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THE TRANSITION FROM THE LATTICE TO THE CONTINUUM:
 CP^{N-1} MODELS AT LARGE N

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A B S T R A C T

We study in detail the large N solution for the CP^{N-1} models on an Euclidean lattice and their phase structure in D dimensions. In the two-dimensional case, we evaluate some physical quantities and discuss their behaviour in the continuum limit. In particular, for the internal energy, we found the analytic expression for the perturbative tail which, at large β , obscures the predicted renormalization group behaviour; such a tail, which, as we show, is reproduced by a mean field calculation at $N = \infty$, must be subtracted in order to define a suitable scaling quantity. We also investigated the behaviour of the topological charge and we found no perturbative tail in agreement with previous Lüscher's results. Finally, some aspects of the supersymmetric case are discussed.

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1. - INTRODUCTION

It has been recognized, in the last ten years, that the study of non-perturbative features of a gauge field theory can be conveniently performed by introducing a discrete space-time¹⁾ characterized by a lattice spacing a . The theory is then formulated on the lattice with the requirement of reproducing the continuum theory in the limit of a very small lattice constant a . In this formulation, the lattice structure does not have any physical meaning; it is only introduced in order to regularize the original continuum theory in a non-perturbative way, and moreover it is particularly suitable for Monte Carlo simulations.

Once the lattice theory has been constructed, a criterion should be given on how to extract from the lattice formulation the physical quantities of the continuum theory.

In an asymptotically free theory, these quantities can be computed by extracting the renormalization group behaviour that is obtained when we let both a and the coupling constant $g(a)$ go to zero. In this way, the string tension²⁾ and the glueball mass³⁾ have been evaluated from Monte Carlo data in a four-dimensional lattice gauge theory. For other quantities, such as the gluon condensate⁴⁾ or the topological charge fluctuations⁵⁾, the extraction of the renormalization group behaviour is less obvious because it involves the subtraction of a perturbative tail that hides the exponential fall-off predicted by the renormalization group.

The transition from the lattice to the continuum theory may be explicitly studied in the case of the two-dimensional vector models, such as the $O(N)$ non-linear σ model and the CP^{N-1} model, which share with the four-dimensional gauge theories the important property of being asymptotically free, and which have the big advantage of being exactly solvable in the large N limit.

The extraction of the physical quantities for the $O(N)$ non-linear σ model in the $N \rightarrow \infty$ limit has already been discussed by us in a short note⁶⁾; in this paper we want to discuss the details of the calculations, concentrating our attention mainly on the CP^{N-1} model. Some of the results for this model have already been briefly discussed in a paper⁷⁾ by Lüscher, where the topological susceptibility of the continuum theory is recovered analytically from the lattice formulation at $N \rightarrow \infty$.

Recently, the large N technique in two dimensions has also been applied to study analytically the effect of improving⁸⁾ the lattice action in the $O(N)$ non-linear σ model, in the framework suggested by Symanzik⁹⁾ and to analyze the issue of recovering chiral symmetry near the continuum limit in the Gross-Neveu model on the lattice¹⁰⁾.

Our paper is organized as follows. In Section 2 we introduce the generating functional for the Euclidean Green functions of the CP^{N-1} model in D dimensions on the lattice and we perform the integration over the fundamental fields $z^\alpha(x)$. In particular, we consider two different versions of the model on the lattice that have already appeared in the literature¹¹⁾⁻¹³⁾. We derive then the saddle point equations valid for large N , and we compute explicitly in this limit several quantities such as the internal energy, the propagator for the z field and for some composite fields, and the correlation length.

Section 3 is devoted to the study of the large N saddle point equations in $D \neq 2$ dimension.

The two-dimensional case, physically the most interesting, is discussed in detail in Section 4. We show in particular that one lattice formulation of the model does not have any phase transition, while the other has a phase transition at $\beta \approx 0.956$. We then compute physical quantities and show that in the weak coupling phase the results of the continuum theory are recovered. However, while the magnetic susceptibility and the inverse correlation length show the renormalization group behaviour directly, the same is not true for the internal energy where the renormalization group behaviour is hidden by a perturbative tail. At $N = \infty$, however, this tail may be exactly evaluated and after its subtraction one finds complete agreement with the prediction of the renormalization group.

