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LATTICE QUANTUM CHROMODYNAMICS

P. Hasenfratz
CERN - Geneva

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CONTENTS

1	General introduction to lattice gauge theories	1
1.1	Going to Euclidean space. The formal relation between quantum field theory and classical statistical physics	1
1.2	Gauge theory on the lattice	3
1.3	The continuum limit of lattice field theories	8
1.4	Strong coupling limit, confinement	11
1.5	Renormalizability, dimensional transmutation	14
1.6	Scaling, asymptotic scaling	17
1.7	General, non-perturbative renormalization group arguments on renormalizability and universality	17
1.8	Weak coupling perturbation theory. Freedom in choosing the lattice action	24
2	Non-perturbative methods in the gauge sector	27
2.1	Strong coupling expansion	27
2.2	Monte Carlo simulation	34
2.3	Mean field techniques	37
2.4	Variational methods combined with Monte Carlo simulations	40
2.5	Monte Carlo renormalization group techniques (MCRG)	42
3	How the lattice works: string tension in $SU(2)$	46
4	$SU(3)$ gauge theory at finite physical temperature	52
4.1	How to introduce finite temperature in a quantum field theory?	52
4.2	The order parameter of the deconfining phase transition. Theoretical arguments	54
4.3	MC results on the deconfining phase transition in $SU(3)$	55
5	Universality tests in $SU(2)$ lattice QCD	58
6	Numerical results in $SU(3)$ pure gauge theory	62
7	Fermions on the lattice. General introduction	64
7.1	Defining fermions on the lattice naïvely, the doubling problem	64
7.2	Wilson's method of removing the degeneracy	66
7.3	Technical problems and methods	67
7.4	The procedure of extracting the hadron masses	69
7.5	How the parameters are fixed	71
8	QCD at finite temperature and baryon density	73
8.1	Chemical potential on the lattice	73
8.2	The fate of the deconfining phase transition in the presence of quarks	78

INTRODUCTION

It is generally accepted that relativistic field theory is relevant in high energy physics. It is also recognized that even in QCD, which is asymptotically free (i.e., the interactions become weak at high enough energies) the scope of perturbation theory is very limited. In spite of the tremendous theoretical and experimental effort studying scaling, scaling violations, e^+e^- lepton pair creation, jets and so on, the answer to the question whether and to what extent is QCD the theory of strong interactions, is very vague. At present-day energies it is difficult to disentangle perturbative and non-perturbative effects.

QCD is an important and relevant theory in itself. Additionally, many of the basic questions: confinement, spontaneous chiral symmetry breaking, mass generation are interesting beyond the context of this specific theory. The presupposed behaviour of QCD enters everywhere in model building. We cannot afford constructing huge buildings on a shaky foundation. QCD must be understood.

Quantitative non-perturbative methods are needed: methods, where the approximations can be controlled, and where there exists the possibility of improving the calculation by increasing the computational effort. The lattice formulation of field theories is a promising approach of this kind. At present, a large part of this subject can be identified as computational physics.

Computational physics entered theoretical high energy physics. The recent history of different fields - statistical mechanics, atomic and molecular physics, nuclear physics, astrophysics - shows that the different subjects unavoidably arrive to a certain stage, when the systems of interest become highly complex and questions arise whose answer must be found by computational techniques^{*)}.

This event in high energy physics coincides accidentally with another one. It would be hard not to observe that there is a revolution going on around us. We could say in a banal way: computers entered everyday life.

The effect of computers might be different on different subjects. I think that in nuclear physics, the computers did not help this subject to become more fascinating. On the other hand, in statistical mechanics, where computers played an essential rôle in deriving precise predictions on critical behaviour (for instance via long high-temperature expansions) the results prepared the way for a beautiful theoretical development, the renormalization group approach.

The idea of defining QCD - more generally gauge theories - on a lattice has been put forward by Wilson in 1974[1]. Not only the formulation was given at that time, but new notions and different approximations were raised also. In a few years time the lattice established itself as a sensible regularization of QCD, and the new techniques proved to be powerful enough to attack exciting non-perturbative questions (confinement, string tension, mass gap,...) in the theory[2].

What is the greatest thing about the lattice approach? It opened the way towards new, non-perturbative techniques ... yes, this is true. It opened the way towards the rigorous results derived in statistical physics ... yes, that is also true. But, what one might enumerate first is: the lattice approach created a new standard in strong interactions. There is a way to check (and hopefully control) the systematical and statistical errors and the approximations.

*) Quoted freely from "Prospectus for Computational Physics, Report by the Subcommittee on Computational Facilities for Theoretical Research to the Advisory Committee for Physics, Division of Physics, National Science Foundation (1981).

1 GENERAL INTRODUCTION TO LATTICE GAUGE THEORIES

1.1 Going to Euclidean space. The formal relation between quantum field theory and classical statistical physics

The relation between the problems of quantum field theory and those in classical statistical physics played a very important role in all the ideas and methods going beyond perturbation theory in QCD.

In the path integral formulation of quantum mechanics[3] the amplitude of propagation between the points of x_a , t_a and x_b , t_b is given by

$$K(x_b, t_b; x_a, t_a) = \sum_{\text{paths}} e^{i \int_{t_a}^{t_b} \mathcal{L}(x, \dot{x}) dt} \quad (1.1)$$

where the summation is over all paths connecting the initial and final space-time points. The weight of a path is given by e^{iS} , where S is the classical action of the path.

The path integral formulation can be generalized to systems with many or infinite degrees of freedom. In quantum field theory the vacuum functional is defined analogously

$$Z = \sum_{\text{field configurations}} e^{i \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)} \quad (1.2)$$

Here, again, we might consider the amplitude that the field is equal to $f_a(x)$ and $f_b(x)$ at time t_a and t_b respectively (by summing over the connecting configurations). More often, however, $t_a \rightarrow -\infty$, $t_b \rightarrow +\infty$ is taken, some boundary condition is prescribed (the nature of which is expected to be irrelevant, like that of the spatial boundary conditions for large systems), and the system is probed by the introduction of external sources, taking $\mathcal{L} \rightarrow \mathcal{L} + j\phi$.

Both Eqs. (1.1) and (1.2) are formal until some sensible definition is not given to the summations. This definition can be given easily if the field theory is treated perturbatively (when functional integrals of

Gaussian nature are considered only). The usual diagrammatic expansion is obtained in this case.

The vacuum expectation value of the time ordered product of the fields (Green's function) is obtained in the path integral formulation as the expectation value of the product of the fields, using the measure

$$\sim \frac{1}{Z} e^{i \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)} \quad (1.3)$$

This measure is complex. In order to establish a relation with classical statistical physics an analytic continuation is done in the time coordinate

$$X_0 \longrightarrow -iX_4 \quad (1.4)$$

which defines the model in four-dimensional Euclidean space. A real, normalized measure is obtained

$$\sim \frac{1}{Z_E} e^{-\int d^4x \mathcal{L}_E(\phi, \partial_\mu \phi)} \quad (1.5)$$

where $\mathcal{L}_E$ is the Euclidean Lagrangian, where all the indices enter symmetrically with positive metric diag(1,1,1,1). Specifically, for QCD the Euclidean vacuum functional is given as (the zeroth component of a vector field becomes imaginary also):

$$Z = \sum_{\text{all Euclidean gauge field configs}} e^{-\frac{1}{g^2} \int d^4x \sum_{a=1}^3 \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a} \quad (1.6)$$

which can be interpreted as the partition function of a four-dimensional classical Yang-Mills system. The coupling g^2 plays the rôle of temperature in this analogy.

The possibility of the analytic continuation to imaginary times can be established easily in perturbation theory[4] (Appendix A) and there are rigorous theorems beyond that[5]

In practice, in many cases, the results obtained in Euclidean space can be used directly to obtain physical predictions, and an explicit analytic continuation back to Minkowski space is unnecessary. Take for instance the free propagator in Euclidean space

$$G(x) = \frac{1}{(2\pi)^4} \int d^4p e^{ipx} \frac{1}{p^2 + m^2} \quad (1.7)$$

For large $|x| = \sqrt{x_\mu x_\mu}$, $G(x)$ behaves as

$$G(x) \sim e^{-m|x|} \quad (1.8)$$

An analytic continuation back to Minkowski space would lead to the usual oscillatory behaviour for time-like separations. However, in order to recover the mass of the particle from the asymptotic behaviour of the propagator - a problem often encountered in non-perturbative studies - we can use (1.8) directly as well.

1.2 Gauge theory on the lattice

By going to Euclidean space the quantum field theoretical problem is transformed into the statistical physics of classical fields. This is not an easy problem either. In particular, it is plagued by the same kind of divergences as the original system. Regularization is needed. We want to discuss non-perturbative phenomena, therefore this regularization should be more general than a special prescription for Feynman diagrams.

The lattice regularization satisfies not only this criterion, but opens the way towards the non-perturbative techniques of statistical physics.

The continuous Euclidean space is replaced by discrete lattice points. Consider a regular hyper-cubic lattice [6]. The lattice unit is denoted by "a", while a lattice point by $n_\mu = (n_1, n_2, n_3, n_4)$ (integers). The Fourier transform of a function $f(n)$ defined on the lattice points is given as

$$\tilde{f}(p) = a^4 \sum_n e^{i(pn)a} f(n) \quad (1.9)$$

The function $\tilde{f}(p)$ is periodic over $p_\mu = 2\pi/a$, therefore the momentum values can be constrained within the first Brillouin zone

$$-\frac{\pi}{a} \leq p_\mu \leq \frac{\pi}{a} \quad (1.10)$$

The inverse transform is given by

$$f(n) = \frac{1}{(2\pi)^4} \iiint_{-\pi/a}^{\pi/a} d^4p e^{-i(pn)a} \tilde{f}(p) \quad (1.11)$$

As we see from Eqs. (1.10) and (1.11), the lattice provides for a cut-off in momentum space:

$$\text{cut-off momentum} = \frac{\pi}{a} \quad (1.12)$$

Given the continuum Lagrangian, there is a natural way to define a field theory on the lattice: scalar fields are defined on the points, vector fields (characterized by a position and a direction) on the links of the lattice [7]. Derivatives are replaced by discrete differences

$$\partial_\mu f(x) \longrightarrow \Delta_\mu f(n) = \frac{1}{a} (f(n, \hat{\mu}) - f(n)), \quad (1.13)$$

where $\hat{\mu}$ is the unit vector along the μ direction.

In the case of gauge fields, however, the requirement of gauge invariance complicates the matter.

Consider the case of electrodynamics. In Maxwell's theory Gauss' law says: the electric flux coming through a closed surface is equal to the charge inside. There is no charge without its accompanying Coulomb field. It is the same in QED.

Let us denote the vector potential and the electric field by $A_r(x)$ and $\vec{E}_r(x)$ respectively, $r = 1, 2, 3$. The electric field is the momentum conjugate to the vector potential. They satisfy the usual commutation rules, the Hilbert space is restricted to physical states (satisfying Gauss' law). Consider the operator

$$e^{i \int d^3x \vec{A}_r(x) \cdot \vec{c}_r(x)} \quad (1.14)$$

where $c_r(x)$ are three c-number functions. By acting on a state with this operator, the value of the electric field is increased by $\vec{c}_r(x)$. Really

$$\hat{E}_r(y) e^{i \int d^3x \vec{A}_r(x) \cdot \vec{c}_r(x)} = e^{i \int d^3x \vec{A}_r(x) \cdot \vec{c}_r(x)} (\vec{E}_r(y) + \vec{c}_r(y)) \quad (1.15)$$

Consider now a source and a sink of strength g at the points 1 and 2, respectively. An electric flux of g should be led from the point 1 to 2. It is the requirement of Gauss' law. It is a kinematical requirement. Gauss' law does not tell us, what distribution the electric flux takes. It only requires the conservation of the electric flux. A special choice could be to let the electric flux propagate along an infinitesimal flux tube between 1 and 2

In this case c_r is a transversal δ function along the path and the situation is described by

$$e^{ig \int_1^2 \vec{A} \cdot d\vec{s}} \quad (\text{source at 1}) \quad (\text{sink at 2}) \quad (1.16)$$

Let 1 and 2 be two neighbouring points separated by a small distance "a" along the μ direction. We get:

$$\phi(x) e^{iga A_\mu(x)} \phi(x+a\hat{\mu}) \quad (1.17)$$

From small "a" we can expand (1.17) in a Taylor series, and the terms linear in "a" give the usual covariant derivative. For finite "a", however, we must keep the complete expression to preserve fluxes (or equivalently, to preserve gauge invariance).

These considerations suggest that the basic variables of the lattice formulation are not the vector potentials themselves, but their exponentiated forms

$$U_{n\mu} = e^{iga A_{n\mu}} \quad , \quad \text{link: } n \rightarrow n+\hat{\mu} \quad (1.18)$$

associated to the directed link with endpoints n and $n+\hat{\mu}$. The oppositely oriented link is associated with $U_{n\mu}^\dagger$. $U_{n\mu}$ is called - for obvious reasons - a string bit. $U_{n\mu}$ is an element of the local symmetry group, U(1) here, SU(3) for QCD:

$$(U_{n\mu})_{st} = \left(e^{iga A_{n\mu}^b T^b} \right)_{st} \quad b=1,2,\dots,8 \quad s,t=1,2,3 \quad (1.19)$$

where T^b are the SU(3) generators.

If there are no sources and sinks, that is we are dealing with a pure gauge theory, the string bits must form closed loops, like smoke rings.

In the simplest case the smoke rings run along the smallest loops of the lattice: around the plaquettes. Keeping the symmetry between the four directions of the Euclidean lattice one arrives to the action [1.8]

$$S_w = (\text{const.}) \cdot \sum_{\text{plaquettes}} (T_\tau U_\rho + T_\tau U_\rho^\dagger) \quad (1.20)$$

where

$$U_p = U_{\eta\mu} U_{\eta\hat{\mu}\nu} U_{\eta\hat{\nu}\rho}^+ U_{\eta\nu}^+ \quad p: \quad \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \begin{array}{l} \eta\hat{\mu} \\ \eta\hat{\nu} \end{array} \quad \begin{array}{l} \eta\hat{\rho} \\ \eta\nu \end{array} \quad (1.21)$$

U_p and U_p^+ describe oppositely oriented loops. Trace (Tr) is taken to get numbers from matrices. Gauge invariance can be expressed as an invariance under local symmetry transformations - like in the continuum. S_w is invariant under the independent rotations of colour reference frames at every lattice point n by an $SU(3)$ matrix V_n :

$$U_{\eta\mu} \longrightarrow V_n U_{\eta\mu} V_n^+ \quad (1.22)$$

This action respects maximally the symmetries of the lattice: it is invariant under 90° rotations and lattice translations. It is invariant also under parity transformations and charge conjugation (Appendix B).

The constant in Eq.(1.20) should be chosen to reproduce the continuum action in the classical $a \rightarrow 0$ continuum limit. By using repeatedly the Baker-Hausdorff formula

$$e^A e^B = e^{A+B + \frac{1}{2}[A,B] + \dots} \quad (1.23)$$

one obtains[9]

$$U_p \equiv e^{ig a^2 F_{\eta\mu\nu}} = 1 + ig a^2 F_{\eta\mu\nu} + \frac{1}{2!} (ig a^2)^2 (F_{\eta\mu\nu})^2 + \dots, \quad (1.24)$$

where

$$F_{\eta\mu\nu} = \sum_{a=1}^8 F_{\eta\mu\nu}^a T^a$$

and

$$F_{\eta\mu\nu}^a = \Delta_\mu^a A_{\nu\rho} - \Delta_\nu^a A_{\rho\mu} - g f^{abc} (A_{\eta\rho} A_{\mu\nu} + a \Delta_\nu A_{\eta\rho} A_{\mu\nu}^+ + a A_{\eta\rho} \Delta_\mu A_{\nu\rho} - \frac{1}{2} a \Delta_\mu A_{\nu\rho} A_{\eta\rho} + \frac{1}{2} a^2 \Delta_\nu A_{\eta\rho} \Delta_\mu A_{\nu\rho}) + \mathcal{O}(g^2).$$

$F_{\eta\mu\nu}^a$ is gauge covariant on the lattice. In the $a \rightarrow 0$ classical continuum limit it reduces to the continuum field strength tensor [the first three terms in Eq. (1.25)]. By using Eq. (1.24) and $\text{Tr}(T^a T^b) = 1/2(\delta_{ab})$, the continuum action is obtained in Eq. (1.20), if $\text{const.} = -1/g^2$ is chosen.

Let us write down Wilson's action again, now with the correct normalization

$$S_w = -\frac{1}{g^2} \sum_{\text{plaquettes}} (\text{Tr} U_p + \text{Tr} U_p^+) \quad (1.26)$$

In the partition function we sum over the field configurations, that is, we integrate over the $SU(3)$ group on each link of the lattice:

$$Z = \prod_{\text{links}} \int dU_l e^{-S_w}, \quad (1.27)$$

where dU is the invariant Haar measure having the properties

$$\int dU = 1, \quad \int dU f(U) = \int dU f(U_0 U), \quad (1.28)$$

where U_0 is an arbitrary element of the group.

1.3 The continuum limit of lattice field theories

The lattice is only a regularization. It should be removed at the end of the calculation by taking $a \rightarrow 0$ limit.

This limit is subtle. The action in (1.26) violates Euclidean rotation invariance (corresponding to Lorentz invariance in Minkowski space) - it is invariant only under 90° rotations. In the continuum limit the full symmetry should be recovered.

To achieve this goal it is not enough to consider correlations over distances r , which are much larger than a . For a generic g^2 in (1.26) the asymptotic decay of correlation functions will show a direction dependence, defining masses [via relations analogous to (1.8)] which are direction dependent also. It is like in a solid crystal which behaves differently along different directions.

The only dimensionful parameter in pure gauge QCD is the lattice unit a . The lattice predictions are always dimensionless numbers, the correct dimensions are restored with the help of a . For instance, a mass prediction will have the form

$$m = \frac{1}{a} f(g) \quad (1.29)$$

In a naive $a \rightarrow 0$ limit, all the masses would go to infinity. In other words, for a generic g , $f(g)$ is a number of $\mathcal{O}(1)$. Therefore the mass is $\sim 1/a \sim \text{cut-off}$.

