Appendix A
Special Functions

A.1 Bessel Functions and $J$-Functions

The properties of Bessel function are summarized in standard references [1, 3, 8].

A.1.1 Ordinary Bessel Functions

Ordinary Bessel functions, $J_\nu(z)$, have the following properties.

Differential equation:

$$z^2 J''_\nu(z) + z J'_\nu(z) + (z^2 - \nu^2) J_\nu(z) = 0, \quad (A.1.1)$$

where a prime denotes differentiation with respect to $z$.

Power series:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{2k+\nu}. \quad (A.1.2)$$

Recursion relations:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2 \frac{\nu}{z} J_\nu(z), \quad (A.1.3)$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2 J'_\nu(z). \quad (A.1.4)$$

Generating function:

$$e^{iz \sin \phi} = \sum_{n=-\infty}^{\infty} e^{i n \phi} J_n(z). \quad (A.1.5)$$
Sum rules:

\[
\sum_{n=-\infty}^{\infty} J_n^2(z) = 1, \quad \sum_{n=-\infty}^{\infty} n J_n^2(z) = 0, \quad \sum_{n=-\infty}^{\infty} J_n(z) J_n'(z) = 0,
\]

\[
\sum_{n=-\infty}^{\infty} n^2 J_n^2(z) = \frac{1}{2} z^2, \quad \sum_{n=-\infty}^{\infty} J_n^2(z) = \frac{1}{2}.
\] (A.1.6)

A.1.2 Modified Bessel Functions \( I_\nu(z) \)

Differential equation:

\[
I_\nu''(z) + \frac{1}{z} I_\nu'(z) - \left(1 + \frac{\nu^2}{z^2}\right) I_\nu(z) = 0.
\] (A.1.7)

Power series:

\[
I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{2k+\nu}.
\] (A.1.8)

Recursion relations:

\[
I_{\nu-1}(z) - I_{\nu+1}(z) = 2(v/z)I_\nu(z),
\]

\[
I_{\nu-1}(z) + I_{\nu+1}(z) = 2I_\nu'(z).
\] (A.1.9)

Generating function:

\[
e^{z \cos \phi} = \sum_{s=-\infty}^{\infty} I_s(z) e^{\pm is\phi}.
\] (A.1.10)

A.1.3 Macdonald Functions \( K_\nu(z) \)

Differential equation:

\[
\frac{d^2}{dz^2} K_\nu(z) + \frac{1}{z} \frac{d}{dz} K_\nu(z) - \left(1 + \frac{\nu^2}{z^2}\right) K_\nu(z) = 0.
\] (A.1.11)
Recursion relations:

\[ K_{v-1}(z) - K_{v+1}(z) = -2(v/z)K_v(z), \]
\[ K_{v-1}(z) + K_{v+1}(z) = -2K'_v(z). \] \hfill (A.1.12)

The recursion relations imply \( K_{-v}(z) = K_v(z) \) and

\[ \frac{1}{z} \frac{d}{dz} \left[ z^{\pm v} K_v(z) \right] = -z^{\pm v-1} K_{v\mp 1}(z). \] \hfill (A.1.13)

Expansion of \( K_v(z) \) for small \( z \) is

\[ K_v(z) \approx 2^{v-1} \Gamma(v) z^{-v}. \] \hfill (A.1.14)

The asymptotic expansion for large \( z \) is

\[ K_v(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} \left( 1 + \frac{4v^2 - 1}{8z} + \frac{(4v^2 - 1)(4v^2 - 9)}{128z^2} + \cdots \right). \] \hfill (A.1.15)

Integral representation:

\[ K_v(x) = \frac{(x/2)^v \Gamma(\frac{1}{2})}{\Gamma(v + \frac{1}{2})} \int_0^\infty d\chi \sinh^{2v} \chi e^{-x \cosh \chi}. \] \hfill (A.1.16)

The Gamma function satisfies

\[ \Gamma(x + 1) = x \Gamma(x), \quad \Gamma(1) = 1, \quad \Gamma \left( \frac{1}{2} \right) = \pi^{1/2}. \] \hfill (A.1.17)