In Section 5, we compute the $1/N$ corrections and we use them to evaluate the topological charge density correlations. We confirm the results of Ref. 7), showing that the topological susceptibility of the continuum theory¹⁴⁾ is recovered without any perturbative tail.

In Section 6, for the sake of completeness, we briefly repeat the calculations of the previous sections for the $O(N)$ non-linear σ model, and we recall the results already published in Ref. 6).

In Section 7, we perform the mean field approximation for both CP^{N-1} and $O(N)$ models, and we show that in this approximation the non-perturbative effects, preserved in the large N expansion, in fact get lost while the perturbative tail is reproduced exactly. This implies that the mean field theory does not give any information on the continuum limit, at least at $N = \infty$. However, as we show in the case of the internal energy, it can be very useful in order to evaluate perturbative effects that sometimes hide the renormalization group behaviour in the lattice formulation.

Finally, in Section 8, we consider a lattice version of the supersymmetric CP^{N-1} model. Supersymmetry can be recovered explicitly on the lattice in the large N limit, with a particular choice for the action. However, this action is not the natural gauge invariant extension of the action without fermions. Gauge invariance as well as supersymmetry is recovered only at $N = \infty$.

2. - THE CP^{N-1} MODEL ON THE LATTICE AND ITS LARGE N EXPANSION

The CP^{N-1} model¹⁴⁾ is constructed by introducing N complex fields $z^\alpha(x)$ that satisfy the constraint $\bar{z}^\alpha(x)z^\alpha(x) = 1$ and that transform according to the fundamental representation of $SU(N)$. The action, which is invariant under local $U(1)$ transformations and under global $SU(N)$ transformation, is given by

$$S = \frac{1}{g} \int d^D x \quad \bar{D}_\mu z \cdot D_\mu z \quad (2.1)$$

where g is a dimensionless constant. The covariant derivative

$$D_\mu = \partial_\mu + i A_\mu \quad (2.2)$$

contains the auxiliary gauge field A_μ , which, classically, may be eliminated by means of its equation of motion; however, quantum fluctuations restore the gauge field dynamics^{14),15)}.

The CP^{N-1} model on a lattice is constructed by requiring the constraint to hold at each site of the lattice $\bar{z}_n^\alpha \cdot z_n^\alpha = 1$. As usual, there exists a large arbitrariness in choosing an action which reduces to (2.1) in the continuum limit even when one requires the invariance under local $U(1)$ and global $SU(N)$ to hold on the lattice.

In the following we will use the two actions¹¹⁾⁻¹³⁾:

$$-S_1 = \frac{1}{g} \sum_{n,\mu} \left| \bar{z}_{n+\mu} \cdot z_n \right|^2 \quad (2.3)$$

$$-S_2 = \frac{1}{g} \sum_{n,\mu} \left(\bar{z}_{n+\mu} \cdot z_n \lambda_{n\mu} + \bar{z}_n \cdot z_{n+\mu} \bar{\lambda}_{n\mu} \right) \quad (2.4)$$

where, according to standard notations, $n \equiv \vec{n}$ labels the sites and μ the direction. The Abelian field $\lambda_{n\mu}$ satisfies the constraint

$$\bar{\lambda}_{n\mu} \lambda_{n\mu} = 1 \quad (2.5)$$

Apart from irrelevant constant factors, both actions reproduce (2.1) when the lattice spacing $a \rightarrow 0$; moreover, S_1 for $N = 2$ reduces to the action of $O(3)$ Heisenberg model, as also happens in the continuum.

Introducing an auxiliary field $\lambda_{n\mu}$, we may lead the action S_1 to be quadratic in the z fields in a form very similar to (2.4);

$$-S_1 = \frac{1}{g} \sum_{n\mu} \left[\bar{z}_{n+\mu} \cdot z_n \lambda_{n\mu} + z_{n+\mu} \cdot \bar{z}_n \bar{\lambda}_{n\mu} - \bar{\lambda}_{n\mu} \lambda_{n\mu} \right] \quad (2.6)$$

However, in this case, $\lambda_{n\mu}$ does not satisfy any constraint: it is not a compact variable as in (2.4).