In order to keep the mass finite as $a \rightarrow 0$, g , the bare coupling constant, must also be tuned:

$$\left\{ \begin{array}{l} a \rightarrow 0 \\ g \rightarrow g^* \end{array} \right\}, \quad \text{where } f(g^*) = 0. \quad (1.30)$$

In this limit ($am \rightarrow 0$, $1/am \equiv \text{correlation length in lattice units} \rightarrow \infty$). Therefore:

In the continuum limit, the lattice problem approaches a continuous phase transition point.

Close to this critical point, there will be strong fluctuations over many lattice units in the system washing away the original lattice structure, and leading to the possibility of restoring Euclidean rotation invariance.

Let us illustrate these points on an exactly solvable simple system, on the two-dimensional Ising model: classical spins coupled ferromagnetically on a quadratic lattice. The partition function is defined as:

$$Z = \sum_{\{S_i = \pm 1\}} e^{K \sum_{\langle ij \rangle} S_i S_j} \quad (1.31)$$

where $\sum_{\langle ij \rangle}$ denotes a summation over nearest neighbour couplings, and K is proportional to the inverse temperature $1/T$.

Consider the high temperature phase. The correlation function $\langle S_i S_n \rangle$ decays exponentially as $a|n| \equiv r \rightarrow \infty$ in an angle dependent way [$\sin \alpha = n_1/|n|$ here]:

$$\langle S_i S_n \rangle \underset{r \rightarrow \infty}{\sim} e^{-m(\alpha)r} \quad (1.32)$$

For instance,

$$\frac{m(\alpha=0^\circ)}{m(\alpha=45^\circ)} = \frac{1}{\sqrt{2}} \frac{\ln\left(\frac{1+\nu}{1-\nu}\right)}{\ln\left(\frac{2\nu}{1-\nu^2}\right)}, \quad (1.33)$$

where $\nu = \tanh K[10]$. For small K (high temperature) this ratio is $1/\sqrt{2}$, rotation invariance is strongly violated. By decreasing the temperature the situation is improving, and rotation symmetry is restored exactly at the critical point (Curie point) $V_c = \sqrt{2}-1$ (Table 1)*).

*) It is worth observing that when the correlation length becomes of the order of $\sim 2a$, the violation of rotational invariance is well below 1%.

v	$m(\alpha=0^\circ)/m(\alpha=45^\circ)$	Correlation length along the axes
0	$1/\sqrt{2} \approx 0.707$	0
0.05	1.890	0.35
0.1	0.929	0.48
0.2	0.972	0.83
0.3	0.992	1.71
$\sqrt{2}-1 \approx 0.414$	1	∞

Table 1: Restoration of rotation symmetry in d=2 Ising model as the critical point is approached.

In the relation analogous to (1.29)

$$m(\alpha) = \frac{1}{a} f(T, \alpha) , \quad (1.34)$$

$f \rightarrow -(T-T_C)^v$, $v = 1$, independently of α , and a finite mass is obtained if a is going to zero as $-(T-T_C)^v$ in this limit.

What is the critical coupling g^* for QCD ? The coupling g^* is the bare coupling constant of the continuum theory. The bare coupling constant describes the interactions on momentum scales $\sim$ cut-off. In an asymptotically free theory this coupling becomes vanishingly small as the cut-off is increased: $g^* = 0$.

In order to find an asymptotically free theory in the continuum limit of lattice QCD, this continuum limit should occur when the coupling is tuned towards zero.

1.4 Strong coupling limit, confinement

Lattice gauge theories can be solved exactly in the other extreme limit, when $g \rightarrow \infty$. (This is like the high temperature limit in statistical physics.)

The model exhibits confinement at strong coupling: the potential energy between two heavy quark sources increases linearly for large separations. This potential energy can be obtained by studying special

Figure 1

gauge invariant loop expectation values $\langle w \rangle_{[1,1]}$ (Fig. 1). This loop ("Wilson loop") describes the propagation of a $q-\bar{q}$ pair at rest over a "time-period" T . Really, the spacelike parts of the loop describe the creation and annihilation of a gauge invariant $q-\bar{q}$ source as it is discussed in Section 1.2. The presence of an external current leads to an extra term in the continuum action:

$$\int d^4x \, i g V_\mu A_\mu , \quad (1.35)$$

where V_μ is the current of the external source. For a pointlike source it gives a factor in the path integral:

$$e^{i g \int_C dx_\mu A_\mu} \longrightarrow \pi U_c , \quad (1.36)$$

leading to the "timelike" part of the Wilson loop.

Let us denote by $E(L)$ the energy of a source-sink system separated at a distance L . We expect

$$\langle W \rangle \underset{T \rightarrow \infty}{\sim} e^{-TE(L)} \quad (1.37)$$

which leads to an area decay if the sources are confined by a linear potential $E(L) = \sigma L$ (σ is called the string tension). If, for large distances, the binding potential becomes vanishingly small, the energy is independent of L , the decay of the Wilson loop expectation value is governed by the perimeter of the loop.

Consider therefore

$$\langle W \rangle = \frac{\int DU W e^{\frac{1}{g^2} \sum_p (Tr U_p + cc)}}{\int DU e^{\frac{1}{g^2} \sum_p (Tr U_p + cc)}} \quad (1.38)$$

and expand the Boltzmann factor in powers of $1/g^2$. Using the group integrals

$$\begin{aligned} \int DU U &= \int DU U^* = \int DU UU = \int DU U^+ U^* = 0, \\ \int DU U_{ij} U_{kl}^* &= \frac{1}{3} \delta_{ik} \delta_{jl} \end{aligned} \quad (1.39)$$

[which can be deduced from (1.28) and from $UU^* = 1$], we see that in lowest order a U_x variable of the Wilson loop should meet a U_x^* from

Figure 2

the Boltzmann factor, which leads to the arrangement of Fig. 2. Therefore in leading order

$$\langle W \rangle \underset{g \rightarrow \infty}{\sim} \left(\frac{1}{3g^2} \right)^{LT} = e^{-LT \ln(3g^2)} \quad (1.40)$$

W decays exponentially with the area of the loop, signalling confinement. For $g \rightarrow \infty$ the model confines. This conclusion is not spoiled by higher order corrections since it can be shown rigorously that the strong coupling expansion has a finite radius of convergence [12].

After restoring the dimensions, the string tension is given by

$$\sigma = \lim_{g \rightarrow \infty} \frac{1}{a^2} \ln(3g^2) \quad (1.41)$$

Unfortunately, in this limit the model has not much to do with continuum QCD. The masses are much larger than the cut-off, rotation invariance is badly violated (the string tension would increase by rotating the sources away from the axes).

The coupling g should be changed towards $g = 0$, where the continuum limit is to be found. It is a long way to go, and we might meet surprises. If we want an asymptotically free, confining theory at the end, a deconfining phase transition must not be among them.

1.5 Renormalizability, dimensional transmutation

In calculating the string tension in the strong coupling limit the leading contribution is obtained by constructing a minimal surface over the Wilson loop (Fig. 2). It describes the propagation of an infinitely narrow, straight flux tube spanned between the sources.

In higher orders this surface begins to fluctuate, the flux tube acquires a finite transversal width. The dimensionless function $f(g)$ in

$$\sigma^{1/2} = \frac{1}{a} f(g), \quad (f(g) \underset{g \rightarrow \infty}{\rightarrow} (\ln 3g^2)^{1/2}), \quad (1.42)$$

becomes smaller as g is decreased - the string begins to thaw. The relation between the cut-off and $\sigma^{1/2}$, or between the width of the tube

and the lattice distance becomes more and more acceptable

Figure 3

physically (Fig. 3).

In the continuum limit the lattice distance a is small compared to the characteristic physical distances, (like the radius of the flux tube), the lattice becomes fine grained. In this limit the lattice is expected to play no important role anymore. The size of the cut-off becomes irrelevant. We expect that it is possible to change the lattice distance a and the coupling g together keeping the physical properties of the system unchanged. This is the usual requirement of renormalizability.

This requirement uniquely determines the coupling constant dependence of any masses (and similar dimensionful quantities) in this limit. If m is a physical mass, cut-off independence implies

$$a \frac{d}{da} m = 0 \quad (1.43)$$

which is a first order differential equation for the function f of Eq. (1.29):

$$-f(g) + \left(\frac{d}{dg} f(g) \right) a \frac{d}{da} g(a) = 0 \quad (1.44)$$

Of course, not only masses, but other physical quantities should also become independent of the cut-off. For instance, results related to high energy deep inelastic processes. The requirement of cut-off independence determines the function $-\beta(g) \equiv (1/a)(d/da)g(a)$ there, in perturbation theory. $\beta(g)$ is the usual β function

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 \dots \quad (1.45)$$

The first two coefficients, β_0 and β_1 are universal^[13]: they are the same in the continuum (in any renormalization scheme) and in the lattice perturbation theory.

Apart from an integration constant, the function f (and m) is completely determined by Eqs. (1.44) and (1.45) when $g \rightarrow 0$:

$$m = c \frac{1}{a} e^{-\frac{1}{2\beta_0 g^2} \left(\beta_0 g^2 \right)^{-\frac{\beta_1}{2\beta_0^2}} \left[1 + \mathcal{O}(g^2) \right]} \quad (1.46)$$

$$\equiv c \Lambda^{\text{latt}}$$

Λ^{latt} is a cut-off independent mass parameter. This mass parameter sets the scale for QCD. It is an external parameter, like the hyperfine coupling $\alpha = 1/137$ in QED.

The dimension of physical masses and lengths is carried by Λ^{latt} ^[14], and the non-perturbative content of the theory lies in the value of the connecting constants c_i :

$$\begin{aligned} \bar{c}^{1/2} &= c_g \Lambda^{\text{latt}} \\ m_{\text{gap}} &= c_g \Lambda^{\text{latt}} \\ &\vdots \end{aligned} \quad (1.47)$$

1.6 Scaling, asymptotic scaling

If the lattice unit a is finite, there will be corrections to Eq. (1.43). Perturbation theory suggests that the leading corrections to the right-hand side of Eq. (1.43) are of the order of $O(a^2(2na)^P)$. This is an exponentially small correction in $1/g^2$, since a goes to zero like $(1/2\beta_0 g^2)$ [Eq. (1.46)]. Continuum behaviour sets in (cut-off independence is satisfied) if these corrections are negligibly small. In

Figure 4

this coupling constant region (region I+II in Fig. 4) any dimensionless mass should behave in a unique way as the function of g^2 :

$-e^{-\int g(1/\beta(g'))dg}$. This well-defined behaviour is called scaling. Although this behaviour is common for the different masses, it is not universal with respect to the different lattice formulations. The higher order terms of the β function are scheme dependent.

Decreasing g^2 further we arrive at a region (region I in Fig. 4) where not only the exponentially small corrections can be neglected, but even the power corrections in the bracket of Eq. (1.46) can be suppressed. These terms come from the higher order terms of the β function ($-g^n$, $n \geq 7$). In this region the mass scale according to the explicit terms of Eq. (1.46). This behaviour is universal, independent of the lattice action chosen. We might call this functional dependence asymptotic scaling.

1.7 General, non-perturbative renormalization group arguments on renormalizability and universality

This section can be considered as a general introduction to the Monte Carlo renormalization group method which will be discussed in Section 2.5. I hope it will help in a deeper understanding of Section 1.5 also [15]

Continuum physics is expected to be recovered in the long distance behaviour of the lattice problem approaching a phase transition point. One might attempt to integrate out those variables of the theory which describe short distance lattice phenomena one is not interested in. By thinning out the variables systematically in configuration space, (or by integrating over the high momentum components in Fourier space) one arrives to an effective action of the relevant, long distance variables.

RG transformations in a gauge theory are complicated by the requirement of gauge invariance. In order to avoid this problem at the start, consider a simple spin model, like the Ising model before. Take a Hamiltonian with nearest neighbour interactions^{*)}

$$-H = K \sum_{\langle ij \rangle} S_i S_j$$

$$Z = \sum_{\{S_i\}} e^{-H(S)} \quad (1.48)$$

Perform a RG transformation by averaging out the short distance components of the model. Let us work in configuration space. We

Figure 5

define a block spin variable μ_k as illustrated in Fig. 5. The block spin μ_k is constructed from the spin S_i of the corresponding block, $\mu_k = \mu_k(S_i, i \in \text{block } k)$. There are different possibilities for this block transformation, like "decimation" (when the block spin is identified with one of the spins of the block), "majority rule with tie-breaker" (where μ_k takes the value the majority of the spins take in block), and so on. Keeping the block variables $\{\mu_k\}$ fixed and summing over spins one obtains

*)The factor $(1/k_B T)$ is included in H .

$$\sum_{\{S_i\}} \prod_i f(\mu_i, S_i, i \in \text{block } l) e^{-H(S)} = e^{-H(\mu)} \quad (1.49)$$

where $H(\mu)$ is the new Hamiltonian describing the interaction of the block spins on a lattice with $a' = 2a$. The function f describes the way the block spin is constructed of the spins S_i and it is largely arbitrary. If we want H' to have the same symmetries as H , the function f should also respect this symmetry. Additionally, there is a normalization condition on f , coming from the requirement that H' should give the same thermodynamics as the original Hamiltonian:

$$\sum_{\{\mu_i\}} e^{-H(\mu)} = \sum_{\{S_i\}} e^{-H(S)} \rightarrow \sum_{\mu_i} f(\mu_i, S_i) = 1. \quad (1.50)$$

In general, $H'(\mu)$ will contain all kind of different interactions even if we started with the simple Hamiltonian in Eq. (1.48). As we want to repeat this transformation, it is better to start with a general Hamiltonian

$$-H = \sum_{\alpha} K_{\alpha} \theta_{\alpha}(S) \quad (1.51)$$

where θ_{α} ($\alpha = 1, 2, \dots$) describe nearest-neighbour, next-to-nearest-neighbour, four-spin, ... interactions. We consider a multidimensional coupling constant space $\{K_{\alpha}\}$ and the RG transformation moves the system in this space:

$$\{K_{\alpha}^{(1)}\} \xrightarrow{\text{RG}} \{K_{\alpha}^{(2)}\} \xrightarrow{\text{RG}} \dots \{K_{\alpha}^{(n)}\} \rightarrow \dots \quad (1.52)$$

This sequence of transformations might have a fixed point:

$$\{K_{\alpha}^*\} \xrightarrow{\text{RG}} \{K_{\alpha}^*\} \quad (1.53)$$

In the fixed point the (dimensionless) correlation length ξ is either 0 or ∞ . Really, under a RG transformation, $\xi' = 1/2\xi$ (remember the lattice

unit is increased by a factor of 2), on the other hand, at the fixed point $\xi' = \xi$, therefore $\xi = 0$ or $\xi = \infty$. We are interested in those fixed points where $\xi = \infty$ which describe a continuous phase transition.

Critical behaviour is a result of fluctuations on all scales from the cut-off to small momenta. As the transformation $\{K_{\alpha}^{(n)}\} \rightarrow \{K_{\alpha}^{(n+1)}\}$ involves only one scale, it is expected that the critical singularities come from the repeated application of the RG transformation and not from the functional relation $K_{\alpha}^{(n+1)} = K_{\alpha}^{(n)}$ itself. Let us expand this relation around the fixed point. We obtain a linearized equation for the difference $\Delta K_{\alpha} = K_{\alpha} - K_{\alpha}^*$:

$$\Delta K_{\alpha}^{(n+1)} = T_{\alpha\beta} \Delta K_{\beta}^{(n)}, \quad (1.54)$$

where

$$T_{\alpha\beta} = \frac{\partial K_{\alpha}^{(n+1)}}{\partial K_{\beta}^{(n)}} \Big|_{K^*}. \quad (1.55)$$

Consider a two-dimensional parameter space, for simplicity. $T_{\alpha\beta}$ is a 2×2 matrix. Let us assume it has two real eigenvalues $\lambda_1 > 1$, $\lambda_2 < 1$. The operator combination lying along the eigendirection 1 is called relevant, while the other one is called irrelevant. Under repeated RG transformations, the coupling of the relevant (irrelevant) operator

Figure 6

increases (decreases) rapidly. The flow lines are illustrated in Fig. 6. The line starting along the 1 axes is called renormalized trajectory (RT), while the line starting along 2 is the line of criticality: the points of this line (in general it is a surface spanned by the irrelevant variables) end up in the critical point, they describe the $\xi = \infty$ critical system. Starting from a point outside the line of criticality the flow line moves along the RT after a few RG steps.

Let us turn back to lattice QCD. Parametrize the generalized action as

$$-\frac{1}{g^2} \left\{ \theta_1 + c_2 \theta_2 + c_3 \theta_3 + \dots \right\} \quad (1.56)$$

where $\theta_1 = \sum \text{plaquettes} (\text{TrUp}^*cc)$, while $\theta_2, \theta_3, \dots$ are combinations of different kinds of interactions, containing loops and more complicated functions of the trace. We require that $\theta_2, \theta_3, \dots \rightarrow 0$ in the classical $a \rightarrow 0$ limit in order to recover the usual classical action.

In the multiparameter space $(g^2, c_2, c_3, \dots)$ we expect a fixed point somewhere in the $g^2 = 0$ hyperplane: $(g^2 = 0, c_2^*, c_3^*, \dots)$. This fixed point can be found by using perturbation theory. The identical type of gauge invariant expectation values (for instance the plaquette expectation value before and after the transformation) should remain unchanged if we are in the fixed point. Before and after the transformation*) the leading order result has the form $1 - g^{*2} f(c_2^*, c_3^*, \dots) |_{g^{*2}=0}$ and $1 - g^{*2} \tilde{f}(c_2^*, c_3^*, \dots) |_{g^{*2}=0}$, respectively. ($\tilde{f}$ is obtained by expressing the block gauge matrices in terms of the original variables everywhere, and performing perturbation theory in terms of the original variables.) The fixed point condition is $f(c_2^*, c_3^*, \dots) = \tilde{f}(c_2^*, c_3^*, \dots)$, which determines $c_2^*, c_3^*, \dots$ if the procedure is repeated for the different gauge invariant expectation values.