The integral (A.1.16) also applies when \( v \) is negative, and then \( K_{-v}(x) = K_v(x) \) implies

\[ K_v(x) = \frac{(x/2)^{-v} \Gamma(v + \frac{1}{2}) \cos \pi v}{\Gamma(\frac{1}{2})} \int_0^\infty d\chi \frac{e^{-x \cosh \chi}}{\sinh^{2v} \chi}, \] \hfill (A.1.18)

\[ \Gamma \left( \frac{1}{2} + v \right) \Gamma \left( \frac{1}{2} - v \right) = \frac{\pi}{\cos \pi v}. \] \hfill (A.1.19)

An integral identity due to Schwinger is

\[ \int_0^\infty d\xi \xi^2 K^2_\mu(\xi) = \frac{\pi^2(1 - 4\mu^2)}{32 \cos \pi \mu}. \] \hfill (A.1.20)
A.1.4 Airy Functions

The two Airy functions that appear are defined by

\[ \text{Ai}(z) = \frac{1}{\pi} \int_0^\infty dt \cos \left( zt + \frac{1}{3}t^3 \right), \quad \text{Gi}(z) = \frac{1}{\pi} \int_0^\infty dt \sin \left( zt + \frac{1}{3}t^3 \right). \]  

(A.1.21)

For \( z > 0 \) one has

\[ \text{Ai}(z) = \frac{1}{\pi} \left( \frac{z}{3} \right)^{1/2} K_{1/3}(\zeta), \quad \text{Ai}'(z) = -\frac{z}{\pi \sqrt{3}} K_{2/3}(\zeta), \]  

(A.1.22)

with \( \zeta = 2z^{3/2}/3 \).

The approximations available for \( \text{Gi}(z) \) are for large and small \( z \). The leading terms in the asymptotic expansion for \( z \gg 1 \) are [5]

\[ \text{Gi}(z) \sim \frac{1}{\pi} \left( \frac{1}{z} + \frac{2}{z^4} + \cdots \right), \quad \text{Gi}'(z) \sim \frac{1}{\pi} \left( -\frac{1}{z^2} + \cdots \right), \]  

\[ \int_0^\zeta d\zeta' \text{Gi}(\zeta') \sim \frac{1}{\pi} \left( \ln z + \frac{2C + \ln 3}{3} - \frac{2}{3z^3} + \cdots \right), \]  

(A.1.23)

where \( C = 0.577 \cdots \) is Euler’s constant. The expansion for \( z \ll 1 \) gives

\[ \text{Gi}(z) = \frac{1}{\pi} \left[ \frac{3^{1/3}}{2} \Gamma(4/3) + \frac{3^{2/3}}{4} \Gamma(5/3) z - \frac{z^2}{2} + \cdots \right], \]  

\[ \text{Gi}(0) = 0.205, \quad \text{Gi}'(0) = 0.149. \]  

(A.1.24)

Rothman [5] found that the asymptotic expansion is accurate for \( z \gtrsim 8 \) and tabulated the functions for lower \( z \).

A.1.5 J-Functions

Definition

The \( J \)-functions used here are defined by, for \( v \geq 0 \),

\[ J_v^n(x) = \left( \frac{n!}{(n + v)!} \right)^{1/2} e^{-x/2} x^{v/2} L_n^v(x). \]  

(A.1.25)
By requiring $J_{\nu}^n(x) = (-)^\nu J_{\nu}^{n+\nu}(x)$, for $\nu < 0$ one has

$$J_{\nu}^n(x) = (-)^\nu \left( \frac{n - |\nu|!}{n!} \right)^{1/2} e^{-x/2} x^{\nu/2} L_n^{|\nu|}(x), \quad (A.1.26)$$

with $L_n^{|\nu|}(x)$ the generalized Laguerre polynomial, defined by

$$L_n^{|\nu|}(x) = \frac{e^x x^{-\nu}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\nu}) = \sum_{k=0}^n \frac{(n + \nu)! (-x)^k}{(n - k)! (k + \nu)! k!}. \quad (A.1.27)$$