Any physical quantity of the models may be obtained from the generating functional of the Euclidean Green functions:

$$Z_N(J, \bar{J}) = \frac{\int \prod_{n,\mu} \left(\frac{dz_n^\mu d\bar{z}_n^\mu}{2\pi i} \right) \prod_n \left[\Gamma(N) \delta(\bar{z}_n \cdot z_n - 1) \right] D\mathcal{A} e^{-S + \sum_n (\bar{J}_n \cdot \bar{z}_n + \bar{J}_n \cdot z_n)}}{Z_N(0,0)} \quad (2.7)$$

where $D\mathcal{A}$ stands for the integration over the auxiliary fields that have been previously introduced, and the factors $\Gamma(N)$ are needed in order to have

$$Z_N(0,0) = \int \prod_{n,\mu} \left(\frac{dz_n^\mu d\bar{z}_n^\mu}{2\pi i} \right) \prod_n \left[\Gamma(N) \delta(\bar{z}_n \cdot z_n - 1) \right] e^{-S} \quad (2.8)$$

properly normalized to be $Z_N(0,0) = 1$ in the limit $g \rightarrow \infty$.

For any finite value of N , the two models (2.4) and (2.6) are, at present, insoluble. In order to obtain information on physical quantities, one must either use approximate methods, such as strong and weak coupling expansions or perform Monte Carlo simulations. They become solvable in the limit $N = \infty$.

To perform the large N expansion^{*)}, it is convenient to exponentiate the δ functions of the constraints introducing the auxiliary fields α_n and then to rescale in the following way:

$$z_n \rightarrow \sqrt{g} z_n ; \bar{z}_n \rightarrow \sqrt{g} \bar{z}_n ; \alpha_n \rightarrow \frac{\alpha_n}{g} \quad (2.9)$$

It is easy to perform the Gaussian integration over the variables z_n , and one gets the following expression for the generating functional:

^{*)} We follow closely Ref. 11).

$$Z_N(J, \bar{J}) = \frac{\int DA e^{-S^{\text{eff}} + \frac{1}{\beta N} \sum_{n,m} \bar{J}_m \bar{B}_{nm}^{-1} J_m}}{Z_N(0,0)} \quad (2.10)$$

where DA now contains also $\prod_n (d\alpha_n / 2\pi i)$. The effective actions are given by:

$$-S_1^{\text{eff}} = N \left\{ -\beta \sum_n \left(i\alpha_n + \sum_\mu \lambda_{n\mu} \bar{\lambda}_{n\mu} \right) - \text{Tr} \ln B + V \frac{N-1}{N} \ln \frac{1}{\beta N} + \frac{V}{N} \ln \Gamma(N) \right\} \quad (2.11)$$

$$-S_2^{\text{eff}} = N \left\{ -\beta \sum_n i\alpha_n - \text{Tr} \ln B + V \frac{N-1}{N} \ln \frac{1}{\beta N} + \frac{V}{N} \ln \Gamma(N) \right\} \quad (2.12)$$

where the matrix B_{nm} can be written as:

$$B_{nm} = -i\alpha_n \delta_{nm} - \sum_\mu \left(\lambda_{n\mu} \delta_{m+\mu, n} + \bar{\lambda}_{n\mu} \delta_{m, n+\mu} \right) \quad (2.13)$$

We have also introduced the lattice volume V in units of a^D , D being the space-time dimension, and $\beta \equiv 1/gN$ that will be kept fixed as $N \rightarrow \infty$.

For future purposes, it is useful to introduce the Fourier transform of the fields:

$$\alpha_n = \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \alpha(p) e^{i p \cdot n} \quad \alpha(p) = \sum_n \alpha_n e^{-i p \cdot n} \quad (2.14)$$

$$\lambda_{n\mu} = \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \lambda_\mu(p) e^{i p \cdot n} \quad \lambda_\mu(p) = \sum_n \lambda_{n\mu} e^{-i p \cdot n}$$

and of the matrix B

$$B_{nm} = \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \int_{-\pi}^{\pi} \frac{d^D q}{(2\pi)^D} e^{-i p \cdot n} e^{i q \cdot m} B(p, q) \quad B(p, q) = \sum_{n,m} e^{i p \cdot n} e^{-i q \cdot m} B_{nm} \quad (2.15)$$

Inserting the expression of B into (2.15), we get the following expression for $B(p, q)$

$$B(p, q) = -i\alpha(q-p) - \sum_\mu \left[\lambda_\mu(q-p) e^{i q_\mu} + \bar{\lambda}_\mu(p-q) e^{-i q_\mu} \right] \quad (2.16)$$

The large N expansion can be performed by finding the stationary points of the effective actions (2.11) and (2.12) and by evaluating the fluctuations around those points, since, in terms of the collective variables, only a finite number of stationary configurations will now contribute.