Although - to my best knowledge - no explicit perturbative calculation has been performed to investigate the properties of this fixed point in lattice QCD, it is expected that at this fixed point there

*) An explicit gauge invariant block transformation will be discussed in Section 2.5.

is only one relevant operator*), the other combinations are irrelevant. For a simple scalar field theory the appropriate calculation is easy to perform in the continuum using a RG transformation in momentum space. The result is that the relevant (marginal) operators with respect to the Gaussian fixed point are those whose engineering dimension is ≤ 4 , the higher dimension operators are irrelevant. Taking over this result to the gauge theory we find that there is only one operator with dimension ≤ 4 , which has the basic symmetries (gauge symmetry, 90° rotations, reflections, C parity) of the Euclidean lattice theory: $\sum_{\mu, \nu, a} F_{\mu, \nu}^a F_{\mu, \nu}^a$. Therefore we expect a single renormalized trajectory coming out of the hyperplane $g^2 = 0$ and attracting all the

Figure 7

flow lines starting in the neighbourhood of the fixed point (Fig. 7). This is the basic starting point in expecting renormalizability and universality (i.e., the independence of the physical predictions of the specific lattice action chosen).

Renormalizability of the Wilson action [Eq. (1.26)]

Consider the standard Wilson action, Eq. (1.26). It is defined along the line $c_2 = 0$, $c_3 = 0, \dots$, $g^2 \in (0, \infty)$ in the multiparameter space. Perform a RG transformation starting from a small g^2 . After this transformation we arrive to the point 1 in the multidimensional parameter space (in Fig. 8, only one extra co-ordinate, c_2 , is kept). After a few RG steps we land on the RT and continue to move along it. Let us repeat the same procedure starting from the point

*) Actually it is marginal having an eigenvalue 1 in the linear approximation.

Figure 8

$g'^2 > g^2 (c_2=c_3=\dots=0)$. By tuning g'^2 it can be arranged that those points of this second sequence which lie on the RT coincide with the corresponding points of the first sequence, but one step behind (Fig. 8). Therefore the Wilson action with coupling g^2 and g'^2 describes identical long-distance (continuum) physics, the only difference is in their scale:

$$\xi(g^2) = 2 \xi(g'^2)$$

or

$$a(g^2) = \frac{1}{2} a(g'^2)$$

Therefore it is possible to change the cut-off and the coupling appropriately without changing the physical content of the theory. For very small g^2 , the relation $g^2 \rightarrow g'^2$ is determined by the known, one-loop β function (asymptotic scaling, Section 1.8). If g^2 is not so small (but small enough for the procedure described to go through), this β function has higher order perturbative and non-perturbative pieces.

Universality

As an example, consider the Wilson action S_A ($c_2=c_3=\dots=0$, $g_A^2 \in (0, \infty)$) and another action S_B ($c_2=\text{const}$, $c_3=c_4=\dots=0$, $g_B^2 \in (0, \infty)$) (Fig. 9). Let us fix g_A^2 . By tuning the coupling g_B^2 it can be arranged that the two sequences of points coincide along the RT. Therefore the two theories become identical after a few steps of RG transformation (which keeps the long-distance properties untouched) if g_B^2 is chosen appropriately: $g_B^2 = g_B^2(g_A^2)$. For very small g^2 perturbation theory gives

Figure 9

$$g_B^2 = g_A^2 + C_{18} g_A^3 + \dots$$

which leads to the universality of the first two coefficients of the β function [13].

1.8 Weak coupling perturbation theory. Freedom in choosing the lattice action

In the limit $g \rightarrow 0$, the factor $1/g^2$ in front of the action [Eq. (1.26)] becomes very large, which tends to suppress the fluctuations in the plaquette variable. Apart from the gauge freedom (which can be handled by fixing the gauge) the U matrices will be close to 1:

$$U_{\eta\mu} = e^{iga A_{\eta\mu}} = 1 + iga A_{\eta\mu} + \dots \quad (1.57)$$

Using this expansion, a systematic perturbation theory (in g) can be constructed.

Lattice perturbation theory is rather involved, because ever more new types of interaction terms enter as the order of the calculation is increased. Expanding the Wilson action we obtain [16]:

$$S_W = a^4 \sum_n \sum_{\mu, \nu} \left\{ -\frac{1}{4} (\Delta_\mu A_{\nu\rho} - \Delta_\nu A_{\mu\rho})^2 + g f^{abc} [A_{\eta\mu}^a A_{\nu\rho}^b \Delta_\mu A_{\nu\rho}^c + \dots \right\}$$

$$+\frac{1}{2}aA_{\eta\mu}^a\Delta_\nu A_{\eta\mu}^b\Delta_\mu A_{\eta\nu}^c]-\frac{1}{4}\int\int A_{\eta\mu}^aA_{\eta\nu}^bA_{\eta\mu}^cA_{\eta\nu}^d+ \tag{1.58}$$
$$+\frac{1}{2}\int\int\int a\Delta_\mu A_{\eta\mu}^aA_{\eta\mu}^bA_{\eta\mu}^cA_{\eta\mu}^d+\dots\}$$

Although the new type of vertices are all proportional to a^k (k positive), they cannot be neglected, since they contain extra derivatives and fields leading to divergences cancelling the a^k factor in front.

Nevertheless, it is true that these extra contributions can be absorbed into the definition of the coupling and fields and the lattice and continuum formulations of QCD lead to identical physical predictions in perturbation theory^[17]. In this case the lattice is just one of the possible regularizations (a rather inconvenient one). It is a special scheme, like the dimensional regularization with minimal subtraction, the MOM scheme, and so on. In different renormalization or regularization schemes the Λ parameter is defined as

$$\Lambda = (\text{some mass } \mu) e^{-\frac{1}{2\beta_0 g^2(\mu)}} \left(\beta_0 g^2(\mu) \right) \left[1 + O(g^2(\mu)) \right], \tag{1.59}$$

where $g^2(\mu)$ is the coupling corresponding to this mass. For instance

- μ mass, introduced in n-dim. reg. with min. subtr. Λ^{DR}
- m_{PV} PV regulator mass Λ^{PV}
- M where $\Gamma_{(2)}$ and $\Gamma_{(3)}$ are fixed (MOM scheme) Λ^{MOM}
- $1/a$ on the lattice Λ^{latt}

Let us choose these masses to be very large (then the corresponding couplings are very small), and equal to each other. Then the ratio between the Λ parameters of schemes A and B is given by

$$\Lambda^A / \Lambda^B = e^{-\frac{1}{2\beta_0} \left(\frac{1}{g_A^2} - \frac{1}{g_B^2} \right)} \tag{1.60}$$

The difference in the exponent is a constant for weak couplings, and can be determined in perturbation theory at the one-loop level

$$\frac{1}{g_A^2} - \frac{1}{g_B^2} = C_{AB} + O(g^2) \tag{1.61}$$

After calculating C_{AB} , the ratio between the scales is known and the predictions can be converted from one scheme to the other. For example, the relation between the scale parameter of the Wilson action (1.26) Λ_w^{latt} and Λ^{mom} is given by ^[16,18]

$$\Lambda^{mom} / \Lambda_w^{latt} = \frac{57.5}{83.5} \frac{SU(2)}{SU(3)} \tag{1.62}$$

In particular, the different lattice formulations of QCD can be converted the same way. There is a large freedom in choosing the lattice action. Even after satisfying the requirement of gauge invariance and of other symmetries, there are many possibilities to write down a lattice action which reproduces the continuum action in the $a \rightarrow 0$ classical continuum limit. Different functions of $Tr U$ can be chosen and/or other types of loop (rectangular, or more complicated) can be included. In perturbation theory these formulations are identical. Additionally, a one-loop calculation predicts the relation between their scales, connecting trivially their non-perturbative predictions.

It should be emphasized that the $O(g^2)$ corrections indicated in Eqs. (1.59) and (1.61) are neglected in calculating the connecting ratios of Eq. (1.62). Therefore these ratios are applicable only in the asymptotic scaling region (Section 1.6).

Does the full non-perturbative theory respect these considerations? We cannot answer this question in relation with the lattice and continuum formulations, since there are no quantitative non-perturbative methods in the continuum. However, different lattice formulations can be compared and the basic assumption of universality can be investigated (Section 5).

2 NON-PERTURBATIVE METHODS IN THE GAUGE SECTOR

By going to Euclidean space and using lattice regularization, the non-perturbative methods developed and tested in statistical physics become available for the study of quantum field theoretical problems.

Although the non-perturbative techniques we shall discuss are all familiar in statistical physics, there are special new features due to the fact that a $d = 4$ locally invariant gauge system is a rather unusual statistical problem.

The essential elements of strong coupling expansion, Monte Carlo simulation, mean field techniques, variational Monte Carlo studies and Monte Carlo renormalization group techniques will be discussed briefly. Some of these methods have not become quantitative yet in QCD. However, they influence our intuition and they are part of the everyday vocabulary.

The strong coupling expansion and the mean field method are discussed in great detail in a recent paper by Drouffe and Zuber.[2]. Similarly, there is a recent summary on the specific features of Monte Carlo simulation in gauge theories by Creutz, Jacobs and Rebbi[2].

2.1 Strong coupling expansion

As we discussed earlier, lattice QCD can be solved exactly in the strong coupling, $g \rightarrow \infty$ limit. Gauge theories exhibit confinement in this limit. Unfortunately, continuum QCD is to be recovered in the other extreme region, when $g \rightarrow 0$ (which is "weak coupling", but not perturbative!)

The strong coupling expansion is a systematic expansion in $1/g^2$, starting from the exact, confining solution at $g = \infty$. The idea is to derive a reasonably long series for the physical quantity in question, and then to extrapolate this power series towards the continuum point at $g = 0$ [19]. The technique is in complete analogy with the high temperature expansion in statistical physics.

Consider, for definiteness, the calculation of the string tension, which is related to the expectation value of large Wilson loops $\langle W(C) \rangle$, Eq. (1.38). The Boltzmann factor is expanded in powers of $1/g^2$:

$$e^{\frac{1}{g^2} \sum_p (\text{Tr} U_p + cc)} = \prod_p e^{\frac{1}{g^2} (\text{Tr} U_p + cc)} \quad (2.1)$$

$$= \prod_p \left\{ 1 + \frac{1}{g^2} (\text{Tr} U_p + cc) + \frac{1}{2!} \left(\frac{1}{g^2} \right)^2 (\text{Tr} U_p + cc)^2 + \dots \right\}$$

After resolving the brackets, the result is a sum of terms, each term being the product of N_p factors (N_p is the number of plaquettes of the lattice) of the form:

$$\frac{1}{\ell_p!} \left(\frac{1}{g^2} \right)^{\ell_p} (\text{Tr} U_p + cc)^{\ell_p}, \quad \ell_p = 0, 1, 2, \dots \quad (2.2)$$

Up to any given order in $1/g^2$, the number of terms is finite (on a finite lattice).

This expansion is inserted into (1.38) and then the group integrals are performed. This last step is simplified significantly by replacing the expansion in (2.1) by the character expansion. The Boltzmann factor

$$e^{\frac{1}{g^2} (\text{Tr} U_p + cc)}$$

is defined over equivalence classes:

$$\frac{1}{g^2} (\text{Tr} (V U_p V^{-1}) + cc) \quad \frac{1}{g^2} (\text{Tr} U_p + cc) \\ e = e \quad (2.3)$$

where V is an arbitrary group element, therefore it can be expanded in terms of the characters χ_r of the group

$$e^{\frac{1}{g^2} (\text{Tr} U_p + cc)} = \sum_r \nu_r p_r(1/g^2) \chi_r(U_p) \quad (2.4)$$

Here the summation is over the irreducible representations r , ν_r is a conveniently chosen number (= dimension of the representation r). The basic orthogonality relation is given by

$$\int du \chi_r(su) \bar{\chi}_s(\tau u) = \frac{\delta_{rs}}{v_r} \chi_r(s\tau^{-1}), \quad s, \tau \in G \quad (2.5)$$

As an explicit example, consider the strong coupling expansion of the string tension in a $d=3$, $U(1)$ gauge theory. The expectation

Figure 10

value of an $N \times N$ ($N \rightarrow \infty$) Wilson loop is calculated (Fig. 10). The character expansion (2.4) has the simple form:

$$e^{\frac{1}{g^2}(u_p + cc)} = \sum_{r=-\infty}^{\infty} \beta_r \left(\frac{1}{g^2} \right) (u_p) \quad (2.6)$$

where $U_p = e^{i\alpha_p}$, $\alpha_p \in (-\pi, \pi)$, and

$$\beta_r \left(\frac{1}{g^2} \right) = \beta_{-r} \left(\frac{1}{g^2} \right) = I_r \left(\frac{2}{g^2} \right),$$

$I_r(x)$ being the r th modified Bessel function. Equation (2.6) can be written as

$$I_0 \left(\frac{2}{g^2} \right) \left\{ 1 + \frac{I_1 \left(\frac{2}{g^2} \right)}{I_0 \left(\frac{2}{g^2} \right)} (u_p + u_p^{-1}) + \frac{I_2 \left(\frac{2}{g^2} \right)}{I_0 \left(\frac{2}{g^2} \right)} (u_p^2 + u_p^{-2}) + \dots \right\} \quad (2.7)$$

For small $1/g^2$, we have:

$$\frac{I_r \left(\frac{2}{g^2} \right)}{I_0 \left(\frac{2}{g^2} \right)} \sim \left(\frac{1}{g^2} \right)^r$$

Lowest order

As we discussed previously, the lowest order contribution is obtained by taking the "1" of (2.7) on all the plaquettes, except on those covering the Wilson loop, where

$$\frac{I_1 \left(\frac{2}{g^2} \right)}{I_0 \left(\frac{2}{g^2} \right)} u_p$$

is taken (Fig. 2). The resulting group integrals are trivial and we obtain

$$\langle W \rangle = \left(\frac{I_1}{I_0} \right)^{N^2} \rightarrow \zeta = -\ln \left(\frac{I_1}{I_0} \right), \quad (2.8)$$

where the (dimensionless) tension is denoted by σ .

Next order

The minimal covering surface is distorted by pushing it out of the plane, as shown in Fig. 11.

The contribution is $2N^2(I_1/I_0)^{N^2+4}$, and we get

$$\langle W \rangle = \left(\frac{I_1}{I_0} \right)^{N^2} \left(1 + 2N^2 \left(\frac{I_1}{I_0} \right)^4 \right) \rightarrow \zeta = -\ln \left(\frac{I_1}{I_0} \right) - 2 \left(\frac{I_1}{I_0} \right)^4 \quad (2.9)$$

The string begins to fluctuate, the tension becomes smaller.

Figure 11

Next-to-next order

There are three types of contributions giving:

$$(N_p - 2N^2) 2 \left(\frac{I_1}{I_0} \right)^{N+6}, \quad (\text{Fig. 12a}) \quad (2.10)$$

$$4N^2 \left(\frac{I_1}{I_0} \right)^{N+6}, \quad (\text{Fig. 12b}) \quad (2.11)$$

and

$$2N^2 \left(\frac{I_1}{I_0} \right)^{N+4} \left(\frac{I_2}{I_0} \right)^{N+2}, \quad (\text{Fig. 12c}) \quad (2.12)$$

N_p in (2.10) is the total number of plaquettes on the lattice. This extensive contribution is cancelled by the normalizing partition function in the denominator of (1.38)

Figure 12a

Figure 12b

$$Z = 1 + 2N_p \left(\frac{I_1}{I_0} \right)^6 \quad (2.13)$$

However, there is a non-zero, "excluded volume" contribution from these type of graphs. Terms, proportional to the perimeter of the loop are neglected in (2.11). Collecting everything we get, for the string tension

Figure 12c

$$\zeta = -\ln\left(\frac{I_1}{I_0}\right) - 2\left(\frac{I_1}{I_0}\right)^4 - 2\left(\frac{I_2}{I_0}\right)^4 \left(\frac{I_3}{I_0}\right) \quad (2.14)$$

By this order we met all the main features of the expansion: the correct, order by order exponentialization

$$\langle W \rangle \sim e^{-(\text{area}) \cdot \zeta} \quad (2.15)$$

the cancellation of the extensive contributions and the way the higher characters enter.

Although the procedure is straightforward, the calculation becomes more and more laborious in higher orders. In QCD applications, the method waits for new ideas. The series derived for the string tension and for different glueball masses are "hand-made" and rather short. The analytic continuation towards the continuum limit is ambiguous [20]. In the case of the string tension the probable presence of the roughening transition [21] (where the flux tube begins to fluctuate like a string in the continuum theory) constitutes an extra problem for this continuation.

2.2 Monte Carlo simulation

At present, the Monte Carlo simulation [22] is the most powerful technique for the quantitative study of lattice QCD [23].

Our problem is to calculate expectation values like

$$\langle A \rangle = \frac{\int d\mu_e A(\mu) e^{-S(\mu)}}{\int d\mu_e e^{-S(\mu)}} \quad (2.16)$$

On a finite lattice, this is a well-defined, multidimensional integral. For instance, on a 10^4 lattice there are 4×10^4 links, on every link there is a U_{NP} matrix defined by three integration variables in the case of $SU(2)$. That defines a 120.000 dimensional integral. Monte Carlo simulation is a method to evaluate this huge integral numerically.

In calculating expectation values like that in (2.16), one has to sum over (in the case of a continuous group: one has to integrate over) all the configurations. Random sampling, that is generating configurations randomly, does not work. Due to the Boltzmann factor e^{-S} , the integrand changes very rapidly, and with random sampling most of the time irrelevant configurations would be generated. The relevant configurations should be generated with higher probability: importance sampling.

Let us choose a configuration $\{U\}$ with a probability $P\{U\}$. After generating M configurations we have

$$\langle A \rangle \approx \bar{A} = \frac{\sum_{\nu=1}^M A(\{U_\nu\}) P^{-1}(\{U_\nu\}) e^{-S(\{U_\nu\})}}{\sum_{\nu=1}^M P^{-1}(\{U_\nu\}) e^{-S(\{U_\nu\})}} \quad (2.17)$$

A natural possibility is to generate a configuration according to the corresponding Boltzmann factor:

$$P_{eq}(\{u\}) \sim e^{-S(\{u\})} \quad (2.18)$$

In this case (2.17) gives:

$$\bar{A} = \frac{1}{M} \sum_{v=1}^M A(\{u\}_v) \quad (2.19)$$

Of course, the correctly normalized P_{eq} is not known a priori. A Markov process is constructed such that after many iterations the probability of creating a configuration is just the equilibrium probability P_{eq} .

This Markov process is defined by giving the transition probability $W(\{u\}_v \rightarrow \{u\}_{v'})$ from one configuration to another. W is required to satisfy

$$\sum_{v'} W(\{u\}_v \rightarrow \{u\}_{v'}) = 1, \quad a)$$

- b) any finite-action configuration should be reachable (at least after a number of steps).
c) detailed balance condition

$$P_{eq}(\{u\}) W(\{u\}_v \rightarrow \{u\}_{v'}) = P_{eq}(\{u\}_{v'}) W(\{u\}_{v'} \rightarrow \{u\}_v) \quad (2.20)$$

It is easy to show (at least on a non-rigorous level) that this transition probability drives the system towards equilibrium.