Sokolov and Ternov Function

The function defined by Sokolov and Ternov [6, 7] is related to (A.1.25) by

$$I_{n,n'}(x) = J_{n-n'}^n(x). \quad (A.1.28)$$

Recursion Relations

The $J$-functions satisfy recursion relations

$$x^{1/2} J_{\nu+1}^{n-1}(x) = (n + \nu)^{1/2} J_{\nu}^{n-1}(x) - n^{1/2} J_{\nu}^n(x), \quad (A.1.29)$$

$$x^{1/2} J_{\nu-1}^n(x) = -n^{1/2} J_{\nu}^{n-1}(x) + (n + \nu)^{1/2} J_{\nu}^n(x), \quad (A.1.30)$$

and also

$$\nu J_{\nu}^{n-1}(x) = x^{1/2} \left[ (n + \nu)^{1/2} J_{\nu+1}^{n-1}(x) + n^{1/2} J_{\nu-1}^{n-1}(x) \right], \quad (A.1.31)$$

$$\nu J_{\nu}^n(x) = x^{1/2} \left[ n^{1/2} J_{\nu+1}^n(x) + (n + \nu)^{1/2} J_{\nu-1}^n(x) \right]. \quad (A.1.32)$$

A further pair of relations that is similar to the recursion relations for Bessel functions is

$$(x + \nu) J_{\nu}^n(x) = \left[ x(n + \nu) \right]^{1/2} J_{\nu-1}^n(x) + \left[ x(n + \nu + 1) \right]^{1/2} J_{\nu+1}^n(x). \quad (A.1.33)$$

$$2x \frac{d}{dx} J_{\nu}^n(x) = \left[ x(n + \nu) \right]^{1/2} J_{\nu-1}^n(x) - \left[ x(n + \nu + 1) \right]^{1/2} J_{\nu+1}^n(x). \quad (A.1.34)$$

Relations Involving $J$-Functions

With $\nu = n - n' \quad p_n = (2neB)^{1/2}, \ x = k_\perp^2/2eB$, relations (A.1.33) and (A.1.34) become
\[ p_{n'} J_{n'-n}^n(x) = p_n J_{n'-n}^{n-1}(x) + k_{\perp} J_{n'-n}^n(x), \]
\[ p_{n'} J_{n'-n}^{n-1}(x) = p_n J_{n'-n}^n(x) + k_{\perp} J_{n'-n-1}^{n-1}(x). \] (A.1.35)

The following identities result from squares of the relations (A.1.35):

\[ (p_{n'}^2 + p_n^2)(J_{n'-n}^{n-1})^2 + (J_{n'-n}^n)^2) - 4 p_{n'} p_n J_{n'-n}^{n-1} J_{n'-n}^n = k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 + (J_{n'-n-1}^n)^2], \] (A.1.36)
\[ (p_{n'}^2 - p_n^2)(J_{n'-n}^{n-1})^2 - (J_{n'-n}^n)^2) = k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 - (J_{n'-n-1}^n)^2], \] (A.1.37)
\[ (p_{n'}^2 + p_n^2)(J_{n'-n}^{n-1})^2 - (J_{n'-n}^n)^2) = 2 p_{n'} k_{\perp} [J_{n'-n}^{n-1} J_{n'-n}^n - J_{n'-n}^n J_{n'-n+1}^{n-1}] + k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 - (J_{n'-n-1}^n)^2], \] (A.1.38)
\[ (p_{n'}^2 - p_n^2)(J_{n'-n}^{n-1})^2 + (J_{n'-n}^n)^2) = 2 p_{n'} k_{\perp} [J_{n'-n}^{n-1} J_{n'-n}^n + J_{n'-n}^n J_{n'-n+1}^{n-1}] + k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 + (J_{n'-n-1}^n)^2]. \] (A.1.39)

In evaluating the response tensor in the summed form (9.1.20) some tensorial components are multiplied by \((p k)_{n'n'} = \frac{1}{2} (k^2)_{n'n'} + p_{n'}^2 - p_n^2\), and (A.1.37), (A.1.39) allow one to rewrite some of the terms that are multiplied by \(p_{n'}^2 - p_n^2\). Other terms that are multiplied by \(p_{n'}^2 - p_n^2\) can be rewritten using

\[ (p_{n'}^2 - p_n^2) J_{n'-n}^{n-1} = k_{\perp} [p_n J_{n'-n-1}^{n-1} + p_{n'} J_{n'-n}^{n-1}], \] (A.1.40)
\[ (p_{n'}^2 - p_n^2) J_{n'-n}^n = k_{\perp} [p_n J_{n'-n+1}^{n-1} + p_{n'} J_{n'-n}^{n-1}]. \] (A.1.41)

The remaining terms that are multiplied by \(p_{n'}^2 - p_n^2\) involve the square and products of \(J_{n'-n+1}^{n-1}, J_{n'-n-1}^{n-1}\), and these can be rewritten by first expressing these in terms of \(J_{n'-n}^{n-1}, J_{n'-n}^n\) using (A.1.36)–(A.1.39), but no major simplifications occur.