Translational invariance requires the stationary values of α_n and $\lambda_{n\mu}$ to be independent of the lattice site¹⁶⁾. Hence, we can write the expansions

$$\alpha_n = i\alpha + \delta\alpha_n \quad ; \quad \lambda_{n\mu} = \lambda_\mu + \delta\lambda_{n\mu} \quad (2.17)$$

where $\delta\alpha_n$ and $\delta\lambda_{n\mu}$ are the fluctuations around the saddle point; the factor i in front of α (real) is required to insure the convergence of the Gaussian integral over the z 's, and λ_μ is generally complex, i.e.,

$$\lambda_\mu = |\lambda_\mu| e^{i\theta_\mu}$$

The values α and λ_μ are then obtained, minimizing the effective actions

$$-\frac{(S_1^{\text{eff}})^0}{NV} = \beta \left(\alpha - \sum_\mu \lambda_\mu \bar{\lambda}_\mu \right) - \int \frac{d^D p}{(2\pi)^D} \ln \left[\alpha - \sum_\mu (\lambda_\mu e^{ip_\mu} + \bar{\lambda}_\mu e^{-ip_\mu}) \right] + \ln \frac{1}{\beta} - 1 \quad (2.18)$$

$$-\frac{(S_2^{\text{eff}})^0}{NV} = \beta \alpha - \int \frac{d^D p}{(2\pi)^D} \ln \left[\alpha - \sum_\mu (\lambda_\mu e^{ip_\mu} + \bar{\lambda}_\mu e^{-ip_\mu}) \right] + \ln \frac{1}{\beta} - 1 \quad (2.19)$$

where we used the asymptotic expansion of $\Gamma(N)$. In Eq. (2.19) the field λ_μ must satisfy $\lambda_\mu \bar{\lambda}_\mu = 1$ and is therefore of the form $\lambda_\mu = e^{i\theta_\mu}$.

We notice, that because of (2.17), we cannot make a local change in the phase of λ_μ , i.e., at the saddle point ($N = \infty$) we lose local gauge invariance as also happens in the continuum theory^{*)}. We will see later in the discussion of $1/N$ effects how gauge invariance is recovered. Moreover, we see that both the actions (2.18) and (2.19) are independent of the global phase of λ_μ . Therefore, in the following, for simplicity, we choose $\theta_\mu = 0$, i.e., $\lambda_\mu = \bar{\lambda}_\mu = |\lambda_\mu|$ for the first model and $\lambda = 1$ for the second one.

In the case of the action S_1 , the stationary points are determined by the two saddle-point equations:

$$\beta = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\alpha - 2 \sum_\mu \lambda_\mu \cos p_\mu} \quad (2.20)$$

^{*)} See, e.g., Ref. 15).

$$\beta \lambda_\mu = \int \frac{d^D p}{(2\pi)^D} \frac{\cos p_\mu}{\alpha - 2 \sum_\mu \lambda_\mu \cos p_\mu} \quad (2.21)$$

Assuming $\lambda_\mu = \lambda$ for any direction μ , and combining together the two equations, it is easy to show that they are equivalent to:

$$\beta = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\alpha - 2\lambda \sum_\mu \cos p_\mu} \quad (2.22)$$

$$\alpha - 2D\lambda^2 = \frac{1}{\beta} \quad (2.23)$$

For the action S_2 , we get instead the saddle-point equation:

$$\beta = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\alpha - 2 \sum_\mu \cos p_\mu} \quad (2.24)$$

We sketch how to compute the large N expansion of some useful Green functions involving the fields z . It can be done starting with the expression (2.10) and differentiating with respect to the external sources J_n . The leading term in the $1/N$ expansion is obtained by taking:

$$(B_0^{-1})_{nm} = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip \cdot (n-m)}}{\alpha - 2\lambda \sum_\mu \cos p_\mu} \quad (2.25)$$

