STABILITY OF CHARMONIUM SPECTRUM CALCULATIONS FROM QCD SUM RULES

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A B S T R A C T

We show that contrary to recent claims the QCD sum rules for charmonium remain unchanged when dimension six and eight operators are included. These results explain the remarkable accuracy of the mass determinations.

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In two recent papers^{1),2)}, Nikolaev and Radyushkin extended previous calculations^{3),4)} of the polarization function of heavy quark-antiquark systems. Using the fixed-point gauge technique⁵⁾, they calculated the Wilson coefficients of all operators of dimension six and eight in the operator product expansion for S and P wave currents. This is a difficult and important calculation. Their conclusions, however, are misleading as we will show in the following.

At $Q^2 = 0$ (Q^2 being the external reference scale) the n-th derivative of $\Pi^V(Q^2)$, usually called the n-th moment, can be written as

$$M_n^V(Q^2=0) = A_n^V(Q^2=0) \left[1 + a_n^V(Q^2=0) \alpha_s + b_n^V(Q^2=0) \phi^{(2)} + c_n^V(Q^2=0) \phi^{(3)} + d_n^V(Q^2=0) \phi^{(4)} \right] \quad (1)$$

where $A_n^V(Q^2=0)$ is the moment of the bare loop

$$A_n^V(Q^2=0) = \frac{3}{4\pi^2} \cdot \frac{2^n (n+1) (n-1)!}{(2n+3)!!} \cdot \frac{1}{(4m_c^2)^n} \quad (2)$$

and

$$b_n^V(Q^2=0) = - \frac{n(n+1)(n+2)(n+3)}{2n+5} \quad (3a)$$

is the coefficient of the gluon condensate defined as

$$\phi^{(2)} = \frac{4\pi^2}{g} \left\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \right\rangle / (4m_c^2)^2 \simeq 0.14 \times 10^{-2}. \quad (3b)$$

For our analysis we have chosen the vector current, but the behaviour is similar for other currents.

The value quoted for $\phi^{(2)}$ is determined in Ref. 4) from the sum rules for charmonium. It agrees well with the value obtained from massless quark systems^{3),6)}, although in these systems it can vary by about 50%. In charmonium the accuracy is about 20% if higher order corrections are small. In Ref. 4), this was assumed to be the case.

The coefficients a_n^V and c_n^V do not concern us here. The a_n^V vary very slowly with n and Q^2 compared to the power corrections, and $c_n^V \phi^{(3)}$ gives

negligible contributions for $n \leq 10$ already at $Q^2 = 0$ and decreases further with Q^2 . The expressions for a_n^V and c_n^V can be found in Refs. 3) and 2), respectively.

Here, we therefore concentrate on the contribution from dimension eight operators. The general structure of the coefficients is

$$d_n^V(Q^2=0) \phi^{(4)} = \frac{n(n+1)(n+2)(n+3)}{(2n+5)(2n+7)(2n+9)} \times \\ \times \sum_{I=1}^7 \left(\sum_{J=0}^6 A(I,J) n^J \right) \phi_I^{(4)} \quad (4a)$$

where the matrix $A(I,J)$ is given in Table I,

$$\phi_I^{(4)} = \frac{4}{81} \langle O_I^{(4)} \rangle / (4m_c^2)^4 \quad (4b)$$

and the operators $O_I^{(4)}$ are given in Table II.

Using a few models for the vacuum to estimate the vacuum expectation values of $O_I^{(4)}$, Nikolaev and Radyushkin found that at $Q^2 = 0$ the contribution from eight-dimensional operators to the moments (1) is large and rapidly growing with n . In particular, they found that for $n = 6$ this contribution is already about 50% of the gluon condensate contribution. In Refs. 1) and 2) and in a recent paper by the ITEP group⁷⁾, a gloomy view is taken of these results. Nikolaev and Radyushkin remark correctly that when the reference scale Q^2 is equal to zero these coefficients are too large and the original ITEP calculation³⁾ collapses. They then go on to cast doubt on the over-all validity of the method. In Ref. 7), a solution of the problem is suggested by claiming that the vacuum saturation hypothesis [which was used by Nikolaev and Radyushkin to estimate the vacuum expectation values of the operators $O_I^{(4)}$] is not valid for gluonic operators. The argument which is based on the large N_c expansion is not very convincing since the resulting values for the operator matrix elements given in Ref. 2) are of the expected order of magnitude. In fact, as was pointed out in Ref. 4), the original calculation of Ref. 3) was on shaky grounds to start with. $Q^2 = 0$ is not a suitable reference point. For the calculations to be reliable, Q^2 has to be chosen unequal to zero (space-like). Figure 1 which is taken from Ref. 4) shows this very clearly. For $Q^2 = 0$, i.e., $\xi = 0$ there is no stability region in which a reliable determination of the resonance mass (in this case the η_c) is possible.