**Sum Rules**

The sum rules

\[ \sum_{n'=0}^{\infty} J_{n'-n}^{n'}(x) J_{n'-n'}^n(x) = \delta_{n'n''}, \] (A.1.42)
\[ \sum_{n'=0}^{\infty} (n' - n)[J_{n'-n'}^n(x)]^2 = x, \] (A.1.43)

were derived by Quinn and Rodriguez [4] and Sokolov and Ternov [6].
Orthogonality Relation

\[ \int_{0}^{\infty} dx \, J_n^0(x) J_{n'}^0(x) = \delta_{nn'} . \]  

(A.1.44)

Integral Identities

\[ \int_{0}^{\infty} dx \, x^{1/2} [J_n^n(x)]^2 = (n + v + 1)^{1/2} \left( 1 + \frac{n + \frac{1}{2}}{4(n + v + 1)} \right) , \]  

(A.1.45)

\[ \int_{0}^{\infty} dx \, x [J_n^n(x)]^2 = 2n + v + \frac{3}{2} , \]  

(A.1.46)

Particular Values

For \( v \geq 0 \), one has

\[ J_v^0(x) = (-)^v J_{-v}^{v+1}(x) = \frac{x^{v/2}e^{-x/2}}{(v!)^{1/2}} , \]  

(A.1.47)

\[ J_v^1(x) = (-)^v J_{-v}^{v+1}(x) = \frac{x^{v/2}e^{-x/2}}{(v + 1)!^{1/2}}(v + 1 - x) , \]  

(A.1.48)

\[ J_v^2(x) = (-)^v J_{-v}^{v+2}(x) = \frac{x^{v/2}e^{-x/2}}{(2!(v + 2))^{1/2}} \times [(v + 1)(v + 2) - 2(v + 2)x + x^2] . \]  

(A.1.49)

\[ J_v^3(x) = (-)^v J_{-v}^{v+3}(x) = \frac{x^{v/2}e^{-x/2}}{(3!(v + 3))^{1/2}}[(v + 1)(v + 2)(v + 3) - 3(v + 2)(v + 3)x + 3(v + 3)x^2 - x^3] . \]  

(A.1.50)

Expansion in \( x \)

For \( x \ll 1 \), the \( J \)-functions may be approximated by the leading term in their expansion in powers of \( x \):

\[ J_{n'-n}^n(x) = \left( \frac{n'}{n!} \right)^{1/2} \frac{x^{(n'-n)/2}}{(n'-n)!} \left[ 1 - \frac{n' + n + 1}{2(n' - n + 1)} x + \cdots \right] . \]  

(A.1.51)
which applies for \( n' \geq n \). The limit \( x \to 0 \) gives

\[
J^n_0(0) = 1, \quad J^n_\nu(0) = 0 \quad \text{for } \nu \neq 0.
\] (A.1.52)

**Approximation by Bessel Functions**

The expansion of the \( J \)-functions in terms of Bessel functions,

\[
J^\nu_n\left(\frac{z^2}{4n}\right) = \left[\frac{(n + \nu)!}{n!n^\nu}\right]^{1/2} \sum_{a=0}^{\infty} b_a \left(\frac{z}{2n}\right)^a J_{\nu+a}(z),
\]

\[
b_0 = 1, \quad b_1 = -\frac{1}{2}(\nu + 1), \quad b_2 = \frac{1}{8}(\nu + 1)(\nu + 2),
\]

\[
(a + 1)b_{a+1} = -\frac{1}{2}(\nu + 1)b_a + \frac{1}{4}(\nu + a)b_{a-1} - \frac{1}{4}n b_{a-2},
\] (A.1.53)

converges rapidly for sufficiently large \( n \).

