical setting. Let us fix a point, n , on S^{D-1} and also a point on G/H denoted by σ . We consider only those mappings $S^{D-1} \rightarrow G/H$ which map n onto σ . Let X denote the set of all such mappings and ϕ_0^ω the trivial mapping. We are interested in the ℓ th pointed homotopy group (at ϕ_0^ω) of X . Then the following isomorphism is true :

$$\pi_\ell(X, \phi_0^\omega) \simeq \pi_{\ell+D-1}(G/H) \quad (3)$$

(3) can be directly verified by explicitly constructing the representative elements of both homotopy groups. If $\pi_\ell(X, \phi_0^\omega)$ is non-trivial, then there are non-contractible loops ($\ell=1$) or hyperloops ($\ell > 1$) in $\bar{C}$. The next step is to take loops starting and ending at the vacuum in a fixed homotopy class of $\bar{C}$. Then one can apply the minimax procedure of Ljusternik-Snirelman as it was extended for gauge theories by Taubes. The essence of the minimax procedure is to calculate first the suprema of the energy functional, (1), for each loop (one has to remember that points on these loops are finite E configurations) and then take the infimum of

the suprema. If there exists a loop in the fixed homotopy class for which the supremum of E is just this infimum, then this extremal configuration is a saddle point and therefore a solution of the field equations. To actually prove its existence is highly non-trivial, and we make no attempt in this direction.

Thus the possible existence of Taubes-type solutions is signalled by the non-trivial homotopy groups, $\Pi_n(G/H) \simeq \mathbb{Z}$, $n > D-1$. In the following we shall concentrate on $\Pi_D(G/H)$ corresponding to non-contractible loops in configuration space. In some simple cases, like the example given before, one can choose a suitable NCL which goes through the actual saddle point and then of course there is no need for a proof of existence.

We would like to give a survey of models which possess NCLs. First, we consider the one-dimensional case. We take an Abelian gauge theory in $D=1$

$$-\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi^* + \frac{\lambda}{2} (\phi\phi^* - 1)^2$$

It is known that in this model there are Θ vacua, tunnelling "instantons" ⁷⁾. Let us recall the standard description of multiple vacua. First, we fix the $A_0=0$ gauge and characterize the topologically inequivalent ground states as follows :

$$A_1 = i(\partial_1 U_n) U_n^{-1}, \quad \phi = U_n \quad \text{where} \quad U_n = \left(\frac{\alpha + ix}{\alpha - ix} \right)^n$$

The existence of multiple vacua is of course related to the fact that $\Pi_1(S^1) = \mathbb{Z}$. Here, the "instantons" are the Landau-Ginzburg vortices which are responsible for tunnelling between these vacuum states. In this case, it is useful to define an extended configuration space, $\bar{\mathcal{C}}_{(G)}$. $\bar{\mathcal{C}}_{(G)}$ is defined by taking an infinite set of gauge copies of $\bar{\mathcal{C}}$. The vacuum states in the different copies correspond to the states obtained by applying homotopically inequivalent gauge transformations, U_n , to the vacuum of $\bar{\mathcal{C}}$ [large gauge transformations ⁸⁾].

We are interested in loops of static configurations beginning and ending at the same vacuum. Here $\bar{\mathcal{C}}$ is defined by a complete gauge fixing $A_1=0$ which leads us back to the previously analyzed toy model. Obviously, there are also NCLs and here they imply the existence of unstable solutions. The energy of the saddle point solution gives the height of a sort of "potential barrier" between the topologically inequivalent vacua. One can get a "picture" of a section of the $\bar{\mathcal{C}}_{(G)}$ space as a "periodic potential" :

where points on the ϕ axis represent static configurations and the minima and maxima of $E[\phi]$ correspond to the vacua and the previously described saddle point.

This model is a field theoretic analogue of the quantum pendulum discussed by Manton. In that case, one has a periodic potential in the Schrödinger equation, and the configuration space is a circle. For sufficiently large energies, one has to extend the configuration space by taking an infinite covering space of the circle because then the pendulum is rotating. Now the unstable solution gives the threshold above which the pendulum starts to rotate. In field theory, the obvious analogy is the functional Schrödinger equation. There the energy functional, which is periodic in some directions in $\bar{\mathcal{C}}(G)$ plays the rôle of the periodic potential. The existence of such saddle points leads to the breakdown of semiclassical methods at energies comparable to the height of the "potential". However, this interpretation may give a clue to devise improved approximation schemes for the evaluation of the functional integral. One can try to treat separately the finite dimensional subspace of the configuration space in which the tunneling occurs, as a finite dimensional quantum mechanical problem with a periodic potential. Around all other modes, it is sufficient to take Gaussian fluctuations.