Even within the requirements above, there is a large freedom in choosing the transition probabilities. In most of the lattice gauge theory calculations, the Metropolis and the Heat Bath methods were used.

Metropolis method.

First, an initial configuration is chosen (for instance, a completely ordered or disordered state). Then local changes are performed: a link is chosen (randomly or regularly), and the gauge matrix U_μ of this link is replaced by U'_μ by some random process. After this replacement, the change of the action δS is calculated. If δS is negative, the change is accepted, if δS is positive, the change is accepted with the conditional probability $e^{-\delta S}$. This occasional acceptance simulates the thermal fluctuations. Without it, the system would be driven towards a dead, lowest-action state.

Heat bath method.

Successively a local heat bath is touched to the links which are updated. That is, a new U'_μ is chosen (all the other U 's are fixed) with a probability which is proportional to the Boltzmann factor. Of all the possible Monte Carlo algorithms which vary only a single link at a time, this method leads to equilibrium in the least number of iterations.

In both methods, the subsequent configurations are different along a single link only, therefore these configurations are highly correlated. Usually, A is calculated on configurations which are separated by many updates. Additionally, at the beginning, the system is not in equilibrium, therefore a certain number of configurations should not be used in calculating averages.

If the configurations over which A is calculated are independent, the statistical error is given by

$$\delta A = \left\{ \frac{1}{M(M-1)} \sum_{v=1}^M (A(\{u\}_v) - \bar{A})^2 \right\}^{1/2} \quad (2.21)$$

The statistical error decreases like $M^{-1/2}$ for large M , which is very slow - a notorious point of Monte Carlo studies.

The configurations are generated according to the Boltzmann factor e^{-S} . This probability distribution is independent of the operator A considered. For instance, these configurations do not "know" whether and where quark sources will be inserted later. How will a bag or flux tube be formed then? By random fluctuations, there will be configurations in the generated sequence which are relevant in the wave-functional of a $q\bar{q}$ source. Those are enhanced, the other ones

are suppressed in the averaging process. It is clear, however, that we are looking for a needle in a haystack if the Wilson loop is large. It would be nice to advise a generating procedure where the probability distribution would know about the presence of sources in an essential way[24].

2.3 Mean field techniques

The mean field method[1,25-27] is an old, well-known procedure in statistical physics. Its limitations are also well known. For spin models the long-distance properties (for instance, critical indices) are predicted incorrectly by mean field theory in low dimensions. It is a quick, easy method to obtain some information on the system and in many cases it works surprisingly well in predicting the existence and the place of phase transitions.

In lattice gauge theories there is a recent, renewed interest in mean field techniques[26,27]. The original "one-shot" method has been extended to a systematic expansion, and the problem related to gauge invariance has been understood[26].

Consider first the naïve procedure which later became the first approximation of a systematic expansion. In the mean field analysis, one concentrates on a single link in the partition function $\prod_{n,\mu} U_{n\mu}$, and the effect of all the other links is replaced by a mean field value M , which is taken to be proportional to the unit matrix. In this

Figure 13

approximation the action is reduced to (Fig. 13)

$$-S(u) = \frac{\beta}{N} \sum_{\mu} \frac{1}{2} (\text{Tr } U_{\mu} + cc) \rightarrow 2(d-1) \beta M^3 \frac{1}{N} \frac{1}{2} (\text{Tr } U_{\eta\mu} + cc) + \quad (2.22)$$

+ (terms, independent of $U_{\eta\mu}$),
where $\beta = 2N/g^2$ for $SU(N)$, d is the space-time dimension.
Consistency requires

$$\frac{1}{N} \langle \frac{1}{2} (\text{Tr } U_{\eta\mu} + cc) \rangle = M, \quad (2.23)$$

leading to the equation

$$M = \underbrace{\frac{\partial}{\partial \alpha} \ln \left(\int dU e^{\alpha \frac{\text{Tr } U + cc}{2N}} \right)}_{\mathcal{F}(\alpha)} \Big|_{\alpha = 2(d-1) \beta M^3} \quad (2.24)$$

The function $\mathcal{F}$ is 0 and 1 for $\alpha = 0$ and ∞ respectively, and it is changing smoothly between these values. For small β , (2.24) has only the trivial $M = 0$ solution (Fig. 14a). There is a certain $\beta = \beta_0$, however, above which a non-trivial, $M \neq 0$ solution exists also

Figure 14a

Figure 14b

(Fig. 14b). The transition point should be determined by comparing the free energies. At large β , the free energy with $M \neq 0$ is larger. The crossing with the $M = 0$ free energy happens at $\beta_c > \beta_0$ in general.

For simple gauge models (like $Z_2, Z_3, \dots$) which have a deconfining phase transition as β is increased, β_c agrees quite well with the known phase transition points. For $SU(2)$ and $SU(3)$, where no deconfining

phase transition is expected to occur, it is hoped (and there are indications for that) that the phase transition, predicted by the lowest approximation, disappears as corrections are included systematically. One might note that for $SU(2)$ β_c is close to the point where an abrupt change towards the continuum behaviour is observed in Monte Carlo studies.

However, there is an apparently basic theoretical objection against the mean field method. According to a rigorous theorem, local gauge symmetry cannot be broken spontaneously, $\langle U_{n\mu} \rangle = 0$ is true for any coupling [28]. In a system with global symmetry there is a possibility for degenerated ground states separated by an infinite potential barrier. In the case of a local gauge symmetry there are degenerate states close to each other in phase space. By applying a gauge transformation which is different from unity only at a single point of the lattice, a new configuration is created. This configuration has the same action as the original one, they are identical everywhere except on 2d links, and they are separated by a finite - actually zero - potential barrier. The vacuum functional is a superposition of these states. On the links effected, the gauge matrix can be changed arbitrarily by the gauge transformation and $\langle U_{n\mu} \rangle = 0$ occurs unavoidably.

The starting point of the mean field procedure is to assume $\langle U_{n\mu} \rangle = M$ which contradicts Elitzur's theorem for $M \neq 0$. A nice, recent result is the observation that the mean field method can be reconciled with Elitzur's theorem the same way as the soliton quantization steps restore translation invariance [26].

In quantizing a field theory around a soliton solution the tree graph approximation is given by the classical, localized solution itself, violating translational invariance. In the correct quantization procedure (the usual way is to introduce collective co-ordinates related to the centre-of-mass motion of the soliton) translation invariance is restored already at the one-loop level.

The mean field method can be extended into a systematic saddle point approximation, the old mean field result providing the zeroth order result. The mean field solution $U_{n\mu} = M$ and all its gauge transformed forms are equivalent saddle points, which should be integrated over, restoring gauge invariance.

The possibility of a novel systematic expansion method for lattice QCD is an exciting prospect. Unfortunately it turned out that in the strong coupling region this expansion essentially coincides with the usual strong coupling expansion, while in the weak coupling régime it is a rearranged weak coupling expansion.

Although the theoretical status of the mean field method improved significantly, it has not become yet a new quantitative method in lattice QCD.

2.4 Variational methods combined with Monte Carlo simulations

The variational method is very effective in quantum mechanics in determining low-lying energy levels. In field theory, even staying within the simplest type of ansätze, the calculation of the necessary matrix elements is very difficult. It has been suggested by Wilson [29] to use Monte Carlo simulation for this problem. This method has been extensively applied in glueball mass calculations [30].

The glueballs are hypothetical particles formed solely from gluons, without valence quarks. They are colourless and flavourless objects. Their existence is predicted (almost) unavoidably by QCD. The glueball with lowest mass is called the mass gap of the theory.

In searching for the lowest mass with given quantum numbers the usual procedure is to create an excitation with the given quantum numbers and to follow its propagation over large distances. Let be θ a local operator having the required quantum numbers. Then

$$\langle \theta(q,t) \theta(q,0) \rangle \sim_{t \rightarrow \infty} e^{-mt}, \quad (2.25)$$

where m is the lowest mass in the channel. The operator θ is arbitrary to a large extent. The only a priori requirement is that $\theta|0\rangle$ should have a non-zero component in the exact wave function of the lowest-lying state. Of course, by hitting the vacuum with θ , not only the lowest-lying state, but excited single particle states, possible two, three, ... particle states are also created. However, they have a faster exponential decay and at large enough t the propagation of the relevant state is observed.

Assume the lowest glueball is O^{++} . Then a natural possibility is to take

$$\theta(t) = \sum_{\substack{\text{orientations} \\ \text{and } \underline{x}}} \left(\square(\underline{x}, t) + \square(\underline{x}, t) \right), \quad (2.26)$$

where $\square(\underline{x}, t)$ is the product of U matrices around the plaquette at $(\underline{x}, t)$. Summing over $\underline{x}$ assures that the excitations created by $\theta(t)$ are in the rest frame ($\underline{p} = 0$).

In early glueball calculations the correlation

$$\langle \theta(t) \theta(0) \rangle - \langle \theta(t) \rangle \langle \theta(0) \rangle \quad (2.27)$$

was measured by Monte Carlo simulation. The main problem is, however, that the signal decreases rapidly with increasing t , and for $t > 3$ completely disappears into the noise (within acceptable computer time limits).

Combining this method with variational considerations improved the situation. The operator θ is taken to be a combination of different loop products with unknown coefficients

$$\theta = c_1 \sum (\square + \square) + c_2 \sum (\square + \square) + \dots \quad (2.28)$$

The unknown variational parameters are determined by the usual minimization procedure:

$$\begin{aligned} & \min \left[-\frac{1}{t} \ln \frac{\langle 0 | \theta(0) e^{-Ht} \theta(0) | 0 \rangle}{\langle 0 | \theta(0) \theta(0) | 0 \rangle} \right] = \\ & = \min \left[-\frac{1}{t} \ln \frac{\langle 0 | \theta(0) \theta(t) | 0 \rangle}{\langle 0 | \theta(0) \theta(0) | 0 \rangle} \right] \end{aligned} \quad (2.29)$$

If the basis is large enough in (2.28), then even $t = 1$ will result in a good value for the mass gap. In practice, a few operators are taken and $t = 0, 1, 2$ is measured by Monte Carlo simulation.

2.5 Monte Carlo renormalization group techniques (MCRG)

MCRG is a well known, powerful method for spin systems in statistical physics. The procedure was suggested by S.K. Ma[31] and it was significantly developed later by R.H. Swendsen[32]. The method has been used by Wilson in an $SU(2)$ lattice gauge theory study[33]. Although the method is potentially powerful and theoretically beautiful, the expected quantitative results have not shown up yet (in gauge theories). Nevertheless, the underlying ideas govern almost everything we know of the continuum limit of lattice theories.

In setting up a RG transformation, first a block transformation must be given. This is the prescription, how the new gauge variable is obtained as some kind of block average of the old variables. This block transformation should preserve gauge invariance in order to have a gauge invariant transformed action. A possibility is illustrated in Fig. 15[34]. The crosses denote the points of the lattice after the RG transformation. The new gauge field variable associated with the link A-B is taken to be equal with one of the three (in d dimensions $2d-1$) matrix products $U_1 U_2 U_3 U_4$, $U_5 U_6$, $U_7 U_8 U_9 U_{10}$. For $U(1)$ or $Z(N)$ theories, this might be the one whose phase is closest to the average of the three $(2d-1)$ phases. [For a $Z(2)$ gauge theory this is just the majority rule.] A similar rule can be applied for $SU(N)$ using the eigenvalues of the corresponding matrices, or some analogous, less time consuming procedure the reader might invent. This block transformation is gauge invariant: a gauge transformation on the original lattice with gauge matrices V_A and V_B on the sites A and B, respectively, will transform the new link variable U_{AB} into $V_A U_{AB} V_B^*$ as it should[34].

As we discussed in Section 1.7, this RG transformation is expected to have a fixed point in the $g^2 = 0$ hyperplane of the multiparameter space: ($g^2=0, c^*_2, c^*_3, \dots$). There is a renormalized trajectory starting out from this point and attracting the nearby trajectories (Fig. 7).

Let us work with the standard action. Take a small g^2_1 and find the corresponding g^2_2 in such a way that after performing n and $(n-1)$ RG steps from g^2_1 and g^2_2 respectively, the resulting actions are approximately the same if n larger than a certain $n_{\min}$. As both sequences of actions will follow RT after a few steps, this is possible (Fig. 8).

The change $g^2_1 \rightarrow g^2_2$ corresponds to decreasing the cut-off by a factor of two. Similarly, the dimensionless correlation length is decreased by a factor of two also. By repeating the procedure, a sequence of couplings $g^2_1 \rightarrow g^2_2 \rightarrow g^2_3 \rightarrow \dots \rightarrow g^2_k$ is obtained corresponding to systems with identical long distance behaviour, but reduced correlation length $\xi \sim 1/2 \xi \rightarrow 1/4 \xi \rightarrow \dots \sim 1/2^k \xi$, ($\xi \gg 1$). The system is pushed towards the strong coupling regime, and it is expected that beyond a certain g^2 value no matching will be possible anymore. Let g^2_k be the last point for which matching was possible. The coupling g^2_k is the $\sim$ largest g^2 value where the standard action describes continuum physics.

Until g^2 is small, we are deep in the continuum limit, and the matching connection should be given by the known, universal, lowest order terms of the β function (Eq.1.45). In order to show that the correct continuum theory is recovered, one should confirm this behaviour. The matching sequence in the intermediate coupling region is informative for the full β function. This is not universal, it depends on the action we study.

The system at g^2_k is identical to the system at g^2_1 what concerns their long-distance properties. On the other hand, it is easier to work at $g^2 = g^2_k$, where the correlation length is smaller and the numerical techniques are more powerful. By measuring a mass at g^2_k :

$$a(g^2_k) \cdot m = \alpha$$

we obtain

$$a(g^2_1) \cdot m = 2^{-l} \alpha$$

As g^2_1 is small, the one-loop expression for Λ^{latt} can be used and the connecting factor c in the relation $m = c\Lambda^{\text{latt}}$ can be determined:

$$C = 2^{-l} \alpha e^{\frac{1}{2\beta_0 g_1} (p_0 g_1^2)} \frac{p_1}{2\beta_0}$$

But how to find the sequence $g^2_1 \rightarrow g^2_2 \rightarrow \dots \rightarrow g^2_k$? A possibility would be [31] to generate configurations by MC at g^2_1 , to construct the block configuration at every time and calculate the block correlation functions (block Wilson loop expectation values). Then, by trial and error we search for an action in the multiparameter space which gives (via MC) the same correlation functions (without blocking). The procedure is repeated now starting with this new action, and so on. This way, the sequence of points in the multiparameter space (Fig. 8) are found explicitly.

It was the nice idea of Swendsen that this trial and error search is not really needed [32]. The new, multiparameter action is used anyhow in a MC simulation to generate new configurations. We are not interested in the action itself (it has no universal properties, depends on the block transformation applied) but in the configurations. However, simulating the new action, a certain configuration would occur with the same probability as the probability of having this configuration occurring via the block transformation of the configurations generated by the standard action with $g^2 = g^2_1$!

The simplified procedure is then the following. Configurations are generated on an N^4 lattice by MC, using the standard action with $g^2 = g^2_1$. On every configuration the block variables are constructed [they are defined on an $(N/2)^4$ lattice] and different block correlation functions are calculated. Next, block variables are constructed from the first block variables [they will be defined on an $(N/4)^4$ lattice], and so on. The whole procedure is repeated at $g^2 = g^2_2$ starting on an $(N/2)^4$ lattice. The block correlation functions obtained in the first sequence after the k th step are compared with those obtained in the second sequence after the $(k-1)$ th step. The coupling g^2_2 is tuned until matching is found after a few RG steps. By comparing numbers obtained on lattices of the same size, the matching condition will not be distorted by finite size effects.

In practice, N cannot be very large and if the matching requires many steps, we cannot do it. Therefore, one should start with an action which gives early matching. Of course, the points of the

renormalized trajectory are the "perfect" actions: they give matching after the first step. Unfortunately we do not know, where is the RT. However, by keeping a few operators in the generalized action, Eq.(1.56), the approximate fixed point $c_2^*, c_3^*, \dots$ can be determined by leading order perturbation theory, as it was discussed in Section 1.7. Then we can work with the one-parameter action^[33] (Fig. 16):

$$\frac{1}{q^2} [\theta_1 + c_2^* \theta_2 + c_3^* \theta_3 + \dots] ,$$

which has a better chance to give early matching than the standard action.

Let me remark that MCRG is not the only possible method to drive the system towards smaller correlation lengths. Another possibility is, for instance, to measure the short distance part of the q - q potential.

Figure 15

Figure 16

3 HOW THE LATTICE WORKS: STRING TENSION IN SU(2)

The way the string tension has been measured in the SU(2) lattice gauge theory including the effort to control the systematic errors, to check the restoration of rotation symmetry, universality and so on, shows nicely the lattice at work.

On the other hand, one has to admit that even in this simplified SU(2) problem, the existing calculations can be considered only as exploratory. As it is discussed in Section 2.5, a serious calculation should necessarily contain two steps. First, deep in the continuum limit, asymptotic scaling should be confirmed. For this purpose the string tension is not really appropriate, since it is a long-distance property even in the continuum theory. Deep in the continuum limit measuring the tension would require inaccessible large lattices. Practically, it is better to study short distance physical properties, or using MCRG in this region. In the process the system is pushed towards moderate correlation lengths (see section 2.5), and in the second step the long-distance continuum properties are measured in this region.