In taking the nonquantum limit, one takes the limit \( \hbar \to 0 \), with \( n \to \infty \) so that \( p_n = (2\pi eBh)^{1/2} \to p_\perp \) remains finite; the ratio \( a/n = (n - n')/n \) is regarded as of order \( \hbar \). To first order in \( \hbar \) one has

\[
J^n_{n-n'}(x) = J_a(z) - \frac{1}{2}(a + 1) \frac{\hbar k_\perp}{p_\perp} J_a(z).
\] (A.1.54)

The \( J \)-functions with upper index \( n - 1 \) and \( n \) differ at first order in \( \hbar \):

\[
J^{n-1}_{n-n'}(x) - J^n_{n-n'}(x) = -\frac{\hbar k_\perp}{p_\perp} J'_a(z).
\] (A.1.55)

Related identities (with arguments \( x \) and \( z \) omitted) are

\[
(J^{n-1}_{n-n})^2 + (J^n_{n-n})^2 = J^2_a - \frac{2\hbar k_\perp}{p_\perp} J'_a J_a,
\]

\[
J^{n-1}_{n-n} J^n_{n-n} = J^2_a - \frac{\hbar k_\perp}{p_\perp} J'_a J_a,
\]

\[
(J^{n-1}_{n-n+1})^2 + (J^n_{n-n+1})^2 = \sum_{\eta = \pm 1} J^2_{a-\eta} \left(1 + \eta \frac{a(a - \eta) eB}{p_\perp^2}\right) + \frac{2\hbar k_\perp}{p_\perp} J'_a J_a,
\]

\[
J^{n-1}_{n-n+1} J^n_{n-n+1} = J_{a+1} J_{a-1} \left(1 + \frac{eB}{p_\perp^2}\right) - \frac{\hbar k_\perp}{p_\perp} J'_a J_a,
\]

\[
(J^{n-1}_{n-n})^2 - (J^n_{n-n})^2 = -\frac{2\hbar k_\perp}{p_\perp} J'_a J_a.
\]
\[(J_{n'-n+1}^n)^2 - (J_{n'-n-1}^n)^2 = \sum_{\eta = \pm 1} \eta J_{a-\eta}^2 \left( 1 + \eta \frac{a(a - \eta)eB}{p_\perp^2} \right) + \frac{a}{n} J_a^2.\]

(A.1.56)

### A.2 Relativistic Plasma Dispersion Functions

#### A.2.1 Relativistic Thermal Function \(T(z, \rho)\)

The function \(T(z, \rho)\), defined by (2.4.29), has alternative integral representations:

\[
T(z, \rho) = -\rho \int_0^\infty d\chi \sinh \chi e^{-\rho \cosh \chi} \ln \left( \frac{z + \tanh \chi}{z - \tanh \chi} \right)
= 2z \int_0^\infty d\chi \frac{e^{-\rho \cosh \chi}}{(1 - z^2) \cosh^2 \chi - 1}
= -\frac{2\rho}{1 - z^2} \int^z d\xi \frac{K_1(\rho R)}{R},
\]

(A.2.1)

with \(R = [(1 - \xi^2)(1 - z^2)]^{1/2}\).

The function \(T(z, \rho)\) satisfies the partial differential equations [2]:

\[
(1 - z^2) \frac{\partial^2}{\partial \rho^2} T(z, \rho) = 2z K_0(\rho) + T(z, \rho),
\]

(A.2.2)

\[
z(1 - z^2)^3 T''(z, \rho) - (1 - z^2)^2 (1 + 2z^2) T'(z, \rho) - \rho^2 z^3 T(z, \rho)
= 2z^2 \rho^2 K_0(\rho) + 2(1 - z^2) \rho K_1(\rho),
\]

(A.2.3)

\[
z \frac{\partial}{\partial \rho} T(z, \rho) = 2K_1(\rho) + \frac{(1 - z^2)}{\rho} T'(z, \rho),
\]

(A.2.4)

with \(T'(z, \rho) = \partial T(z, \rho)/\partial z, T''(z, \rho) = \partial^2 T(z, \rho)/\partial z^2\).

#### A.2.2 Trubnikov Functions

Trubnikov functions are defined by

\[
t_v^n(z, \rho) = (k\tilde{u})^{n+1} \int_0^\infty d\xi \xi^n \frac{K_v(r(\xi))}{r^v(\xi)},
\]

(A.2.5)

with \(r(\xi)\) given by (2.4.10), and where the power of \(k\tilde{u}\) is included so that the integral is dimensionless. They satisfy the recursion relations
\[ t_{n+1}^n(z, \rho) = \frac{i \rho z^2}{1 - z^2} t_n^n(z, \rho) + \frac{z^2}{1 - z^2} \left\{ \begin{array}{ll} K_v(\rho) & \text{for } n = 0, \\ \frac{\rho^v}{n t_v^{n-1}(z, \rho)} & \text{for } n > 0, \end{array} \right. \]  