Using this expression, it is straightforward to compute the following Green functions:

$$\langle \bar{z}_n^\alpha z_m^\beta \rangle = \frac{\delta^{\alpha\beta}}{\beta N} \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip \cdot (n-m)}}{D(p)} \quad (2.26)$$

$$\begin{aligned} \langle \bar{z}_n^\alpha z_m^\beta \bar{z}_r^\gamma z_s^\delta \rangle &= \left(\frac{1}{\beta N} \right)^2 \left\{ \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip \cdot (r-s)}}{D(p)} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq \cdot (n-m)}}{D(q)} \right. \\ &\quad \left. + \delta^{\gamma\delta} \delta^{\alpha\beta} \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip \cdot (n-s)}}{D(p)} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq \cdot (r-m)}}{D(q)} \delta^{\beta\gamma} \delta^{\alpha\delta} \right\} \end{aligned} \quad (2.27)$$

where

$$D(p) \equiv D(\alpha, \lambda, p) = \alpha - 2\lambda \sum_{\mu} \cos p_{\mu} \quad (2.28)$$

From (2.26), we see that

$$G(p) \equiv \frac{1}{\beta} D^{-1}(p)$$

is the momentum space lattice propagator of the z field, while $\sqrt{\beta}$ gives the wave function renormalization constant. If we put $\lambda = 1$ in the previous expressions, we obtain the corresponding quantities for the theory defined by S_2^{eff} .

It is now easy to evaluate the internal energy per link, defined as:

$$\mathcal{U} = \frac{1}{\sqrt{D}} \frac{d}{d\beta} \ln Z \quad (2.29)$$

For the action S_1 using (2.27):

$$\mathcal{U}_1 = \langle |\bar{z}_{m+\mu} z_m|^2 \rangle = \frac{1}{\beta^2} \left[\int \frac{d^D p}{(2\pi)^D} \frac{\cos p_{\mu}}{D(p)} \right]^2 \quad (2.30)$$

and the saddle-point equations yield:

$$\mathcal{U}_1 = \lambda^2 = \frac{1}{2D} \left(\alpha - \frac{1}{\beta} \right) \quad (2.31)$$

For the action S_2 the internal energy is instead:

$$\mathcal{U}_2 = \langle (\bar{z}_m \cdot z_{m+\mu} \lambda_{\mu} + \bar{z}_{m+\mu} z_m \bar{\lambda}_{\mu}) \rangle \quad (2.32)$$

and since at the saddle point we may choose $\lambda = 1$:

$$\mathcal{U}_2 = 2 \langle \bar{z}_m \cdot z_{m+\mu} \rangle = \frac{2}{\beta} \int \frac{d^D p}{(2\pi)^D} \frac{\cos p_{\mu}}{\alpha - 2 \sum_{\mu} \cos p_{\mu}} \quad (2.33)$$

Using the equations of motion we find:

$$\mathcal{U}_2 = \frac{1}{D} \left(\alpha - \frac{1}{\beta} \right) \quad (2.34)$$

Notice that, due to the definition, U_2 cannot be related to U_1 via $\lambda = 1$. In a similar way, we may evaluate the $SU(N)$ invariant propagator for the composite field $\bar{z}_n \lambda^i z_n$, λ^i being the $SU(N)$ generators in the fundamental representation. We get

$$G(n, m) \equiv \sum_i \langle \bar{z}_n \lambda^i z_n \bar{z}_m \lambda^i z_m \rangle = \frac{2}{\beta^2} \left[\int \frac{d^D p}{(2\pi)^D} \frac{\cos(n-m) \cdot p}{D(p)} \right]^2 \quad (2.35)$$

where we used the standard normalization:

$$\sum_i \lambda_{\alpha\beta}^i \lambda_{\gamma\delta}^i = 2 \delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{2}{N} \delta_{\alpha\beta} \delta_{\gamma\delta}$$

The corresponding result for the action S_2 is again recovered if:

$$D(p) = \alpha - 2 \sum_{\mu} \cos p_{\mu}$$

i.e., $\lambda = 1$.