The string tension in an SU(2) lattice gauge theory has been measured first by Creutz^[23,35] using Monte Carlo simulation. Assuming that the expectation value of large Wilson loops can be parametrized as

$$\langle W \rangle = e^{-\left(\text{tension} \cdot \text{area} + b \cdot \text{perimeter} + c \right)} , \quad (3.1)$$

the tension can be obtained by taking appropriate ratios of Wilson loop expectation values. A possibility is to take

$$\chi(I) = -\ln \frac{\langle W(I,I) \rangle \langle W(I-1, I-1) \rangle}{\langle W(I, I-1) \rangle \langle W(I-1, I) \rangle} , \quad (3.2)$$

or^[36]

$$K(I) = -\ln \frac{\langle W(I,I) \rangle}{\langle W(I+1, I-1) \rangle} , \quad (3.3)$$

and so on. Here $W(I, I)$ is a Wilson loop of size $I \times I$. It is expected that

$$\chi(I), K(I) \xrightarrow{I \rightarrow \infty} (\text{dimensionless}) \text{ string tension} \quad (3.4)$$

The ratio $\chi(I)$ can be written as

$$\chi(I) = \left(-\ln \frac{\langle W(I, I) \rangle}{\langle W(I-1, I) \rangle} \right) - \left(-\ln \frac{\langle W(I, I-1) \rangle}{\langle W(I-1, I-1) \rangle} \right) \quad (3.5)$$

Using Eq. (1.37) (see also Fig. 1) one obtains

$$-\ln \frac{\langle W(K, I) \rangle}{\langle W(K-1, I) \rangle} \xrightarrow{K \rightarrow \infty} V(I), \quad (3.6)$$

where $V(I)$ is the $q\bar{q}$ static potential at a distance I . Although in Eq. (3.5) the condition $K \gg I$ is not satisfied, one might interpret $\chi(I)$ as a crude approximation to $V(I) - V(I-1)$.

By measuring $\chi(I)$ ($K(I)$) for different values of I at a given $\beta \equiv 4/g^2$, we expect a convergence towards a number as I is increased (Fig. 17). By repeating the calculation for different values of β , the limiting points represent the (dimensionless) tension as the function of the coupling. Deep in the continuum limit, renormalization group arguments dictate a very definite behaviour (asymptotic scaling) [Section 1.5, Eq. (1.46)], and if this behaviour is confirmed, the relation between the tension and the QCD scale parameter can be determined.

Figure 17

The data points obtained on a 10^4 lattice by Creutz are given in

Figure 18

Fig. 18. The statistical errors are suppressed in this Figure. Observe that $\chi(3)$ measured at $\beta = 2.3$ and 2.5 is consistent with the renormalization group behaviour (dotted line). On the other hand, there is not much sign of a convergence towards a limiting point as I is increased - clearly there is a need to measure χ for larger separations of the quark sources (larger Wilson loops). At a fixed value of β ,

$\chi(I)$ is a monotonously decreasing function of I , which is expected from its approximate relation to the derivative of the potential^[36]. If $\chi(3)$ can be taken as representing well the true limiting points at $\beta = 2.3$ and 2.5, the relation between the tension and the scale parameter is predicted to be [Eq.(1.47)]

$$\Lambda_{\text{latt}}^{1/2} = (0.013 \pm 0.002) (\text{tension})^{1/2} \quad (3.7)$$

Bhanot and Rebbi^[37] increased the lattice size to 16^4 and studied the potential up to five lattice distances. Their result is given in

Figure 19

Fig. 19. There is a significant improvement^{*)}. There is a sign of convergence at $\beta = 2.25$ and 2.5.

The limiting points are consistent with the renormalization group behaviour and the numerical result agrees with (3.7) within the statistical errors.

*) The case is less convincing if the statistical errors are also included. The errors are quite large for $I = 4$ and 5.

These results tell us that beyond $\beta \approx 2.25$ the continuum behaviour sets in and that the potential is essentially linear after a few lattice distances in this coupling constant region.

However, we cannot think in terms of the value of the bare charge and in terms of lattice units. We can think in terms of Fermi's and MeV's. There are certain questions which we should always ask for and answer in interpreting a lattice result:

- A) Resolution? Take the point $\beta = 2.5$. At this coupling $a^2\sigma \approx 0.1$. Using the experimental value of the tension $\sqrt{\sigma} \approx 400$ MeV, one obtains

$$a_1|_{\beta=2.5} \approx 0.16 \text{ fermi} \quad (3.8)$$

The characteristic distances in hadron physics are of the order of ~ 1 fm. A resolution of 0.16 fm is not extremely good, but acceptable. It is not against our intuition that the continuum behaviour already sets in at this resolution.

- B) Separation? A separation of $4a$, $5a$ corresponds to ~ 0.6 - 0.8 fm. This is a rather small distance, but either by taking the usual Coulomb + linear form of the potential, or by taking the potential derived in the bag model for heavy quarks, we obtain a precocious linear behaviour. Therefore, the lattice prediction saying that at 0.6 - 0.8 fm the potential is essentially linear, is not against our expectation.

- C) $\Lambda_{\text{latt}} = (5.2 \pm 1.0)$ MeV? Equation (3.5) predicts this small value for Λ_{latt} . We do not know very well Λ from deep inelastic experiments, but there is an order of magnitude problem here. However, the lattice is a strange regularization and the value of Λ is scheme-dependent (Section 1.8). Actually, $\Lambda_{\text{latt}} = 5.2$ MeV corresponds to $\Lambda_{\text{MOM}} = 280$ MeV^[16], which is quite reasonable. Of course, we do not know Λ_{MOM} precisely, especially not in an SU(2) world without quarks. But this is an acceptable number.

- D) Rotation invariance (Lorentz invariance)? Rotation invariance is expected to be restored only in the continuum limit. The string tension was measured along the co-ordinate axes. We would like to see that the results do not depend on that.

In Fig. 20a, the equipotential lines of two heavy quarks at $\beta = 2.0$ are given, as it was measured by Lang and Rebbi [38]. According to the tension calculations, $\beta = 2.0$ is not in the continuum limit yet, and really, the equipotential lines clearly remember the lattice structure.

The coupling $\beta = 2.25$ is expected to be in the continuum régime. Figure 20b gives the equipotential lines at this coupling.

Figure 20a

Figure 20b

The restoration of rotation invariance is quite reassuring.

E) Universality? Let us postpone the discussion on this point to Section 5.

The story of the string tension in the SU(2) gauge theory illustrates nicely the strength of the lattice method. Hopefully, we should not wait long for a serious study of this relatively simple problem in the sense of the discussion at the beginning of this section.

4 SU(3) GAUGE THEORY AT FINITE PHYSICAL TEMPERATURE

In answering the question whether QCD describes correctly the hadronic world, the spectroscopical investigations (discussed in section 7) are very important. There is a large amount of quantitative experimental information available which can be confronted with the predicted numbers. Of course, it is true that these investigations do not influence present or future experiments in high energy physics.

On the other hand, the study of QCD under extreme conditions [39] (high temperature or baryon density) might have an important effect on future heavy ion accelerators, or laboratory experiments. Additionally, measuring different thermodynamical properties (at least in the gauge sector) is somewhat easier than to measure spectroscopical data - like the tension or the glueball mass. The results on the deconfining critical temperature are presumably the most trustworthy predictions obtained on the lattice until now.

The existence of a deconfining phase transition in a gauge theory (as the physical temperature is increased) is a longstanding conjecture [40]. Its presence has been demonstrated by analytic methods in the strong coupling limit of lattice gauge theories [41]. This proof has been extended to arbitrary fixed g^2 recently [42]. The MC measurements for SU(2) [43,44] and the subsequent SU(3) results [45,46] show a clear sign of a deconfining phase transition and predict the critical temperature in the continuum limit. Additionally, recent Monte Carlo results on SU(3) indicate convincingly that the phase transition in SU(3) is of first order [46] and there are quantitative results for the latent heat also [47]. These are important parameters in judging on the feasibility of producing gluon matter in laboratory experiments.

Of course, these experiments would deal with the complete theory: gauge and quark fields in interaction. While in many spectroscopical problems the effect of (virtual) quarks is believed to be small, in the case of the deconfining phase transition it is expected to be relevant and quantitative. The recent situation is summarized in Section 8.2.

4.1 How to introduce finite temperature in a quantum field theory?

Consider first a quantum mechanical particle, described by the Hamiltonian H . Consider the behaviour of this simple system under the influence of a heat bath of temperature $T \equiv 1/\beta$. The partition function is given by

$$Z = \text{Tr} (e^{-\beta H}) \quad (4.1)$$

The trace can be calculated in any basis, take a co-ordinate basis, for instance:

$$Z = \int dx \langle x | e^{-\beta H} | x \rangle \quad (4.2)$$

This is just the propagation kernel [Eq. (1.1)] continued to the imaginary time $-\beta$. Therefore, in a path integral representation, the partition function is given by

$$Z = \sum_{\text{periodic paths}} e^{-\int_0^\beta dx_4 \mathcal{L}_E(x_4)} \quad (4.3)$$

where the summation is over all paths satisfying $x(\beta) = x(0)$.

By generalization, it is easy to modify the Euclidean prescription of zero temperature Yang-Mills theory to include the effect of finite temperatures. The partition function is given as

$$Z = \pi \int_{\mathcal{L}} du_4 e^{-S_E} \quad , \quad -S_E = \frac{1}{g^2} \sum_p (\text{Tr} u_p + cc) \quad , \quad (4.4)$$

with the constraint:

a) the lattice is finite in the fourth direction $N_\beta \times N_s \times N_s \times N_s$, where $(N_s a) \rightarrow \infty$, $(N_\beta a)$ is fixed, $N_\beta a = 1/T$.

b) the gauge field is periodic in the fourth direction.

Having the partition function we can take derivatives with respect to β to obtain the internal energy and other thermodynamical functions. These functions are expected to behave singularly if a phase transition occurs as the temperature is changed.

4.2 The order parameter of the deconfining phase transition. Theoretical arguments

Consider a gauge theory based on the group G . Apart from the usual local gauge symmetry, the model possesses a special global symmetry if the centre of the group G is not trivial. Really, consider an $n_4 = \text{const.}$ hyperplane and multiply all the U matrices associated with the links orthogonal to this hyperplane (i.e., pointing along the temperature direction) by an element of the centre:

$$U_{n_1, n_2, \hat{n}} \rightarrow z U_{n_1, n_2, \hat{n}} \quad , \quad z \in \text{centre of } G, \quad (4.5)$$

$n_4 \text{ is fixed}$

The action, and actually all the usual (topologically trivial) Wilson loops are invariant. The thermal loop, that is the loop running along the

Figure 21

temperature direction and closing upon itself (Fig. 21) is not invariant:

$$L_q \equiv \text{Tr} \left(\prod_{n_4} U_{n_1, n_2, \hat{n}} \right) \rightarrow z L_q \quad (4.6)$$

Therefore the thermal loop expectation value is an order parameter of this symmetry, $\langle L \rangle = 0$ if the symmetry is unbroken.

On the other hand (as it follows from the discussion of Section 1.4), L describes the presence of an isolated quark in the system. Therefore $\langle L \rangle$ is related to the quark free energy F_q :

$$\langle L \rangle = \frac{\int \mathcal{D}U L e^{-S}}{\int \mathcal{D}U e^{-S}} = e^{-\frac{F_q}{T}} \begin{cases} = 0 & \text{in the confining phase} \\ \neq 0 & \text{in the deconfined phase} \end{cases} \quad (4.7)$$

Correspondingly, the spontaneous breakdown of the global symmetry discussed above is just the deconfining phase transition.

It is natural to expect that this phase transition is related to the long-range fluctuations in the order parameter L . One might try - in principle or in a systematic strong coupling expansion [48] - to integrate over the other degrees of freedom and obtain an effective three-dimensional theory for the order parameter L . This effective theory is a three-dimensional spin model with a global symmetry corresponding to the centre of G . Since the variables, which are integrated out, do not show long-range fluctuations at T_c , it is expected that this spin model is described by a Hamiltonian of short range interactions and it belongs to the same universality class as the well-known spin models with the same symmetry [49]. This way the deconfining phase transition in $SU(2)$ and $SU(3)$ gauge theories is related to the phase transition of the three-dimensional Ising and Z_3 Potts models respectively. Therefore, the theoretical expectation is that $SU(2)$ has a second order while $SU(3)$ has a first order deconfining phase transition.

4.3 MC results on the deconfining phase transition in $SU(3)$

Take a lattice of fixed size $N_t \times N_s^3$ ($N_t \ll N_s$). The temperature on this lattice is given by

$$T = \frac{1}{N_t a(q^2)} \quad (4.8)$$

Since N_t is fixed, the temperature can be changed by changing $a(q^2)$ via tuning g^2 . If g^2 is decreased, the temperature is increased and the other way around. If we want to study the model in the continuum limit, the critical g^2 value must lie in the scaling region. This requirement puts a constraint on the minimal temperature size N_t required, since decreasing N_t will increase the critical coupling.

A possible way of identifying the first order phase transition is the following [46]. One performs MC sweeps at a fixed temperature starting from a completely ordered and, separately, from a disordered configuration. After sufficiently many iterations, the averages arising from the two different sequences will converge to one stable equilibrium value (the sequences will forget the starting point), if we are not in the critical region. At the critical temperature, however, the expectation values will attain two different values depending on whether they are calculated in the "hot" or "cold" sequence. In a finite system, there will be occasional phase flips with a small probability if the volume is large.

Figure 22 shows how this works on a 3×8^3 lattice, as it was measured by the Bielefeld group [46]. The critical g^2 value in this system was found at $6/g^2 = 5.55$. Somewhat below and above, the critical temperature, $\langle L \rangle$ ($\approx 3 \cdot \bar{L}$) settles rapidly to its unique, equilibrium value (zero and non-zero respectively). At the critical temperature the two-phase structure is stable over several thousands iterations.

The numerical results are summarized in Section 6.

5 UNIVERSALITY TESTS IN SU(2) LATTICE QCD

As we discussed in Section 1, there is a considerable freedom in defining a lattice action. The requirement of obtaining the continuum action in the classical continuum limit, and the requirement of gauge invariance, translation and 90° rotation invariance, parity and charge conjugation invariance are not very restrictive. The Wilson action in Eq. (1.26) is built up from the simplest gauge invariant products of the $U_{\mu\nu}$ matrices: Tr Up [Eq. (1.21)]. Consider $SU(2)$ gauge theories and define

$$U_p = e^{i\theta_p^a \tau^a} = \cos\theta_p + i\tau^a \left(\frac{\theta_p^a}{\theta_p}\right) \sin\theta_p, \quad (5.1)$$

where $\theta_p^2 = \theta_p^a \theta_p^a$. Tr Up , and therefore θ_p is gauge invariant. [Actually θ_p^a is proportional to the lattice field strength tensor defined in Eqs. (1.24) and (1.25). In terms of θ_p the Wilson action can be written as:

$$S_W = -\frac{4}{g_w^2} \sum_p \cos\theta_p. \quad (5.2)$$

There are many other functions of θ_p satisfying the basic requirements. The Manton action is defined as [50]

$$S_M = \frac{2}{g_m^2} \sum_p \bar{\theta}_p^2, \quad \bar{\theta}_p = \theta_p \pmod{2\pi}, \quad (5.3)$$

while the Villain action has the form [51]:

$$e^{-S_V} = \prod_p \left\{ \sum_{\ell=0}^{\infty} (\ell+1) \frac{\sin(\ell+1)\theta_p}{\sin\theta_p} e^{-\ell(\ell+2) \frac{g_v^2}{8}} \right\}. \quad (5.4)$$

(The Villain action corresponds to a simple, special choice of β_r in the character expansion $e^{f(\text{TrUp})} = \sum_r \beta_r X_r$). Additionally, other loop products, higher representations, different lattice structures, and so on, can be considered.

Figure 22

The independence of the physical predictions on the specific action chosen is a basic requirement for this approach to make sense. It is expected (and can be checked explicitly in perturbation theory) that the different formulations define identical renormalized theories after adjusting a single parameter (the coupling constant) appropriately^{*)}. This expectation is based on the accumulated knowledge and experience concerning universality in statistical physics and discussed briefly in Section 1.7.

Consider the actions defined in Eqs. (5.2)-(5.4). In renormalized perturbation theory the three models are identical in what concerns the physical predictions. The only difference, predicted by perturbation theory, is in their respective scale parameter. One-loop calculation gives the following result (exact in the $g \rightarrow 0$ continuum limit)

$$\frac{\Lambda_M^{\text{latt}}}{\Lambda_W^{\text{latt}}} = 3.07, \quad (5.5)$$

$$\frac{\Lambda_M^{\text{latt}}}{\Lambda_V^{\text{latt}}} = 2.45 \quad (5.6)$$

On the other hand, Monte Carlo studies give the following non-perturbative predictions^[54]

$$\begin{aligned} \frac{1}{2} (\text{string tension}) &= \frac{\Lambda_W^{\text{latt}}}{\Lambda_V^{\text{latt}}} = \frac{(83 \pm 14) \Lambda_W^{\text{latt}}}{(46.2 \pm 0.5) \Lambda_V^{\text{latt}}} = \frac{\Lambda_W^{\text{latt}}}{\Lambda_V^{\text{latt}}} \quad \begin{array}{l} \text{Wilson action} \\ \text{Manton action} \\ \text{Villain action} \end{array} \end{aligned} \quad (5.7)$$

*) For Green's functions, which are not renormalization group invariant, wave function renormalization is needed also. It is assumed that the symmetries of the specific lattice formulation reduce the number of dimension-four, gauge invariant operator combinations to 1. If this condition is not satisfied (like in the Hamiltonian formulation^[52], where the absence of the symmetry of 90° rotations between space and time allows two combinations: $F_{0i}^a F_{0i}^a$ and $F_{ik}^a F_{ik}^a$) then a careful tuning process is required in a multidimensional parameter space in order to get a correct Lorentz and gauge invariant continuum theory^[53]

where the MC data were interpreted assuming asymptotic scaling (see Section 1.6). Compare the theoretical prediction with these results. The tension should be independent of the action giving

	TH	"EXP"
$\frac{\Lambda_M^{\text{latt}}}{\Lambda_W^{\text{latt}}}$	3.07	5.14 ± 0.87
$\frac{\Lambda_M^{\text{latt}}}{\Lambda_V^{\text{latt}}}$	2.45	2.99 ± 0.19

(5.8)

There is some discrepancy beyond the statistical errors. A possible source of the problem is that the MC data do not come from the asymptotic scaling region^{*)}

Until we wait for a full, two-step calculation (Section 2.5), there is another possibility: measure a different physical quantity, T_c for instance, and form the ratio:

$$\frac{(\text{tension})^{1/2}}{T_c} = \text{universal} \quad (5.9)$$

From this ratio the unknown corrections coming from the higher order terms of the β functions cancel. The Monte Carlo results for T_c are^{*)}[55]

$$T_c = \frac{\Lambda_W^{\text{latt}}}{\Lambda_M^{\text{latt}}} = \frac{42.8 \Lambda_W^{\text{latt}}}{10.5 \Lambda_M^{\text{latt}}} = \frac{27.3 \Lambda_V^{\text{latt}}}{\Lambda_M^{\text{latt}}} \quad \begin{array}{l} \text{Wilson action} \\ \text{Manton action} \\ \text{Villain action} \end{array} \quad (5.10)$$

*) There exists the possibility also that the statistical and systematic errors are underestimated in the Monte Carlo calculations.
*) Unfortunately, no errors are quoted by the authors.