(A.2.6)

Two further identities are

\[ \frac{\partial t_n^n(z, \rho)}{\partial \rho} = -i \rho t_v^n(z, \rho) - i t_{v+1}^{n+1}(z, \rho). \]  

(A.2.7)

The relation to \( T(z, \rho) \) follows from

\[ t_0^0(z, \rho) = i z \frac{\partial T(z, \rho)}{\partial \rho} = \frac{i}{2} \left[ 2 K_1(\rho) + \frac{(1 - z^2)}{\rho} T'(z, \rho) \right], \]  

(A.2.10)

\[ t_1^0(z, \rho) = -\frac{iz}{2\rho} T(z, \rho). \]  

(A.2.11)

The functions for higher \( n \) are generated from these using (A.2.6).

### A.2.3 Shkarofsky and Dnestrovskii Functions

The generalized Shkarofsky functions are defined by (2.5.28) for real \( q \), integer \( r \geq 0 \) and complex \( z, a \) with \( \text{Im}(z - a) > 0 \) by

\[ F_{q,r}(z, a) = -i \int_0^\infty dt \frac{(it)^r}{(1 - it)^q} \exp \left[ izt - \frac{at^2}{1 - it} \right] \]  

\[ = -i e^{-a} \int_0^\infty dt \frac{(it)^r}{(1 - it)^q} \exp \left[ i(z - a)t + \frac{a}{1 - it} \right]. \]  

(A.2.12)

The definition is extended to \( \text{Im}(z - a) < 0 \) by analytic continuation. Generalized Dnestrovskii functions are defined by (2.5.34), viz. \( F_{q,r}(z) = F_{q,0}(z, 0) \). The usual Shkarofsky functions, \( F_q(z, a) = F_{q,0}(z, a) \), and Dnestrovskii functions, \( F_q(z) = F_{q,0}(z) \), are the special cases \( r = 0 \).

The Shkarofsky functions and the Dnestrovskii functions are related by an expansion in modified Bessel functions:
Recursion Relations and Differential Equations

Recursion relations satisfied by the Shkarofsky functions are

\[ a \mathcal{F}_q(z, a) = 1 + (a - z) \mathcal{F}_q(z, a) - q \mathcal{F}_{q+1}(z, a), \quad (A.2.14) \]

\[ \mathcal{F}'_q(z, a) = \mathcal{F}_q(z, a) - \mathcal{F}_{q-1}(z, a), \quad (A.2.15) \]

\[ \mathcal{F}''_q(z, a) = \mathcal{F}_q(z, a) - 2 \mathcal{F}_{q-1}(z, a) + \mathcal{F}_{q-2}(z, a), \quad (A.2.16) \]

where a prime denotes a derivative with respect to \( z \). Eliminating \( \mathcal{F}_{q-1}(z, a) \) and \( \mathcal{F}_{q-2}(z, a) \) between these gives a second order differential equation satisfied by the Shkarofsky functions:

\[ (a - z) \mathcal{F}''_q(z, a) - [2(a - z) - q - 2] \mathcal{F}'_q(z, a) - (z + q - 2) \mathcal{F}_q(z, a) + 1 = 0. \quad (A.2.17) \]

Recursion relations for the Dnestrovskii functions follow from (A.2.14) and (A.2.15) for \( a = 0 \):

\[ (q - 1) F_q(z) = 1 - z F_{q-1}(z), \quad (A.2.18) \]

\[ F'_q(z) = F_q(z) - F_{q-1}(z). \quad (A.2.19) \]

Eliminating \( F_{q-1}(z) \) between these gives a first order differential equation satisfied by the Dnestrovskii functions:

\[ z F'_q(z) = (z + q - 1) F_q(z) - 1. \quad (A.2.20) \]

The function \( F_q(z) \) also satisfies (A.2.17) with \( a = 0 \). Equation (A.2.19) integrates to give

\[ F_q(z) = z^{q-1} e^z \Gamma(1 - q, z), \quad \Gamma(q, z) = \int_z^\infty d\zeta \zeta^{q-1} e^{-\zeta}, \quad (A.2.21) \]

where \( \Gamma(q, z) \) is the incomplete gamma function.