Finally, we want to compute the lowest-lying generated mass (i.e., at zero space-momentum) which can be obtained using (2.26) and summing over all directions orthogonal to the one defined by the two sites n and m . This sum gives the following result independent of n and μ

$$\Gamma(l) \equiv \sum_{\alpha} \langle \bar{z}_{n+l\mu}^{\alpha} z_n^{\alpha} \rangle = \frac{1}{\beta} \int \frac{d^D p}{2\pi} \frac{\cos l p}{\alpha - 2\lambda(D-1) - 2\lambda \cos p} \quad (2.36)$$

The integral may be computed (Appendix B) and we get:

$$\Gamma(l) = \frac{1}{\beta} \frac{1}{\sqrt{[\alpha - 2\lambda(D-1)]^2 - 4\lambda^2}} \cdot \left[\frac{\alpha - 2\lambda(D-1) - \sqrt{[\alpha - 2\lambda(D-1)]^2 - 4\lambda^2}}{2\lambda} \right]^l \quad (2.37)$$

Since for large distances

$$\Gamma(l) \underset{l \rightarrow \infty}{\sim} e^{-a m l}$$

we have the following expression for the generated mass

$$am = \ln \frac{\gamma}{1 - \gamma(D-1) - \sqrt{[1 - \gamma(D-1)]^2 - \gamma^2}} \quad (2.38)$$

where $\gamma = 2\lambda/\alpha$ for the action S_1 and $\gamma = 2/\alpha$ for the action S_2 .

3. - CP^{N-1} MODELS IN ARBITRARY DIMENSIONS

CP^{N-1} models have so far been considered in any (Euclidean) space time dimension D . Starting from the results presented in Section 2, we shall discuss the general features of these models in the large N limit with a special emphasis on the dependence on D . The quantities that are relevant for the discussion of the thermodynamical properties of these models are the free energy (with unconventional sign):

$$F = - \frac{1}{NV} \ln Z = \frac{S^{eff}}{NV} \quad (3.1)$$

and the internal energy per unit link (2.29). In the large N limit these quantities can be expressed in terms of the saddle-point values of the auxiliary fields α and λ . For the action S_1 this leads to Eq. (2.31) for U_1 and to

$$F_1 = \frac{1 - \beta\alpha}{2} + \ln \beta\alpha + \int \frac{d^D p}{(2\pi)^D} \ln \left(1 - \gamma \sum_{\mu} \cos p_{\mu} \right) \quad (3.2)$$

where $\gamma = 2\lambda/\alpha$. Action S_2 in turn leads to Eq. (2.34) for U_2 and to

$$F_2 = 1 - \beta\alpha + \ln \beta\alpha + \int \frac{d^D p}{(2\pi)^D} \ln \left(1 - \gamma \sum_{\mu} \cos p_{\mu} \right) \quad (3.3)$$

where $\lambda = 2/\alpha$.

The crucial difference between S_1 and S_2 is that the local symmetry

$$z_n \rightarrow e^{i\theta_n} z_n; \quad \bar{z}_n \rightarrow e^{-i\theta_n} \bar{z}_n; \quad \lambda_{n\mu} \rightarrow e^{i(\theta_{n+\mu} - \theta_n)} \lambda_{n\mu} \quad (3.4)$$

which is lost at $N = \infty$, as we already noticed in Section 2, may be restored for the action S_1 since λ may take the value $\lambda = 0$; whereas that is not the case for the action S_2 where $\lambda_{n\mu}$'s are constrained to $\bar{\lambda}_{n\mu} \lambda_{n\mu} = 1$.

The possibility of a spontaneous breakdown^{*)} of this symmetry leads to the appearance of two sharply-separated phases for the model described by S_1 . In the strong coupling phase ($\beta \leq 1$) the symmetry is unbroken and therefore $\lambda = 0$

^{*)} This is not in contrast with the Elitzur theorem since $N = \infty$.

$$\gamma = 0 \quad ; \quad \alpha = \frac{1}{\beta} \quad (3.5)$$

One can easily check that Eq. (3.5) is indeed a solution of the saddle point equations.