Equations (5.7) and (5.10) give the following predictions for the ratio Eq. (5.9):

$$\frac{(tension)^{1/2}}{T_c} =$$

Wilson action

Manton action

Villain action,

1.94 ± 0.33

1.54 ± 0.05

1.38 ± 0.10

(5.11)

which is consistent with universality within the statistical errors.

6 NUMERICAL RESULTS IN SU(3) PURE GAUGE THEORY

The number of papers containing numerical predictions on SU(3) gauge theory is large and steadily increasing. However, the quality of the results is rather low and the works can be considered as exploratory only. The reason is not that the available non-perturbative techniques or computers do not allow more serious investigations (in the sense of the discussions of Sections 2.5 and 3). The main reason is that the resources available in this field are meagre and their distribution is badly organized.

The results are expressed usually in the form of Eq. (1.47), where Λ_{latt} is identified with the explicit lowest order expression of Eq. (1.46), without the higher order corrections. This procedure assumes that the numerical study is performed in the asymptotic scaling region (Section 1.6). In most (or all?) of the cases, this assumption is not valid, and it results in observing scaling violations, i.e., the connecting factors c_i in Eq. (1.47) change as the coupling is changed. Often it is not clear whether this reflects the lack of continuum behaviour or the presence of higher order terms in the β function (corresponding to Region II in Fig. 4).

String Tension

The new results obtained on larger lattices and with better statistics differ significantly from those of the earlier measurements. By studying X ratios (Section 3), the result is presumably an upper bound for $(tension)^{1/2}/\Lambda_{latt}$, while the results obtained from thermal loop corrections estimate the string tension at a finite temperature. The present situation is summarized in Table 2.

References	$(tension)^{1/2}/\Lambda_{latt}$	Method	$\beta \equiv 6/g^2$	Lattice size
R.W. Ardill et al. ⁵⁶⁾	170 ± 30	X ratios	5.6 - 6.3	8 ⁴
F. Gutbrod et al. ⁵⁷⁾	≤ 145 ± 10	X ratios	5.7	16 ⁴
G. Parisi et al. ⁵⁸⁾	85 ± 5	Thermal loop corr.	6.0	10 ³ × 20
J.D. Stack ⁵⁹⁾	91 ± 25	Elongated Wilson loops	6.0 - 7.0	8 ³ × 12

Table 2

Mass Gap

The results refer to the coupling constant region $\beta \in (5.0-5.9)$ which is rather close to the strong coupling region. In Refs. 60 and 61, the Monte Carlo variational method was used (Section 2.4), while a special source method was applied in Ref. 62. The lattice size is rather small in these calculations, typically $4^3 \cdot 8$. The results are consistent with [60]

$$m(0^{++}) = (280 \pm 50) \Lambda_w^{\text{latt}}$$

Critical Temperature

The presence of a first-order phase transition is clearly identified on lattices with $N_t = 2, 3$ and 4, giving [46]

$$T_c = \begin{matrix} (78 \pm 1) \Lambda_w^{\text{latt}} & , & \beta = 5.11 \\ (86 \pm 1) \Lambda_w^{\text{latt}} & , & \beta = 5.55 \\ (76 \pm 1) \Lambda_w^{\text{latt}} & , & \beta = 5.70 \end{matrix}$$

7 FERMIONS ON THE LATTICE. GENERAL INTRODUCTION

By introducing dynamical quarks, the number of questions for which an answer might be attempted is significantly increased. The basic question is whether QCD is the theory of strong interactions. Is QCD capable of describing the large wealth of spectroscopical data?

During the last year, exciting results were published on hadron spectroscopy [63]. The merit of these early results is rather controversial [64]. One of the main problems is the very small box size in these calculations. Similarly, the low order hopping parameter expansions could not possibly describe physics correctly in the relevant coupling constant region.

In the following we shall discuss a few introductory topics only. Concerning the new results on hadron spectroscopy on significantly enlarged lattices, the reader is referred to the literature [65], as for the very interesting results concerning the nature of chiral symmetry breaking in QCD [66].

7.1 Defining fermions on the lattice naïvely, the doubling problem

Let us try to put fermions on the lattice following the usual recipe. In the continuum action of a free Dirac particle, the derivatives are replaced by differences giving the lattice action:

$$S = a^4 \sum_n \left\{ \sum_\mu \frac{1}{2a} \left[\bar{\psi}(n) \gamma_\mu \psi(n+\hat{\mu}) - \bar{\psi}(n+\hat{\mu}) \gamma_\mu \psi(n) \right] + m \bar{\psi}(n) \psi(n) \right\} \quad (7.1)$$

By inserting the gauge matrices in a gauge invariant way and by adding the gauge field action, a candidate prescription of QED on the lattice is obtained

$$S = a^4 \sum_n \left\{ \sum_\mu \frac{1}{2a} \left[\bar{\psi}(n) \gamma_\mu U_{n\mu} \psi(n+\hat{\mu}) - \bar{\psi}(n+\hat{\mu}) \gamma_\mu U_{n\mu}^\dagger \psi(n) \right] + m \bar{\psi}(n) \psi(n) \right\} + S_{gf} \quad (7.2)$$

where $U_{n\mu} = e^{iagA_{n\mu}} \in U(1)$. For $m = 0$ the model is invariant under the transformations

and

$$\begin{aligned} \psi(n) &\rightarrow e^{i\alpha} \psi(n), & \bar{\psi}(n) &\rightarrow \bar{\psi}(n) e^{-i\alpha} \\ \psi(n) &\rightarrow e^{i\alpha \gamma_5} \psi(n), & \bar{\psi}(n) &\rightarrow \bar{\psi}(n) e^{i\alpha \gamma_5} \end{aligned} \quad (7.3)$$

The symmetry group is $U(1)_{\text{vector}} \otimes U(1)_{\text{axial}}$. This symmetry is exact for any value of the lattice constant, therefore it is there in the $a \rightarrow 0$ continuum limit also.

However, there is a general theorem due to Adler[67] claiming that although the classical theory is $U(1) \times U(1)$ invariant, there is no regularization which could respect both of these symmetries. In the quantum theory the chiral symmetry is necessarily explicitly broken. The axial vector current is not conserved[68], its divergence receives a non-zero contribution from the triangle graph

The above construction seemingly contradicts this theorem. The resolution of this paradox is the following. Though we wanted to describe the interaction of a single fermion with the electromagnetic field, actually our action describes 16 identical fermion species. Running around the triangle graph all of these fermions gives a contribution to the axial anomaly, but - as it was shown by Karsten and Smit[69] - their contribution alternates in sign and adds up to zero.

It is easy to see that this action describes 16 fermion species. The free fermion part of the action

$$\sim \bar{\psi}(n) \gamma_\mu \psi(n+\hat{\mu}) - \bar{\psi}(n+\hat{\mu}) \gamma_\mu \psi(n)$$

Gives the following propagator in momentum space:

$$\sim \frac{1}{\sum_\mu \gamma_\mu \sin p_\mu}, \quad p_\mu \in (-\pi, \pi) \quad (7.4)$$

This propagator has 16 poles at the points $p_\mu = (0000), (\pi, 0, 0, 0), \dots, (\pi, \pi, \pi, \pi)$.

Chiral symmetry implies species doubling on the lattice. If we want to describe a single fermion, the $U(1)$ chiral symmetry must be explicitly broken. We should accept it, it cannot otherwise be due to Adler's theorem.

Similarly, for the general case with n_f massless fermions, we accept that the flavour singlet $U(1)$ axial symmetry is explicitly broken, since it is a phenomenon which is general, independent of the lattice. However, we would like to keep the $SU(n_f)$ axial symmetry. That is the point where the solutions suggested until now are not satisfactory.

7.2 Wilson's method of removing the degeneracy

In order to avoid species doubling, Wilson suggested adding a new term to the action[70]. This term is chosen in such a way that:

-it gives large $(-1/a)$ masses to the unwanted 15 fermion species. They disappear from the theory in the continuum limit;

-it goes to zero in the formal $a \rightarrow 0$ limit, therefore we expect that it will not affect the behaviour of the remaining single fermion at the end.

The action has the following form:

$$\begin{aligned} -S = \sum_n \left\{ -\bar{\psi}_\alpha^{a,i}(n) \gamma_\alpha^{a,i} \psi_\alpha^{a,i}(n) + \sum_\mu K_i \bar{\psi}_\alpha^{a,i}(n) (1 - \gamma_\mu)_{\alpha\beta} U_{\eta\mu}^{ab} \psi_\beta^{b,i}(n+\hat{\mu}) + \right. \\ \left. + \sum_\mu K_i \bar{\psi}_\alpha^{a,i}(n+\hat{\mu}) (1 + \gamma_\mu)_{\alpha\beta} U_{\eta\mu}^{+ab} \psi_\beta^{b,i}(n) \right\} + \frac{1}{g^2} \sum_p (\text{Tr } U_p + \text{cc.}), \quad (7.5) \end{aligned}$$

where a, b, α, β and i are colour, Dirac and flavour indices, respectively.

By finding the coefficient of the quadratic term $\bar{\psi}(n)\psi(n)$, K_i can be related to the bare (classical) quark mass m_i . In the limit $(m_i a) \rightarrow 0$ one obtains:

$$K_i = \frac{1}{8 + 2m_i a}, \quad i = u, d, s, c, \dots \quad (7.6)$$

There is no species doubling in this prescription. Unfortunately, the new terms explicitly break chiral symmetry. Before adding these terms, the classical symmetry is (for massless quarks)

$$SU(n_f)_{\text{vector}} \times SU(n_f)_{\text{axial}} \times U(1)_{\text{vector}} \times U(1)_{\text{axial}},$$

which is reduced by the new terms to

$$SU(n_f)_{\text{vector}} \times \cancel{SU(n_f)_{\text{axial}}} \times U(1)_{\text{vector}} \times \cancel{U(1)_{\text{axial}}}.$$

This is the symmetry for any finite lattice distance a . We can only hope that the $SU(n_f)_{\text{axial}}$ symmetry will be recovered in the continuum limit (and it will be broken spontaneously).

The amplitude of moving a quark by one lattice unit is proportional to K_1 . Hence the name hopping parameter. If $g = 0$, $K = 1/8$ defines a massless fermion. For $g \neq 0$, there are mass corrections (there is no chiral symmetry which would prevent the occurrence of mass counter terms) and the value of K giving a massless fermion is not $1/8$, but receives perturbative and non-perturbative corrections. K_1 should be renormalized. $K_1 = K_1(g^2)$ is not known a priori.

7.3 Technical problems and methods

In a path integral formulation the fermion fields are represented by anticommuting c numbers, by Grassmann variables. There is no effective way of representing them on a computer. In every method the first step is to integrate over the fermion fields.

The action is quadratic in the fermion fields. In a concise notation it has the form:

$$S = \sum_{ij} \bar{\psi}_i \Delta_{ij}(u) \psi_j + S_{\text{gf}}, \quad (7.7)$$

where i, j represent all kind of indices. By integrating over the fermion variables in the vacuum functional or in any expectation value of fermion fields one obtains:

$$\int D\psi D\bar{\psi} e^{-S} = \det(\Delta(u)) e^{-S_{\text{gf}}},$$

$$\int D\psi D\bar{\psi} \bar{\psi}_i \psi_j e^{-S} = \Delta_{ji}^{-1}(u) \det(\Delta(u)) e^{-S_{\text{gf}}}, \quad (7.8)$$

The second example describes the propagation of a quark in the background field U . This background field is generated with a probability distribution governed by $S_{\text{eff}} = S_{\text{gf}} - \text{Tr}(\ln \Delta(U))$. The second term in S_{eff} represents the effect of virtual quark loops.

Until now, most of the results were obtained under the approximation of neglecting the virtual quarks [71]. The valence quarks are treated dynamically. There are qualitative arguments (large N limit, Zweig rule,...) supporting this approximation. Including the virtual quark contribution would require an updating process in Monte Carlo with S_{eff} - a non-local action.

The hadron propagators are constructed from the quark propagators. According to (7.8) this requires the inversion of a very large matrix $\Delta_{ij}(U)$. There are different suggestions for this problem.

A) Direct numerical methods

Δ_{ij} depends on the background gauge field and contains the hopping parameter K as an external parameter. One might try to invert this matrix numerically for a given value of K . In spectrum calculations the Gauss-Seidel iteration method was used but there are other methods suggested.

The matrix $\Delta(U)$ can be written as:

$$\Delta(u) = 1 - KB(u) \quad (7.9)$$

The equation

$$\chi = (1 - KB(u))^{-1} \phi \quad (7.10)$$

can be rearranged as

$$\chi = KB(u)\chi + \phi, \quad (7.11)$$

and this form is ready for iteration.

B) Hopping parameter expansion

The hopping parameter expansion [70,72] is analogous to the high temperature expansion in statistical physics in many respects. The amplitude of moving a quark by one lattice unit is proportional to the hopping parameter K . An expansion in K is equivalent to an expansion in the length of quark paths in configuration space. In their propagation the quarks are constrained by the maximum order of the expansion; nevertheless they should still gather the essential information on the hadron's structure. This defines the conditions under which the expansion to a given order will be reliable. The size of the hadron in lattice units should not be too large, it has to be comparable to the regions covered by possible quark paths. Or, alternatively, the lattice distance a cannot be too small.

The quark propagator can be written:

$$\Delta_{ij}^{-1}(u) = \sum_{\ell=0}^{\infty} c_{\ell}(u; j, i) K^{\ell} \quad (7.12)$$

In this method the coefficients $c_{\ell}(u; j, i)$ are calculated.

7.4 The procedure of extracting the hadron masses

The procedure is rather different for the methods A) and B). Let us discuss them separately. In both methods gauge field configurations are generated by MC using $S = S_{gf}$. Next:

A) On a given background field $\{U_{n\mu}\}$ the quark propagator $\Delta_{ij}^{-1}(U)$ is calculated by some direct numerical method. Then the meson and baryon propagators are built up from the quark propagator. More precisely, the "time slice propagators" are constructed

$$D(n_i) = \sum_{\vec{n}} D(\vec{n}, n_i) \quad , \quad n_i = 0, \pm 1, \dots \quad (7.13)$$

corresponding to a hadron in the rest frame. The procedure is repeated for different K values on the same background field. After that, the next (independent) background field is taken and so on.

For a given K (quark mass), the hadron propagator $D(n_i)$ is expected to decay exponentially. At shorter distances excited states are also present. Their effect might be taken into account by fitting the data points with a sum of exponentials.

B) On a given background field the (quark and) hadron propagators are expanded in the hopping parameter K . By taking the average over many background fields one obtains

$$D(n) = \sum_{\ell=0}^{\infty} c_{\ell}(n) K^{\ell} \quad (7.14)$$

The propagator is expected to have a pole in momentum space. Close to the pole we have, in the continuum limit,

$$D(p_0 = iE, \vec{p} = \vec{0}; K, g) \sim \frac{1}{E^2 - M^2(K, g)} \quad (7.15)$$

Consider now D as a function of K at a fixed E (and g). Let us define K^* by the equation $E^2 = M^2(K^*, g)$. Then, by using the expansion

$$M^2(K, g) = E^2 + \frac{\partial}{\partial K} M^2(K, g) \Big|_{K=K^*} (K - K^*) + \dots \quad (7.16)$$

one finds that a pole in momentum space implies a pole in K .

Of course, a finite expansion cannot produce a pole. The situation is similar to the high temperature expansion, where we have a power series (in $1/T$), and we have some knowledge about the singularity structure. In our case the nearest singularity is expected to be a pole. Padé approximants might be used to identify this pole.

7.5 How the parameters are fixed

The calculation is done at a given value of g . In order to check the renormalization group behaviour the calculation should be repeated for different g values.

Let us fix g and consider an idealized case. Assume that the lowest (dimensionless) meson mass has been determined as the function of K

Figure 23

both in the pion and the rho channels (Fig. 23). K_u^c is the hopping parameter value where the pion is massless. It corresponds to massless quarks and to spontaneously broken chiral symmetry. Increase the quark mass (decrease K) until the m_ρ/m_π ratio is predicted correctly. This defines the physical value of K (and the quark mass). Then the requirement $m_\pi^{\text{phys}} = 140 \text{ MeV}$ gives the value of the lattice distance in physical units at this coupling.

The lattice unit "a" obtained this way should be consistent with the pure gauge theory results (we neglected the virtual quark loops!)

K_s^{phys} and K_c^{phys} are fixed similarly by using the experimental values of m_K and m_ψ , respectively. From now on everything is predicted!

8 QCD AT FINITE TEMPERATURE AND BARYON DENSITY

In Section 4 a brief discussion on the deconfining phase transition in a pure gauge theory was given. As we already mentioned, the effect of quarks on this phenomenon is expected to be relevant. First we discuss the way a finite chemical potential can be introduced on the lattice [73], then the first results on the fate of the phase transition in the presence of quarks are summarized.

8.1 Chemical potential on the lattice

As we discussed in Section 4.1, there is a straightforward way of introducing finite temperature in lattice calculations. On the other hand, the introduction of finite particle density, or, equivalently, finite chemical potential is less trivial. The naive generalization of the continuum prescription leads to quadratic divergences even for free fermions: in the continuum limit ($a \rightarrow 0$) the energy density ε is proportional to $(\mu/a)^2$ instead of the correct finite result $\varepsilon \sim \mu^4$ (for massless fermions), with μ being the chemical potential.

Let H be the Hamiltonian of the system and N some conserved quantity. In the following, we shall take N to be the fermion number, the generalization is straightforward. The partition function is given by

$$Z = e^{-\beta\Omega} = \text{Tr} e^{-\beta(H - \mu N)} \quad (8.1)$$

Going to functional integral representation, the presence of a non-zero chemical potential implies a new term in the Lagrangian: μ times the fourth component of the current the space integral of which gives the conserved quantity N .