We now consider SBGTs in $D=2$. $\Pi_2(G/H)$ contains $\mathbb{Z}$ as a subgroup if H has a $U(1)$ factor. So, in every theory in which there are 't Hooft-Polyakov monopoles in $D=3$, we can exhibit NLCs in $D=2$. In order to avoid complications associated with gauge invariance in all gauge models, we fix the radial gauge, $A_r=0$. The simplest model is $G=SU(2)$, $H=U(1)$; that is, the scalar, ϕ , is in the adjoint representation. We define a family of maps (with parameter μ , $0 \leq \mu \leq \pi$) :

$$\phi^\infty(\mu): S^1 \rightarrow S^2; \quad \phi^\infty(\mu) = (\sin\mu \cos\varphi; \sin^2\mu \sin\varphi + \cos^2\mu; \sin\mu \cos\mu(\sin\varphi-1))$$

$$0 \leq \varphi \leq 2\pi$$

Just in analogy with the one-dimensional case, this family of maps is topologically equivalent to a single $S^2 \rightarrow S^2$ map. We exhibit the following NCL :

$$\begin{aligned}\Phi(\mu, \underline{x}) &= (1-f) (0, \cos^2 \mu, 0) + f \phi^\infty(\mu) \\ A_\varphi &= g \partial_\varphi \phi^\infty(\mu) \wedge \phi^\infty(\mu) \\ A_\vartheta &= 0\end{aligned}\tag{4}$$

where g^{-1} , f^{-1} are smooth, bounded and have compact supports.

In $D=2$ there is a tunnelling effect due to the presence of instantons (monopoles) ⁹⁾. It is interesting that the existence of NCLs and instantons in these models is related to the same homotopy group $\pi_2(G/H)$. This strengthens our arguments for the significance of the NCLs (and the unstable solutions, if they exist) in the tunnelling phenomenon in $D=3$. This is, of course, the most interesting dimension from the physical point of view. Consider first fully broken theories. Since $\pi_3(\tilde{G}) \simeq \mathbb{Z}$ for

$$\tilde{G} = \{ SU(n) \ n \geq 1 ; SO(n) \ n \geq 2 ; Sp(n) \}$$

all such non-Abelian gauge theories possess NCLs. It is likely that unstable solutions also exist and we will give a simple example below. Similarly, as $\pi_3(G/H) \simeq \mathbb{Z}$ when $H=U(1)$ and G is one of the groups $\tilde{G}$ or, more generally, when G is a product of groups with at least one group from $\tilde{G}$, NCLs (and possibly the saddle points) lurk around.

To illustrate the actual existence of unstable solutions, we recall the work of Dashen, Hasslacher and Neveu (DHN) ¹⁰⁾ where $\tilde{G}=SU(2)$ and ϕ is in the fundamental representation. DHN argued that the spherically symmetric ansatz

$$A_i^a = \epsilon_{iak} \frac{x^k}{r} g(r) , \quad \phi = i \frac{\underline{x} \cdot \underline{\sigma}}{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix} f(r)\tag{5}$$

gives rise to a finite energy solution. They did not look at the classical stability of this solution due to the computational complexity of the problem. They pointed out that the solution could decay quantum mechanically anyway by tunnelling.

We observe that Manton's NCL passes through the ansatz (5). To see this recall that his NCL is given by :

$$\begin{aligned}\phi(\mu, x) &= f(r) U^\infty(\mu, x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-f) \begin{pmatrix} e^{i\mu} \cos \mu \\ 0 \end{pmatrix} \\ H_i &= -(\partial_i U^\infty(\mu, x)) (U^\infty(\mu, x))^{-1} g(r)\end{aligned}\quad (6)$$

where

$$U^\infty(\mu, x) = \begin{pmatrix} e^{i\mu} (\cos \mu - i \frac{x_3}{r} \sin \mu) & -\frac{x_1 - ix_2}{r} \sin \mu \\ \frac{x_1 + ix_2}{r} \sin \mu & e^{-i\mu} (\cos \mu + i \frac{x_3}{r} \sin \mu) \end{pmatrix}$$

which really gives (5) for $\mu = \pi/2$. So we can establish, without more work, that the DHN solution is unstable even classically. In a recent work, Burzlaff ¹¹⁾ proved rigorously the existence of the DHN solution and also showed that it is unstable. However, even though it is unstable, its existence is interesting due to its rôle in the tunnelling process. In this model, the instantons are the BPST instantons ¹²⁾ and the scalar fields are covariant constants with respect to the instanton configuration.