Limiting Cases

The expansion of the Dnestrovskii functions for small arguments \( z \) follows from (A.2.21) and the relevant expansion of the incomplete gamma function:
\[ F_q(z) = z^{q-1} e^z \Gamma(1 - q) - \sum_{j}^{\infty} \frac{z^j \Gamma(1 - q)}{\Gamma(j + q - 1) j!} \]

\[ = z^{q-1} e^z \Gamma(1 - q) - e^z \sum_{j}^{\infty} \frac{(-z)^j \Gamma(1 - q)}{\Gamma(j + 2 - q)} . \tag{A.2.22} \]

For real, positive \( z \) there is an expansion in generalized Laguerre polynomials:

\[ F_q(z) = \sum_{j=0}^{\infty} \frac{L_j^{(1-q)}(z)}{j + 1} . \tag{A.2.23} \]

For large argument, \( |z| \gg 1 \), the limit

\[ F_q(z) \sim \sum_{j=0}^{\infty} (-1)^j z^{-1-j} \Gamma(q + j) \tag{A.2.24} \]

applies for \( \arg(z) < 3\pi/2 \).

**Half-Integer \( q \)**

In evaluating (2.5.27) in terms of Shkarofsky functions, the function and its derivative with \( q = 5/2 \) appear. The expansion (2.5.38) then leads to Dnestrovskii functions with half-integer \( q \). For \( q \) a positive half-integer, the Dnestrovskii functions are expressible in terms of the plasma dispersion function

\[ Z(y) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - y} = -\frac{\phi(y)}{y} + i\pi^{1/2} e^{-z^2}, \tag{A.2.25} \]

The relevant form is

\[ \Gamma(q) F_q(z) = \sum_{j=0}^{q-3/2} (-z)^j \Gamma(q - 1 - j) + \pi^{1/2} (-z)^{q-3/2} [i z^{1/2} e^z Z(i z^{1/2})] . \tag{A.2.26} \]

Expansions for small and large arguments are

\[ \Gamma(q) F_q(z) = \begin{cases} 
\sum_{j=0}^{\infty} (-z)^j \Gamma(q - 1 - j) - i \pi (-z)^{q-1} e^z & \text{for } |z|^2 \ll 1, \\
- \sum_{j=0}^{\infty} \Gamma(q + j)(-z)^{-1-j} - i \sigma \pi (-z)^{q-1} e^z & \text{for } |z| \gg 1, 
\end{cases} \tag{A.2.27} \]
with $\sigma = 0$ for $\arg z < \pi$, $\sigma = 1$ for $\arg z = \pi$ and $\sigma = 2$ for $\pi < \arg z < 2\pi$.

## A.3 Dirac Algebra

In this section some results associated with the properties of Dirac matrices are summarized.

### A.3.1 Definitions and the Standard Representation

The Dirac matrices are defined to satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^\mu\nu,$$  \hspace{1cm} (A.3.1)

where the unit Dirac matrix is implicit on the right hand side. The Dirac Hamiltonian is

$$\hat{H} = \alpha \cdot \hat{p} + \beta m, \; \alpha = \gamma^0 \gamma, \; \beta = \gamma^0.$$  \hspace{1cm} (A.3.2)

The requirement that the Dirac Hamiltonian be self-adjoint implies

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0.$$  \hspace{1cm} (A.3.3)

### Standard Representation

The specific choice for the Dirac matrices used here is referred to as the standard representation. It corresponds to

$$\begin{align*}
\gamma^0 &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}, &
\gamma^1 &= \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
\end{pmatrix}, &
\gamma^3 &= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}.
\end{align*}$$  \hspace{1cm} (A.3.4)

A convenient way of writing these and other $4 \times 4$ matrices is in terms of block matrices. Let 0 and 1 be the null and unit $2 \times 2$ matrices. One writes

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad \rho_\chi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
\[
\rho_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]  
(A.3.5)

where the \(2 \times 2\) matrices

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]  
(A.3.6)

are the usual Pauli matrices. In this representation one has

\[
\gamma^\mu = [\rho_z, i \rho_y \Sigma], \quad \alpha = \rho_x \sigma, \quad \beta = \rho_z.
\]  
(A.3.7)