Using the expressions at $Q^2 = 0$ [Eqs. (1)-(4)], the moments at $Q^2 \neq 0$ can easily be calculated⁴⁾. For the bare loop we obtain

$$A_n^V(\xi) = \frac{3}{4\pi^2} \frac{2^n (n+1)(n-1)!}{(2n+3)!!} (4m_c^2)^{-n} (1+\xi)^{-n} {}_2F_1\left(n, \frac{1}{2}, n+\frac{5}{2}; \rho\right) \quad (5)$$

where $\xi = Q^2/4m_c^2$, $\rho = \xi/(1+\xi)$, and $F(a, b, c; x)$ is the hypergeometric function ${}_2F_1(a, b, c; x)$. The coefficients b_n^V become

$$b_n^V(\xi) = - \frac{n(n+1)(n+2)(n+3)}{(2n+5)} (1+\xi)^{-2} \frac{{}_2F_1\left(n+2, -\frac{1}{2}, n+\frac{7}{2}; \rho\right)}{{}_2F_1\left(n, \frac{1}{2}, n+\frac{5}{2}; \rho\right)}. \quad (6)$$

And for $d = 8$, we get

$$d_n^V(\xi) \phi^{(4)} = \frac{n(n+1)(n+2)(n+3)}{(2n+5)(2n+7)(2n+9)} (1+\xi)^{-4} \times \\ \times \sum_{I=1}^7 \left(\sum_{J=0}^6 A(I, J) \frac{G(J, n)}{{}_2F_1\left(n, \frac{1}{2}, n+\frac{5}{2}; \rho\right)} \right) \phi_I^{(4)} \quad (7a)$$

where

$$G(J, n) = \frac{(n+1+J)!}{(n+1)!} {}_2F_1\left(n+4, \frac{7}{2}-J, n+\frac{5}{2}; \rho\right) - \sum_{k=0}^{J-1} \alpha^J(k) G(k, n) \quad (7b)$$

with $\alpha^J(0) = (J+1)!$, $\alpha^J(J-1) = (J+1) + \alpha^{J-1}(J-2)$ ($J \geq 2$), and $\alpha^J(k) = (J+1)\alpha^{J-1}(k) + \alpha^{J-1}(k-1)$ ($k < J-1$).

In Ref. 4), we have discussed at length the problem of choosing Q^2 for the case of charmonium and emphasized the role of the reference scale Q^2 in determining the validity region of the method for a particular problem. More precisely, this reference scale is arbitrary; the predictions for physical observables cannot depend on its value provided the following conditions are satisfied: a) asymptotic freedom is valid, b) the lowest lying resonance saturates the sum rule, c) the Wilson coefficients which are functions of Q^2 must always be small in the sense that the next order correction can be neglected.

Condition a) is the basic starting point of the theory. Figure 1 shows that for the case of charmed quarks, $Q^2 = 0$ is not large enough. There is essentially no plateau where on the one hand the QCD expression is reliable ($n = 6$ is already outside the fiducial region) and on the other hand the lowest lying resonance saturates the sum rule. Condition b) expresses the limitation on choosing Q^2 too large. The larger Q^2 the larger is the weight of higher resonances in the sum rule. The contribution to the moment $M_n^V(Q^2)$ of a higher resonance with mass m_R' , relative to the ground state m_R is

$\sim((m_R^2+Q^2)/(m_R^2+Q^2))^n$ and tends to one for large Q^2 . Condition c) is required since the Wilson coefficients are calculated perturbatively. In the case of open charm⁸⁾, for instance, there is open conflict between conditions b) and c). The power correction proportional to $m_c \langle \bar{q}q \rangle$ (m_c the charmed quark mass and $\langle \bar{q}q \rangle$ the light quark chiral condensate) is so large that single resonance saturation becomes impossible when Q^2 is chosen large enough to make the correction small.