NCLs also arise in chiral gauge models with the gauge group $SU_L(n) \times SU_R(n)$ broken to $SU_V(n)$ ¹³⁾ with a real Higgs multiplet in the fundamental representation of both $SU_L(n)$ and $SU_R(n)$. This class of theories is useful as it provides effective models reproducing the low energy results of current algebra. Here $G/H \simeq SU(n)$ and $\Pi_3(SU(n)) \simeq \mathbb{Z}$ ($n > 1$).

However, $\Pi_3(G/H)$ is trivial for the following coset spaces :

$$\begin{aligned}a) \quad & \frac{SU(n)}{SU(m)} \quad \begin{matrix} n > m \\ n > 2 \end{matrix} & b) \quad \frac{SU(n+m)}{SU(n) \times SU(m) \times U(1)} \quad n+m > 2 \\ c) \quad & \frac{SO(n)}{SO(m)} \quad \begin{matrix} n > m \\ n > 3 \end{matrix} & d) \quad \frac{SO(n+m)}{SO(n) \times SO(m)} \quad \begin{matrix} n+m > 3 \\ n, m > 1 \end{matrix}\end{aligned}$$

Case b) corresponds to theories broken by the Higgs fields in the adjoint representation, which necessarily contain monopoles. For GUTs corresponding to $G/H \simeq SU(n)/SU(3) \times U(1)$ the situation is exactly the same. Nonetheless, as Taubes' solution [in $SU(2)$] can be embedded into any model of b), we would expect that in these models saddle points exist and their existence is signalled by the presence of higher dimensional non-contractible "hyper-loops". E.g.,

$$\pi_4 \left(\frac{SU(n+m)}{SU(n) \times SU(m) \times U(1)} \right) \simeq \mathbb{Z}$$

for $n+m > 3$ in which case the "hyperloop" is a two-dimensional surface in configuration space.

Finally, we would like to deal with non-linear scalar models in $D=2$. In this case the presence of NCLs is signalled by the non-triviality of $\pi_{D+1}(\mathcal{M})$, where by $\mathcal{M}$ we denoted the manifold in which scalars take their values. For the $O(n)$ models $\mathcal{M} = S^{n-1}$. $\pi_3(S^{n-1}) \simeq \mathbb{Z}$ only if $n=3, 4$. For the $O(3)$ model, it has been proved¹⁴⁾ that there are no finite E unstable solutions. This illustrates the point that the presence of NCL's is not sufficient to guarantee the existence of saddle points. In the $O(n)$ case, for $n \geq 4$ a large class of solutions is known¹⁵⁾ in addition to the embeddings of the $O(3)$ solutions. It is very probable that all of these solutions are unstable [see, e.g., Ref. 16]. The $O(n)$ $n > 3$ models strongly indicate the significance of the higher dimensional homotopy groups, $\pi_n(\mathcal{M})$ $n > D+1$.

For the $\mathbb{CP}^n$ models $\mathcal{M} = (U(n+1))/(U(n) \times U(1))$ corresponding to the case b) (for $m=1$). Since $\pi_2(\mathcal{M}) \simeq \mathbb{Z}$ topologically stable solutions exist¹⁷⁾, however as $\pi_3(\mathcal{M}) \simeq 0$ for $n > 1$ one cannot construct non-contractible loops. However, two-dimensional "hyperloops" are present as $\pi_4(\mathcal{M}) \simeq \mathbb{Z}$, ($n \geq 2$), so we may expect unstable solutions as well. Actually, all finite energy solutions to the field equations of the $\mathbb{CP}^n$ models were found by Din and Zakrzewski¹⁸⁾. The non-self-dual solutions - which are the saddle points of the energy functional - are all unstable. We feel that in these models once again the above-mentioned hyperloops may be important to understand the physical rôle of these unstable solutions.

NOTE ADDED

It seems that, in fact, Mandelstam was the first to realize the instability of the DHN solution¹⁹⁾. Also the first, static, finite energy monopole solution which does not satisfy the Bogomol'nyi equations was found by Burzlaff²⁰⁾ in an $SU(3)$ gauge theory.

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