Let us take free fermions in the continuum. The action is

$$\int_0^{\beta} \left(\frac{1}{2} \bar{\psi}(x) \not{\partial}_\mu \psi(x) + m \bar{\psi}(x) \gamma_4 \psi(x) + \mu \bar{\psi}(x) \gamma_4 \psi(x) \right) dx \quad (8.2)$$

with the usual antiperiodic boundary conditions along the fourth direction. The energy density is defined as

$$\varepsilon = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z \Big|_{\mu \text{ fixed}} \quad (8.3)$$

and is given by the following integral when $T \rightarrow 0$:

$$\varepsilon = \frac{1}{(2\pi)^4} \int d^4p \frac{(p_4 + i\mu)^2}{(p_4 + i\mu)^2 + \vec{p}^2 + m^2} \quad (8.4)$$

ε is a $(\text{mass})^4$ quantity. By power counting, the integral is quadratically divergent. There is a trivial part, the vacuum energy at $\mu = 0$. By subtracting this term we get:

$$\varepsilon = -\frac{1}{(2\pi)^4} \int d^4p \left[\frac{p^2 + m^2}{(p_4 + i\mu)^2 + \vec{p}^2 + m^2} - (\mu = 0) \right] \quad (8.5)$$

This integral is finite (not quadratically divergent) as it should. By performing the p_4 integral first, two poles are encountered in the p_4

Figure 24

plane both for $\mu \neq 0$ and $\mu = 0$ (Fig. 24). The difference is zero, except when both poles lie on the lower half plane for $\mu \neq 0$. This gives a θ function:

$$\varepsilon = \frac{1}{(2\pi)^4} \pi \int d^3p \theta(\mu - \sqrt{\vec{p}^2 + m^2}) \sqrt{\vec{p}^2 + m^2} \quad (8.6)$$

or

$$\mathcal{E}_{|m=0} = \frac{1}{4\pi^2} \mu^4,$$

which is the correct result.

The naive generalization of this prescription would lead to the lattice action

$$S = \sum_n \left\{ \frac{1}{2} \sum_\mu \left(\bar{\psi}(n) \delta_\mu \psi(n+\hat{\mu}) - \bar{\psi}(n+\hat{\mu}) \delta_\mu \psi(n) \right) + m a \bar{\psi}(n) \psi(n) + \mu a \bar{\psi}(n) \delta_4 \psi(n) \right\}, \quad (8.7)$$

and yields the following result for the energy density ($T=0$)

$$\mathcal{E} = a^{-4} \left\{ -\frac{1}{4\pi^4} \int_{-\pi}^{\pi} d^4 q \frac{\sum_{j=1}^3 \sin^2 q_j + (ma)^2}{(\sin q_4 - i\mu a)^2 + \sum_{j=1}^3 \sin^2 q_j + (ma)^2} \right\} - a^{-4} \left\{ \mu = 0 \right\} \quad (8.8)$$

What was a potential problem before, is present here explicitly: $\mathcal{E}$ is quadratically divergent

$$\mathcal{E} \sim \left(\frac{1}{a} \right)^2 \mu^2 \quad (8.9)$$

The problem is not related to the species doubling implied by Eq. (8.2). The 16-fold degeneracy can be removed by Wilson's prescription, but the problem discussed above, remains. Replacing the current $\bar{\psi} \gamma_4 \psi_X$ by a point-split form does not help either. In order to obtain a finite result, non-covariant counterterms should be introduced in Eq. (8.2). Although their presence is understandable (there is no Euclidean symmetry for $\mu \neq 0$), it would be an awkward way to proceed.

What is the reason that no similar problems occurred in the continuum formulation? The key to understanding is the observation that in the Euclidean formulation of thermodynamics the chemical potential acts like the fourth component of an imaginary, constant vector potential. In continuum QED, for instance, the chemical potential is introduced exactly as a photon field:

$$e \bar{\psi}(x) \delta_\mu A_\mu(x) \psi(x) - i\mu \bar{\psi}(x) \delta_4 \psi(x) \quad (8.10)$$

For this reason an expansion in powers of μ is equivalent to inserting external, zero momentum photon lines to the amplitude. For instance, the contribution of order μ^2 to the thermodynamic potential $\Omega = -(BV)^{-1} \ln Z$ is proportional to

This graph is proportional to the (potentially quadratically divergent) photon mass renormalization which, however, is zero in a gauge invariant formulation [74] at zero temperature (or, due to the plasmon effect, finite at finite temperature). Similarly, the finiteness of contributions of the order μ^2 to the amplitude $\Gamma^{(n,m)}$ (n and m being the number of external photon and electron lines, respectively) follows from the renormalizability of the amplitude $\Gamma^{(n+l,m)}$. Although l factors of the electromagnetic coupling constant e are replaced by the chemical potential μ and l factors of wave function renormalization are also absent, the missing factors for charge and wave function renormalizations just cancel. This is again the consequence of gauge invariance*.)

In Eq. (8.2) this Abelian gauge invariance is violated: μ does not enter like the fourth component of a gauge field on the lattice. The correct solution is

$$S = a^3 \sum_n \left\{ m a \bar{\psi}(n) \psi(n) + \frac{1}{2} \sum_{j=1}^3 \left(\bar{\psi}(n) \delta_j \psi(n+\hat{j}) - \bar{\psi}(n+\hat{j}) \delta_j \psi(n) \right) + \frac{1}{2} \left(e \bar{\psi}(n) \delta_4 \psi(n+\hat{4}) - e^{-i\mu a} \bar{\psi}(n+\hat{4}) \delta_4 \psi(n) \right) \right\} \quad (8.11)$$

*) The importance of this generalized gauge invariance in the renormalization of theories with finite chemical potentials is emphasized by Baluni [75].

Expand this form in terms of $a\mu$. The first term is the kinetic term. The second term is the point-split current. However, there are higher order extra contributions - just the appropriate non-covariant counterterms we discussed before. Of course, we shall not expand in $a\mu$ - like we do not expand $U_{n\mu}$ in terms of $A_{n\mu}$.

By this prescription, in Eq. (8.8) in the integrand $(\sin q_4 - i\mu)^2$ is replaced by $\sin^2(q_4 - i\mu)$ like in the continuum theory, and performing the q_4 integration we get

$$\mathcal{E}a^4 = \frac{1}{2\pi^3} \int_{-\pi}^{\pi} d^3q \, \theta(e^{i\mu} - b - \sqrt{b^2 + 1}) \frac{b}{\sqrt{b^2 + 1}}, \quad (8.12)$$

with

$$b^2 = \sum_{j=1}^3 \sin^2 q_j, \quad (8.13)$$

Therefore, we see from Eq. (8.12) that in every corner of the Brillouin zone the q_4 integration leads in the $a \rightarrow 0$ limit to the expected, correct result for the momentum cut-off $-\theta(\mu - \sqrt{q^2 + m^2})$, and the resulting energy density is 16 times the usual finite energy density of free fermions at zero temperature.

Using Wilson fermions, the degeneracy is removed and the factor 16 disappears for any $r \neq 0$ as it should ($0 < r \leq 1$ is the usual arbitrary parameter in the Wilson action).

Equation (8.11) can be immediately generalized to the case of QCD. At finite temperature and chemical potential the Wilson action with one flavour has the form

$$S = a^3 \sum_n \left\{ \bar{\psi}(n) \gamma(n) - K \sum_{j=1}^3 \left[\bar{\psi}(n) (\tau \cdot \delta_j) U_{nj} \gamma(n_j) + \bar{\psi}(n_j) (\tau \cdot \delta_j) U_{nj}^{\dagger} \gamma(n) \right] - K \left[e^{i\mu} \bar{\psi}(n) (\tau \cdot \delta_4) U_{nn} \gamma(n) + e^{i\mu} \bar{\psi}(n) (\tau \cdot \delta_4) U_{nn}^{\dagger} \gamma(n) \right] \right\} + \frac{2N}{g^2} \sum_p \left(1 - \frac{1}{N} \text{Re Tr } U_p \right), \quad (8.14)$$

where the gauge fields U are periodic, while the fermion fields $\psi, \bar{\psi}$ are antiperiodic along the "temperature" direction. The prescription is very simple: the hopping parameter K , related to the quark propagation by one lattice unit along the positive (negative) imaginary time axis is replaced by $e^{i\mu} K$ ($e^{-i\mu} K$).

Considering the thermodynamic potential at finite temperature, there are quark paths wrapping around the lattice in the imaginary time direction. Only these paths can lead to chemical potential dependence - from ordinary closed paths the μ dependence cancels. This is understandable: ordinary loops describe virtual pair creation and annihilation, and the chemical potential of quarks and antiquarks is of opposite sign. It follows that it is not advisable to study this system at zero temperature exactly, even if we want to discuss the effect of the finite particle density alone. The hopping parameter expansion (and related iterative methods) breaks down in this case. This fact can also be seen from the explicit result for the energy density of free quarks. At zero temperature ϵ is proportional to $\theta(\mu - m(K))$, and this distribution cannot be expanded in terms of the hopping parameter. No similar problem occurs at finite temperature.

There exists preliminary MC results concerning the effect of finite chemical potential on the deconfining phase transition in the quenched approximation (second Ref. in 73). Apart from the usual statistical and systematical problems, there is an extra question here: is it possible to shift a first-order phase transition point observed in the gauge sector via fermionic expectation values?

8.2 The fate of the deconfining phase transition in the presence of quarks

The presence of quarks modifies the action $S_g \rightarrow S_{\text{eff}}$, where S_{eff} includes the logarithm of the fermionic determinant obtained in the process of integrating over the quark fields (Section 7.3).

Starting from Eq. (7.5), one arrives to the effective action:

$$-S_{\text{eff}} = \frac{1}{g^2} \sum_p (\text{Tr } U_p + cc) + (-1) \sum_n K^n \left\{ \sum_{\substack{\text{all paths} \\ \text{of length } n}} \right\}, \quad (8.15)$$

this case the long paths are suppressed, the effective action can be truncated, and the problem becomes manageable. The fate of the deconfining phase transition can be investigated for heavy quarks. This study will be more than academic, if (by decreasing the quark mass) all the important changes in the nature of the phase transition occur before the approximations break down.

There are some preliminary studies along these lines. In Ref. [78], the lattice size was taken to be $8^3 \times 2$ and the following, truncated effective action was simulated:

$$-S_{\text{eff}} = \frac{1}{g^2} \sum_p (\text{Tr} U_p + c.c.) + 8K \eta_f \sum_{\text{sites}} (\bar{Q} + c.c.), \quad (8.16)$$

where η_f is the number of flavours, and the $O(K^4)$ and higher order terms are neglected in Eq. (8.16). This is a valid approximation until K^2 is very small, (the quark is heavy).

The position and the strength of the first order phase transition was studied on the plane ($\beta \equiv 6/g^2$, $H \equiv 8\eta_f K^2$). The phase transition line starts from the point ($\beta = 5.11, H = 0$) corresponding to $m_q = \infty$ (pure gauge theory) Fig. 25. In the MC procedure, a value of β (≈ 5.11) was fixed, then H was tuned in trying to locate the first order phase transition.

By measuring the jump in the thermal loop (L) and in the plaquette (p) expectation values, it is found that the first order transition weakens rapidly as the quark mass is decreased ($-H$ is increased). Figure 26 shows the rapid decrease of the jumps in $\langle L \rangle$ and $\langle p \rangle$ as the phase transition line dives into the (β, H) plane. The first-order transition seems to disappear at $H \approx 0.055$ describing quite heavy quarks. On this scale, the u , d and s quarks can be thought degenerated, $\eta_f = 3$ can be taken, which gives $K \approx 0.05$. In this coupling constant region, a massless quark corresponds to $K_c \approx 0.2$. The value $K \approx 0.05$ would correspond (in a crude estimate) to a quark with mass $m_q \sim 4T_c$, where T_c is the critical temperature in the pure gauge theory.

Due to the small size of the lattice along the temperature direction, this result can be taken only as an indication concerning the expected behaviour of continuum $SU(3)$ QCD. There is a recent, similar

where the contribution of a given path is obtained by taking the trace in Dirac, flavour and colour space. (We consider here η_f species of equal mass.)

At finite temperature the lattice is finite along the fourth direction. Therefore, there are special paths in Eq. (8.15) which wrap around the lattice. This contribution to S_{eff} is not invariant under the global symmetry transformation $U_{n,n_4,4} \rightarrow z U_{n,n_4,4}$, $z \in \text{centre of } G$, related to the deconfining phase transition (Section 4.2). Really, a loop wrapping around z times receives a factor z^2 under this transformation.

The presence of these symmetry breaking terms in Eq. (8.15) leads to symmetry breaking terms in the effective, 3d spin model obtained for L_n after integrating out the other variables [76]. It is like switching on an external magnetic field in the Ising model or in the Z_3 Potts model in the case of $SU(2)$ and $SU(3)$, respectively.

For the Ising model, it is known that no phase transition remains after introducing a magnetic field, even if it is weak. Therefore, in an $SU(2)$ theory no deconfining phase transition is expected to occur - at least according to this argument. At low temperature the spectrum contains hadrons, at high temperature it is a quark gluon plasma, and between them there is a smooth transition*. Of course, some rapid change - like a spectacular bump in the specific heat, for instance - is not excluded by these general considerations. This is a difficult quantitative problem, just like the question of what happens with the first order phase transition present in a pure $SU(3)$ gauge theory. A very weak magnetic field is not expected to destroy a first order phase transition. The strength of the external magnetic field in the effective spin model is related to the quark mass (or hopping parameter) in Eq. (8.15). Decreasing the quark mass increases the effect of quarks, leading to a stronger magnetic field in the spin model.

Simplified model studies indicate that sufficiently light quarks might wipe out even a first order phase transition [77, 78, 79]. Unfortunately, including light quarks in a MC calculation is notoriously difficult: the effective action in Eq. (8.15) is highly non-local. One might, however, consider a hypothetical world, where the quarks are very heavy - in

*) Of course, there might be singularities from other sources. For instance, a phase transition is expected when the chiral symmetry is restored.

calculation on an $8^3 \times 3$ lattice [80]. Although the results are consistent with those obtained in Ref. 78, the authors have a somewhat different interpretation. Clearly, more work is needed to settle this exciting problem.

Figure 25

Figure 26

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APPENDIX A
ANALYTIC CONTINUATION TO IMAGINARY TIME IN PERTURBATION THEORY [4]

Consider a Green's function in a scalar field theory in Minkowski space

$$G(x_1, x_2, \dots, x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \dots \int \frac{d^4 p_n}{(2\pi)^4} e^{i p_1 x_1} \dots e^{i p_n x_n} \tilde{G}(p_1, p_2, \dots, p_n). \quad (A.1)$$

G is given as a sum of Feynman diagrams, built up from vertices and propagators. The propagators remember the time ordered product of configuration space by their special $i\epsilon$ prescription:

$$\frac{1}{q^2 - q^2 - m^2 + i\epsilon}, \quad (A.2)$$

describing a pole slightly above and below the negative and positive part of the real q^0 axes respectively on the complex q^0 plane

It follows that if all the zero component integration contours are rotated identically anticlockwise, these singularities are avoided. This leads to the possibility of defining the function

$$G(y^0_1, x_1, \dots, y^0_n, x_n), \quad y = g e^{i\theta} \quad (A.3)$$

which is given by the right-hand side of (A.1), where all the zero component integration contours are rotated by $-\theta$ ($-\pi < \theta < 0$) and $x^0_1, \dots, x^0_n$ are replaced by $x^0_1, \theta, \dots, x^0_n, \theta$. This function is analytic in the lower φ plane and the original Green's function can be rediscovered as $\varphi \rightarrow +1$.

After introducing the new integration variables $p_i \rightarrow p_i^0/\varphi, \dots$ (everywhere including the internal loop integrations), in the particular case of $\varphi = -i$ we obtain

$$G(-ix_1, x_1, \dots, -ix_n, x_n) \sim \int_{-\infty}^{\infty} \frac{d^4 p_1}{(2\pi)^4} \dots \int_{-\infty}^{\infty} \frac{d^4 p_n}{(2\pi)^4} e^{-i p_1 x_1} \dots e^{-i p_n x_n} \bar{G}(p_1^0/\varphi, \dots, p_n^0/\varphi) \quad (A.4)$$

Therefore in the perturbative expansion the result of the Euclidean field theory [right-hand side of Eq. (A.4)] is analytically connected to the Minkowski result. They are uniquely determined by each other.

APPENDIX B REFLECTION AND CHARGE CONJUGATION

Under total reflection, the link $n \rightarrow -n$ is transformed into $-n \rightarrow -n$. The corresponding transformation $U_{n\mu} \rightarrow U_{-n-\hat{\mu}, \mu}^+$ is a symmetry of the action in Eq. (1.20). In terms of the vector potentials it leads to the relation $A_\mu \rightarrow -A_{-n-\hat{\mu}, \mu}$.

The action in Eq. (1.20) is also invariant under changing the directedness of all the contributing loops. The corresponding transformation $U_{n\mu} \rightarrow (U_{n\mu}^+)^T$ [or $A_{n\mu}^a \rightarrow -(A_{n\mu}^a)^T$] is the usual charge conjugation.

Therefore, under charge conjugation the real and imaginary part of the trace of a closed loop is even and odd, respectively. In an SU(2) gauge theory, where the trace is real, there is no negative C parity gluonic excitation.