In Ref. 4), we have calculated the S- and P-wave levels of charmonium, including only the power corrections from the four-dimensional operator $G_{\mu\nu}^a G_{\mu\nu}^a$. Figure 1 shows that at $\xi = 0$ the calculation is marginal. This is the reason why we explored the dependence on ξ . At $\xi = 1$, there is a large plateau for determining the resonance mass. A large region of overlap between the resonance and QCD representation of the amplitude is crucial. In the region of stability the two representations agree and resonance parameters can be extracted. Therefore, the mass of the state at the plateau should not change when ξ is varied. This is non-trivial since the coefficients of the operators vary substantially with ξ .

The breakdown of the plateau at large n and at very small n can also be understood. At large n ($n > 10$ for $\xi = 1$) condition c) is not valid, and for small n ($n < 5$) condition b) on saturation by the lowest-lying resonance is not satisfied. However, as expected, saturation starts at lower n for smaller Q^2 (or ξ).

The calculations of Nikolaev and Radyushkin give us the opportunity to check whether condition c) is fulfilled in the plateau region by comparing the contribution (7a) from eight-dimensional operators with the gluon condensate contribution. To do this we have to know the matrix elements of the operators $O_I^{(4)}$. Unfortunately, there exist no reliable models of the vacuum for calculating these matrix elements. One of the models used in Refs. 1) and 2) is to express these matrix elements in terms of lower dimensional ones using the vacuum saturation hypothesis. The results have been tabulated in Table III.

In Fig. 2, we have plotted the $d = 4$ and $d = 8$ contributions to the moments $M_n^V(\xi)$ at $\xi = 0$ and $\xi = 1$. As can be seen, at $\xi = 0$, the $d = 8$ contribution is not small compared to the $d = 4$ contribution. Already at $n = 6$, the $d = 8$ contribution is larger than the uncertainty in the $d = 4$ contribution which is $\sim 20\%$ in the determination of $\phi^{(2)}$. This is not

surprising since the calculation with only dimension-four operators was marginal at $Q^2 = 0$. The conclusion is that $Q^2 = 0$ is too close to threshold to be considered in the asymptotic freedom region.

For $\xi = 1$ the situation is changed drastically. Both contributions are suppressed but the $d = 8$ contribution by a much larger factor than the $d = 4$ one. At $n = 6$ the $d = 8$ terms are essentially zero and up to moment $n = 11$ their contribution is less than 20% of the $d = 4$ one. This confirms the result of Fig. 1 where it can be seen that at $\xi = 1$ the plateau extends up to $n = 10$. Including dimension eight operators can only extend the stability to higher n . The conclusion of the authors of Refs. 1) and 2) on the general failure of the sum rules is therefore unwarranted. In fact, we consider this as a strong corroboration of the results and procedure of Ref. 4) for the S and P levels of charmonium as well as for the standard value of the gluon condensate.

J I	0	1	2	3	4	5	6
1	-396	$\frac{-1075}{4}$	$\frac{-811}{24}$	$\frac{38}{3}$	$\frac{29}{8}$	$\frac{1}{4}$	0
2	-42	$\frac{-257}{56}$	$\frac{3011}{105}$	$\frac{7891}{480}$	$\frac{2203}{672}$	$\frac{671}{3360}$	$\frac{-1}{224}$
3	1638	$\frac{15703}{10}$	$\frac{35209}{60}$	$\frac{1643}{15}$	$\frac{217}{20}$	$\frac{1}{2}$	0
4	- 510	$\frac{6541}{140}$	$\frac{46569}{140}$	$\frac{12229}{80}$	$\frac{46519}{1680}$	$\frac{3443}{1680}$	$\frac{23}{560}$
5	-2232	$\frac{-161939}{70}$	$\frac{-35121}{35}$	$\frac{-27883}{120}$	$\frac{-8157}{280}$	$\frac{-1367}{840}$	$\frac{-3}{280}$
6	0	$\frac{-6344}{35}$	$\frac{-6686}{35}$	$\frac{-1171}{15}$	$\frac{-533}{35}$	$\frac{-289}{210}$	$\frac{-3}{70}$
7	1080	$\frac{14709}{14}$	$\frac{8797}{21}$	$\frac{10801}{120}$	$\frac{1937}{168}$	$\frac{247}{280}$	$\frac{9}{280}$