### Dirac Matrices \(\sigma^{\mu \nu}\) and \(\gamma^5\)

Two additional Dirac matrices that play an important role in the theory are

\[
\sigma^{\mu \nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu],
\]  
(A.3.8)

which plays the role of a spin angular momentum, and

\[
\gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3,
\]  
(A.3.9)

which satisfies the relations

\[
\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0, \quad (\gamma^5)^2 = 1, \quad (\gamma^5)^\dagger = \gamma^5.
\]  
(A.3.10)

One also has

\[
\gamma^\mu \gamma^5 \gamma^\rho \gamma^\sigma \gamma^5 = -i \varepsilon^{\mu \nu \rho \sigma}.
\]  
(A.3.11)

In the standard representation one has \(\gamma^5 = -\rho_x\). The spin 4-tensor \(\sigma^{\mu \nu}\), defined by (A.3.8), has components

\[
\sigma^{\mu \nu} = \begin{pmatrix}
0 & \alpha_x & \alpha_y & \alpha_z \\
-\alpha_x & 0 & -i\sigma_z & i\sigma_y \\
-\alpha_y & i\sigma_z & 0 & -i\sigma_x \\
-\alpha_z & -i\sigma_y & i\sigma_x & 0
\end{pmatrix}.
\]  
(A.3.12)

### A.3.2 Basic Set of Dirac Matrices

There are 16 independent \(4 \times 4\) matrices and for the Dirac matrices it is sometimes convenient to choose a set of 16 basis vectors. A specific choice of 16 independent
matrices is the set
\[ \gamma^A = [1, \gamma^\mu, i\sigma^{\mu\nu}, i\gamma^\mu\gamma^5, \gamma^5]. \]  
(A.3.13)

This choice involves a scalar and a pseudo scalar \((1, \gamma^5)\), a 4-vector and a pseudo 4-vector \((\gamma^\mu, i\gamma^\mu\gamma^5)\) and an antisymmetric second rank 4-tensor \((\sigma^{\mu\nu})\). These have 1, 1, 4, 4, and 6 components, respectively. This set is chosen such that the analogous set, \(\gamma_A\) with indices down, satisfies
\[ \gamma^A\gamma_A = 1 \quad \text{(no sum)}, \quad \gamma^A\gamma_B = \delta^A_B. \]  
(A.3.14)

The expansion of an arbitrary Dirac matrix, \(O\) say, in this basis gives
\[ O = \sum_A c_A \gamma^A, \quad c_A = \frac{1}{4} \text{Tr} [\gamma_A O]. \]  
(A.3.15)

**Traces of Products of \(\gamma\)-Matrices**

The traces of products of \(\gamma\)-matrices are important in detailed calculations in QED. Consider
\[ T^{\alpha_1\alpha_2...\alpha_n} = \text{Tr} \left( \gamma^{\alpha_1}\gamma^{\alpha_2}...\gamma^{\alpha_n} \right). \]  
(A.3.16)

The trace of \(\gamma^\mu\) is zero, as are the traces of \(\sigma^{\mu\nu}, \gamma^\mu\gamma^5\) and \(\gamma^5\). The trace of a product of an odd number of \(\gamma\)-matrices is also zero: \(T^{\alpha_1\alpha_2...\alpha_n} = 0\) for \(n\) odd. The trace of a product of two \(\gamma\)-matrices is nonzero. This trace is evaluated as follows. First the invariance of the trace of a product of matrices under cyclic permutations of the matrices implies \(T^{\mu\nu} = T^{\nu\mu}\). The trace of (5.1.1) implies \(T^{\mu\nu} = 4g^{\mu\nu}\), where the factor of 4 arising from the trace of the unit 4 \(\times\) 4 matrix. Using the invariance of the trace under cyclic permutations and (5.1.1) allows one to evaluate the traces (A.3.16) for all even \(n\). One finds
\[ T^{\mu\nu} = 4g^{\mu\nu}, \quad T^{\mu\nu\rho\sigma} = 4\left[ g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} \right], \]  
(A.3.17)
\[ T^{\mu\nu\rho\sigma\alpha\beta} = 4\left[ g^{\mu\nu}T^{\rho\sigma\alpha\beta} - g^{\mu\rho}T^{\nu\sigma\alpha\beta} + g^{\mu\sigma}T^{\nu\rho\alpha\beta} - g^{\mu\alpha}T^{\nu\rho\sigma\beta} + g^{\mu\beta}T^{\nu\rho\sigma\alpha} \right], \]  
(A.3.18)

and so on.

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