REFERENCES

- [1] K.G. Wilson, Phys. Rev. D10 (1974) 2445.
- [2] A partial list of summary papers on the subject:
 J.-M. Drouffe and C. Itzykson, Phys. Reports 38C (1978) 133.
 L.P. Kadanoff, Rev. Mod. Phys. 49 (1977) 267.
 J.B. Kogut, Rev. Mod. Phys. 51 (1979) 659.
 G. Parisi, XX Int. Conf. on High Energy Physics, Madison, Wisconsin, 1980 (Amer. Inst. Phys., New York, 1981), p. 1531.
- [3] P. Hasenfratz, International Conference on High Energy Physics, Lisbon, 1981, Proceedings p. 620.
 C. Rebbi, XXI Int. Conf. on High Energy Physics, Paris (1982).
- [4] P. Hasenfratz, in the Proceedings of the Johns Hopkins Workshop, Florence (1982).
- [5] J.B. Kogut, Rev. Mod. Phys. 55 (1983) 775.
 J.-M. Drouffe and J.-B. Zuber, Physics Reports, to be published.
- [6] M. Creutz, L. Jacobs and C. Rebbi, Physics Reports 95 (1983) 201.
- [7] R.P. Feynman, Rev. Mod. Phys. 20 (1948) 367.
 R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, New York, (Mc Graw-Hill Book Comp.) (1965).
- [8] J. Schwinger, Phys. Rev. 115 (1959) 721.
 G.C. Wick, Phys. Rev. 96 (1954) 1124.
- [9] K. Osterwalder and R. Schrader, Commun. Math. Phys. 42 (1975) 281.
- [10] B. Simon, The $p(\phi)_2$ Euclidean (Quantum) Field Theory (Princeton University, Princeton, N.J.) (1974).
- [11] E. Nelson, in Constructive Quantum Field Theory, (Springer Verlag, Berlin) (1973).
- [12] The independence of the continuum limit on the lattice structure is an important question. Unfortunately not much has been done until now in this field. An example is discussed by:
 W. Gelmaster, Phys. Rev. D26 (1982) 2955.
 The exciting attempt of defining field theories on a random lattice will not be discussed here. We refer the interested reader to the literature:
- [13] N.H. Christ, R. Friedberg and T.D. Lee, Nucl. Phys. B202 (1982) 89, B210[FS6] (1982) 310, 337.
- [14] Half-integer spin fermion fields will be defined on lattice points also (Section 7). This is not very natural in the above sense. The consequences of a different procedure are discussed in:
 P. Becker and H. Joos, DESY preprint, DESY-82-031 (1982).
 T. Banks, Y. Dothan and D. Horn, Phys. Lett. 117B (1982) 413.
- [15] For a detailed discussion on the non-Abelian case see:
 R. Balian, J.-M. Drouffe and C. Itzykson, Phys. Rev. D10 (1974) 3376.
- [16] B.E. Baaquie, Phys. Rev. D16 (1977) 2612.
- [17] L. Onsager, Phys. Rev. 65 (1944) 117.
- [18] M.E. Fisher and R.J. Burford, Phys. Rev. 156 (1967) 583.
- [19] F. Wegner, J. Math. Phys. 12 (1971) 2259.
- [20] K. Osterwalder and E. Seiler, Ann. Phys. (NY) 110 (1978) 440.
- [21] See for instance:
 D.J. Gross, Lectures at the Les Houches Summer School, 1975, (North Holland P.C., ed. R. Balian).
- [22] Of course, this is not true for those dimensionful quantities which depend explicitly on momenta or co-ordinates like a scattering cross-section.
- [23] The literature on the renormalization group and on its applications to critical phenomena is enormous. The reader who wants to go beyond the level of this introduction is referred to the literature:
 K.G. Wilson and J. Kogut, Physics Reports 12C (1974) 75.
 K.G. Wilson, Physica 73 (1974) 119.
 S.K. Ma, Modern Theory of Critical Phenomena (Benjamin, Reading, Mass., 1976).
- [24] M.E. Fischer, Rev. Mod. Phys. 46 (1974) 597.
- [25] Phase Transitions and Critical Phenomena, eds. C. Domb and M.S. Green, (Academia, London), Vol. 6 (1971).
- [26] K.G. Wilson, Rev. Mod. Phys. 47 (1975) 773, ibid 55 (1983) 583.
- [27] A. Hasenfratz and P. Hasenfratz, Phys. Lett. 93B (1980) 165.
- [28] Actually, no explicit proof, valid in every order, exists in the literature. At the one loop level it is easy to produce a general argument and the statement has been checked explicitly (see [16]). Although it is a common lore that the renormalized perturbation theory on the lattice is identical to that of the continuum prescriptions in every order it would be nice to have an explicit demonstration.
- [29] R. Dashen and D. Gross, Phys. Rev. D23 (1981) 2340.

- H. Kawai, R. Nakayama and K. Seo, Nuclear Phys. B189 (1981) 40.
- A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B193 (1981) 210.
- P. Weisz, Phys. Letters. 100B (1981) 331.
- A. Gonzales-Arroyo and C.P. Korthals Altes, Nucl. Phys. B205 [FS5] (1982) 46.
- H.S. Scharatchandra, H.J. Thun and P. Weisz, Nucl. Phys. B192 (1981) 205.
- F. Karsch, Nucl. Phys. B205 [FS5] (1982) 285.
- [19] A partial list of references:
 J.B. Kogut, R.B. Pearson and J. Shigemitsu, Phys. Rev. Lett. 43 (1979) 484.
 J.B. Kogut and J. Shigemitsu, Phys. Rev. Lett. 45 (1980) 410.
 J.B. Kogut, R.B. Pearson and J. Shigemitsu, Phys. Lett. 98B (1981) 63.
 G. Münster, Phys. Lett. 95B (1980) 59.
 G. Münster, Nucl. Phys. B190 [FS3] (1981) 439, E.: B205 [FS5] (1982) 648.
 G. Münster and P. Weisz, Phys. Lett. 96B (1980) 119.
 K. Seo, Enrico Fermi Inst. preprint, EFI 82-10 (1982).
 J. Smit, Nucl. Phys. B206 (1982) 309.
- [20] A. Hasenfratz, E. Hasenfratz and P. Hasenfratz Nucl. Phys. B181 (1981) 353.
- [21] C. Itzykson, M. Peskin and J.B. Zuber, Phys. Lett. B95 (1980) 259.
 M. Lüscher, G. Münster and P. Weisz, Nucl. Phys. B180 (1980) 1.
- [22] K. Binder, in Phase Transitions and Critical Phenomena (eds. C. Domb and S. Green), Academic Press, New York (1976), Vol. 5B, Monte Carlo Methods in Statistical Physics, Springer Verlag (1979), (ed. K. Binder).
 After the pioneering work of M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Lett. 42 (1979) 1390, where the $d = 4$, Z_2 gauge theory was investigated, and of M. Creutz, Phys. Rev. Lett. 43 (1979) 553, measuring for the first time the string tension in an $SU(2)$ gauge theory, a large number of works has been published. For a detailed list of references, the reader is referred to the summary works in [2].
- [24] For an interesting attempt which works for Abelian gauge theories, see:
 T. Sterling and J. Greensite, Nucl. Phys. B220 [FS8] (1983) 327.
- [25] R. Balian, J.-M. Drouffe and C. Itzykson, Phys. Rev. D11 (1975) 2104.
- [26] J.-M. Drouffe, Nucl. Phys. B170 [FS1] (1980) 211.
 E. Brezin and J.-M. Drouffe, Nucl. Phys. B200 [FS4] (1982) 93.
- [27] V. Alessandrini, V. Hakim and A. Krzywicki, Nucl. Phys. B215 [FS7] (1983) 109.
 V. Alessandrini, CERN preprint TH.3336 (1982).
 J. Greensite and B. Lautrup, Phys. Lett. 104B (1981) 41.
 H. Flyvbjerg, B. Lautrup and J.B. Zuber, Phys. Lett. 110B (1982) 279.
 B. Lautrup, Niels Bohr Institute preprint NBI-HE-82-8 (1982).
 V.F. Müller and W. Rühl, Nucl. Phys. B210 [FS6] (1982) 289.
 S. Elitzur, Phys. Rev. D12 (1975) 3978.
- [28] K. Wilson, Lecture presented at the Abingdon Meeting on Lattice Gauge Theories (1981).
- [29] B. Berg, A. Billoire and C. Rebbi, Ann. Phys. 142 (1982) 185.
- [30] B. Berg and A. Billoire, Phys. Lett. 113B (1982) 65.
 B. Berg, CERN Preprint TH.3327 (1982).
 M. Falcioni, E. Marinari, M.L. Paciello, G. Parisi, F. Rapuano, B. Taglienti and Zhang Yi-cheng, Phys. Lett. 110B (1982) 295.
- [31] K. Ishikawa, M. Teper and G. Schierholz, Phys. Lett. 110B (1982) 399, ibid 116B (1982) 429, and DESY preprint 83-004 (1983).
- [32] S.K. Ma, Phys. Rev. Lett. 37 (1976) 461.
- [33] R.H. Swendsen, Phys. Rev. Lett. 42 (1979) 859.
 K.G. Wilson, in Recent Developments of Gauge Theories, Eds. G. 't Hooft et al. (Plenum Press, 1980).
 A nice explanation of the MCRG ideas together with an application to the $d = 2$ σ model is given by:
 S.H. Shenker and J. Tobochnik, Phys. Rev. B22 (1980) 4462.
- See also:
 A. Hasenfratz and A. Margaritis, CERN Preprint TH. 3683 (1983).
 R.H. Swendsen, Phys. Rev. Lett. 47 (1981) 1775.
- [34] M. Creutz, Phys. Rev. Lett. 45 (1980) 313.
- [35] It can be shown rigorously that the potential $V(l)$ is a concave function:
- [36] C. Bogs and E. Seiler, to be published.
 G. Bhanot and C. Rebbi, Nucl. Phys. B180 [FS2] (1981) 469.
- [37] C.B. Lang and C. Rebbi, Phys. Lett. 115B (1982) 137.
- [39] For recent reviews, see:

- Q. Shafi, in *International Conference on High Energy Physics*, Lisbon (1981), ed. J. Dias de Deus.
- L. Van Hove, in *Quark Matter Formation and Heavy Ion Collisions*, Bielefeld (1982), eds. M. Jacob and H. Satz.
- N. Cabibbo and G. Parisi, *Phys. Lett.* 59B (1975) 67.
- J.C. Collins and M.J. Perry, *Phys. Rev. Lett.* 34 (1975) 1353.
- P.D. Morley and M.B. Kislinger, *Physics Reports* 51C (1979) 63.
- M.B. Kislinger and P.D. Morley, *Phys. Rev. D* 13 (1976) 2771.
- B.A. Freedman and L.D. McLerran, *Phys. Rev. D* 16 (1977) 1130, 1147, 1169.
- J.I. Kapusta, *Nucl. Phys. B* 148 (1979) 461.
- V. Baluni, *Phys. Rev. D* 17 (1978) 2092.
- E.V. Shuryak, *Physics Reports* 61C (1980) 71.
- D.J. Gross, R.D. Pisarski and L.G. Yaffe, *Revs. Mod. Phys.* 53 (1981) 43.
- A.M. Polyakov, *Phys. Lett.* 72B (1978) 477.
- L. Susskind, *Phys. Rev. D* 20 (1979) 2610.
- J.B. Kogut, *CLNS-402* (1978).
- C. Bogs and E. Seiler, *Nucl. Phys. B* 215 [FS7] (1983) 125.
- L.D. McLerran and B. Svetitsky, *Phys. Lett.* 98B (1981) 195.
- J. Kuti, J. Polónyi and K. Szlachányi, *Phys. Lett.* 98B (1981) 199.
- J. Engels, F. Karsch, I. Montvay and H. Satz, *Nucl. Phys. B* 205 [FS5] (1982) 545, *Phys. Lett.* 101B (1981) 89.
- K. Kajantie, C. Montonen and E. Pietarinen, *Z. Phys.* C9 (1981) 253.
- I. Montvay and E. Pietarinen, *Phys. Lett.* 115B (1982) 151.
- J. Kogut, M. Stone, H.W. Wyld, W.R. Gibbs, J. Shigemitsu, S.H. Shenker and D.K. Sinclair, *Phys. Rev. Lett.* 50 (1983) 393.
- T. Celik, J. Engels and H. Satz, *Phys. Lett.* 125B (1983) 411.
- J. Kogut, H. Matsuoka, M. Stone, H.W. Wyld, S.H. Shenker, J. Shigemitsu and D.K. Sinclair, Illinois preprint ILL-(TH)-83-9 (1983).
- T. Celik, J. Engels and H. Satz, Bielefeld preprint BI-TP 83/07 (1983).
- B. Svetitsky and F. Fucito, Cornell University preprint, CLNS 83/571 (1983).
- J. Polónyi and K. Szlachányi, *Phys. Lett.* 110B (1982) 395.
- B. Svetitsky and L.G. Yaffe, *Phys. Rev. D* 26 (1982) 963.
- N.S. Manton, *Phys. Lett.* 96B (1980) 328.
- J.-M. Drouffe, *Phys. Rev. D* 18 (1978) 1174.
- P. Menotti and E. Onofri, CERN preprint TH.3026 (1981).
- J.B. Kogut and L. Susskind, *Phys. Rev. D* 11 (1975) 395.
- For an example, see the third reference in 18.
- C.B. Lang, C. Rebbi, P. Salomonson and B.-S. Skagerstam, *Phys. Lett.* 101B (1981) 173, *Phys. Rev. D* 26 (1982) 2028.
- R.V. Gavai, F. Karsch and H. Satz, *Nucl. Phys. B* 220 [FS8] (1983) 223.
- R.W.B. Ardill, M. Creutz and K.J.M. Moriarty, *Phys. Rev. D* 27 (1983) 1956.
- F. Gutbrod, P. Hasenfratz, Z. Kunszt and I. Montvay, *Phys. Lett.* 128B (1983) 415.
- G. Parisi, R. Petronzio and F. Rapuano, *Phys. Lett.* 128B (1983) 418.
- J.D. Stack, Santa Barbara preprint, to be published.
- B. Berg and A. Billoire, *Nucl. Phys. B* 221 (1983) 109.
- K. Ishikawa, A. Sato, G. Schierholz and M. Teper, DESY preprint, 83-061 (1983).
- C. Michael and I. Teasdale, *Nucl. Phys. B* 215 (1983) 433.
- H. Hamber and G. Parisi, *Phys. Rev. Lett.* 47 (1981) 1792.
- E. Marinari, G. Parisi and C. Rebbi, *Phys. Rev. Lett.* 47 (1981) 1795.
- D.H. Weingarten, *Phys. Lett.* B109 (1982) 57.
- H. Hamber, E. Marinari, G. Parisi and C. Rebbi, *Phys. Lett.* B108 (1982) 314.
- A. Hasenfratz, P. Hasenfratz, Z. Kunszt and C.B. Lang, *Phys. Lett.* B110 (1982) 282 and *Phys. Lett.* B117 (1982) 81.
- F. Fucito, G. Martinelli, C. Omero, G. Parisi, R. Petronzio and F. Rapuano, *Nucl. Phys. B* 210 (1982) 407.
- G. Martinelli, C. Omero, G. Parisi and R. Petronzio, *Phys. Lett.* 117B (1982) 434.
- G. Martinelli, G. Parisi, R. Petronzio and F. Rapuano, *Phys. Lett.* 116B (1982).
- D.H. Weingarten, Indiana University preprint IUHET-82 (1982).
- C. Bernard, T. Draper and K. Olynyk, *Phys. Rev. Lett.* 49 (1982) 1076.
- H. Hamber and G. Parisi, *Phys. Rev. D* 27 (1983) 208.
- P. Hasenfratz and I. Montvay, *Phys. Rev. Lett.* 50 (1983) 309.
- C. Bernard, T. Draper and K. Olynyk, *Phys. Rev. D* 27 (1983) 227.
- R. Gupta and A. Patel, CALTECH preprint CALT-68-966 (1982).
- G. Martinelli, G. Parisi, R. Petronzio and F. Rapuano, *Phys. Lett.* B122 (1983) 283.

- K.C. Bowler, E. Marinari, G.S. Pawley, F. Rapuano and D.J. Wallace, Nucl. Phys. B220 [FS8] (1983) 137.
M. Fukugita, T. Kaneko and A. Ukawa, KEK preprint TH58 (1982).
- [65] H. Lipps, G. Martinelli, R. Petronzio and F. Rapuano, Phys. Lett. 126B (1983) 250.
P. Hasenfratz and I. Montvay, DESY preprint, 83-072 (1983).
K.C. Bowler, G.S. Pawley and D.J. Wallace, paper submitted to the EPS Conference, Brighton (1983).
- [66] J. Kogut, M. Stone, H.W. Wyld, J. Shigemitsu, S.H. Shenker and D.K. Sinclair, Phys. Rev. Lett. 48 (1982) 1140
See also the first reference in 46 and:
J. Engels, F. Karsch and H. Satz, Phys. Lett. 113B (1982) 398.
- [67] J. Engels and F. Karsch, Phys. Lett. 125B (1983) 481.
S.L. Adler, Brandeis University Summer Institute in Theoretical Physics, eds. S. Deser, M. Grisaru and H. Pendleton, MIT Press, Cambridge, MA, and London (1970).
S.L. Adler, Phys. Rev. 177 (1969) 2426.
- [68] J.S. Bell and R. Jackiw, Nuovo Cimento 60A (1969) 47.
L.H. Karsten and J. Smit, Nucl. Phys. B183 (1981) 103.
- [70] K.G. Wilson, in New Phenomena in Subnuclear Physics (Erice 1975), ed. A. Zichichi, Plenum Press, New York (1977).
The stochastic method proposed by:
- [71] J. Kuti, Phys. Rev. Lett. 49 (1982) 183. has been used in a study on the effect of virtual quarks on the deconfining phase transition:
J. Kuti and J. Polonyi, in the Proceedings of the sixth Johns Hopkins Workshop (1982) 15.
Hopping parameter expansion is used by:
H. Joos and I. Montvay, DESY preprint 83-046 (1983), to estimate the effect of virtual quark loops on the $q-\bar{q}$ potential in $SU(2)$, and by:
I. Montvay, DESY preprint 83-074 (1983), to study the effect of virtual quarks on meson masses. in $SU(2)$.
A. Hasenfratz and P. Hasenfratz, Phys. Lett. 104B (1981) 489.
C.B. Lang and H. Nicolai, Nucl. Phys. B200 [FS4] (1982) 135.
I.O. Stamatescu, Phys. Rev. D25 (1982) 1130.
- [73] P. Hasenfratz and F. Karsch, Phys. Lett. 125B (1983) 308, and in the Proceedings of the 1983 Les Houches Meeting.
J. Kogut, H. Matsuoka, M. Stone, H.W. Wyld, S. Shenker, J. Shigemitsu and D.K. Sinclair, Illinois preprint, ILL-(TH)/83-10 (1983).
A different prescription is suggested by:

- N. Bilic and R.V. Gavai, Bielefeld preprint, BI-TP83/05 (1983).
On the lattice this problem is discussed in detail by:
B.E. Baaquie, Phys. Rev. D16 (1977) 2612.
V. Baluni, Phys. Rev. D17 (1978) 2092.
T. Banks and A. Ukawa, Tokyo University preprint, INS/Rep.-465 (1983).
T.A. DeGrand and C.E. DeTar, Colorado University preprint COLO HEP 66 (1983).
P. Hasenfratz, F. Karsch and I.O. Stamatescu, CERN preprint TH.3636 (1983).
F. Green and F. Karsch, CERN preprint, to be published.
T. Celik, J. Engels and H. Satz, Bielefeld preprint BI-TP83/15 (1983).
- [74] [75] [76] [77] [78] [79] [80]