TABLE I - The values of the coefficients A(I,J) in Eq. (4a)
[taken from Ref. 2)].

	$O_I^{(4)}$
1	$g^4 (d^{abc} G_{\mu\nu}^a G_{\alpha\beta}^b)^2 + \frac{2}{3} g^4 (G_{\mu\nu}^a G_{\alpha\beta}^a)^2$
2	$g^4 (f^{abc} G_{\mu\nu}^a G_{\alpha\beta}^b)^2$
3	$g^4 (d^{abc} G_{\mu\alpha}^a G_{\alpha\nu}^b)^2 + \frac{2}{3} g^4 (G_{\mu\alpha}^a G_{\alpha\nu}^a)^2$
4	$g^4 (f^{abc} G_{\mu\alpha}^a G_{\alpha\nu}^b)^2$
5	$g^5 f^{abc} G_{\mu\nu}^a j_\mu^b j_\nu^c$
6	$g^3 f^{abc} G_{\mu\nu}^a G_{\nu\lambda}^b D_\alpha D_\alpha G_{\lambda\mu}^c$
7	$g^4 j_\mu^a D_\alpha D_\alpha j_\mu^a$

TABLE II - The operators $O_I^{(4)}$ expressed in terms of the gluon field tensor $G_{\mu\nu}^a$, $j_\mu^a = \sum_{u,d,s} \bar{\psi} \gamma_\mu \lambda^a / 2 \psi$, and the covariant derivative D_α .

	$\langle 0_I^{(4)} \rangle$	
1	$\frac{65}{144} \langle g^2 G^2 \rangle^2$	$\approx 0.10 \text{ GeV}^8$
2	$\frac{45}{144} \langle g^2 G^2 \rangle^2$	$\approx 0.07 \text{ GeV}^8$
3	$\frac{38}{144} \langle g^2 G^2 \rangle^2$	$\approx 0.06 \text{ GeV}^8$
4	$\frac{9}{144} \langle g^2 G^2 \rangle^2$	$\approx 0.014 \text{ GeV}^8$
5	$-\frac{3}{4} g^4 \langle \bar{q}q \rangle \langle g \bar{q} \sigma_{\mu\nu} G_{\mu\nu} q \rangle$	$\approx -0.7 \times 10^{-3} \text{ GeV}^8$
6	$M^2 \langle g^3 f^{abc} G_{\mu\alpha}^a G_{\alpha\beta}^a G_{\beta\nu}^c \rangle$	$\approx 0.014 \text{ GeV}^8$
7	$-\frac{4}{3} g^4 M^2 \langle \bar{q}q \rangle^2$	$\approx -0.63 \times 10^{-3} \text{ GeV}^8$

TABLE III - The expressions and values of the matrix elements $\langle 0_I^{(4)} \rangle$ using the vacuum saturation hypothesis [Ref. 2)]. $\langle g^2 G^2 \rangle \approx (0.83 \text{ GeV})^4$, the parameter $M^2 \approx 0.3 \text{ GeV}^2$ characterizes the momentum of the vacuum gluons, and $\langle \bar{q}q \rangle$ is the light quark condensate.

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FIGURE CAPTIONS

- Fig. 1 Stability region in n for the pseudoscalar S-state of charmonium for various values of ξ . Figure taken from Ref. 4).
- Fig. 2 The contribution of $d = 4$ and $d = 8$ operators to the moments $M_n^V(\xi)$ (relative to the bare loop) for $\xi = 0$ and $\xi = 1$. We have plotted the absolute value of the $d = 4$ contribution.

Fig. 1

Fig. 2