For sufficiently large β , however, solutions of the equations of motion exist such that $\lambda \neq 0$. In order to give a more complete discussion of these solutions, let us first rewrite the equations in terms of the parameter γ . For the action S_1 we obtain:

$$\alpha \beta = \int \frac{d^D p}{(2\pi)^D} \frac{1}{1 - \gamma \sum_r \cos p_r} \equiv f_D(\gamma) \quad (3.6)$$

where

$$\alpha = \frac{1}{\gamma^2 D} \left(1 \pm \sqrt{1 - \frac{2}{\beta} (\gamma^2 D)} \right) \quad (3.7)$$

while for the action S_2 we have

$$2\beta = \gamma f_D(\gamma) \quad (3.8)$$

We must now carefully study the properties of the function $f_D(\gamma)$. This function is defined only in the domain

$$0 \leq |\gamma| \leq \frac{1}{D} \quad (3.9)$$

In particular when $|\gamma| \rightarrow 1/D$ and $D \leq 2$ the function is divergent, while when $D > 2$

$$f_D\left(\frac{1}{D}\right) = D \int_0^\infty dx \left[I_0(x) e^{-x} \right]^D \equiv f > 1 \quad (3.10)$$

is a number, with a non-trivial dependence on D , and

$$\lim_{D \rightarrow \infty} f_D\left(\frac{1}{D}\right) = 1 \quad (3.11)$$

As a consequence, when $D > 2$ the equations of motion admit two classes of solutions for $\gamma \neq 0$. The "intermediate" regime, which is not physical as we will see, corresponds to

$$0 \leq \gamma < \frac{1}{D} \quad (3.12)$$

and it is defined in the region

$$1 \geq \beta \geq \beta_L \equiv \frac{f^2}{2D(f-1)} > \frac{2}{D} \quad \text{for } S_1 \quad (3.13)$$

$$\beta \leq \beta_L \equiv \frac{1}{2D} f > \frac{1}{2D} \quad \text{for } S_2 \quad (3.14)$$

In writing Eq. (3.13) we have implicitly assumed that $\beta_L < 1$. This condition is only satisfied for $D \geq \bar{D} > 2$; however, since $\bar{D} < 3$ we can forget about this constraint when considering physical dimensions. The "perturbative" weak coupling solution corresponds to

$$\gamma = \frac{1}{D} \quad (3.15)$$

$$\beta \geq \frac{2}{D} \quad \text{for } S_1 \quad \beta \geq \frac{1}{2D} \quad \text{for } S_2 \quad (3.16)$$

Since we are at the boundary of the parameter space, Eqs. (3.6) and (3.8) respectively lose their meaning and the minimum condition is attained automatically.

As a consequence we obtain for action S_1

$$\alpha = D \left(1 \pm \sqrt{1 - \frac{2}{\beta D}} \right) \quad (3.17)$$

We can now substitute our results, Eqs. (3.15) and (3.17) in Eqs. (3.2) and (2.31), and find:

$$F_1 = \begin{cases} 0 & \beta \leq \beta_c \\ \frac{1}{2} - \frac{\beta D}{2} \left(1 + \sqrt{1 - \frac{2}{\beta D}} \right) + \ln \beta D \left(1 + \sqrt{1 - \frac{2}{\beta D}} \right) + \int \frac{d^D p}{(2\pi)^D} \ln \left(1 - \frac{1}{D} \frac{Z \cos p}{r} \right) & \beta > \beta_c \end{cases} \quad (3.18)$$

$$u_1 = \begin{cases} 0 & \beta \leq \beta_c \\ \frac{1}{2} \sqrt{1 - \frac{2}{\beta D}} + \frac{1}{2} - \frac{1}{2\beta D} & \beta > \beta_c \end{cases} \quad (3.19)$$

As one can easily understand, β_c is simply determined by the condition that the weak coupling expression for F_1 vanishes. The meaning of this condition is made clear by considering Fig. 3.1a. It is also apparent that the intermediate phase has only a purely mathematical meaning; from a physical point of view, a first order phase transition (apparent in Fig. 3.1b) occurs at β_c such that the weak and strong coupling phases are directly connected.

From a mathematical point of view, the disappearance of the intermediate phase occurs only for $D \geq \tilde{D} > 2$ but again $\tilde{D} < 3$ and we must not take into account this phenomenon. If we substitute our results in Eqs. (3.3) and (2.34) we find:

$$F_2 = \begin{cases} 1 - \frac{2\beta}{\delta} + \ln \frac{2\beta}{\delta} + \sqrt{\frac{d^D p}{(2\pi)^D}} \ln \left(1 - \gamma \sum_{\mu} \cos p_{\mu} \right) & \beta \leq \beta_c \\ 1 - 2\beta D + \ln 2\beta D + \sqrt{\frac{d^D p}{(2\pi)^D}} \ln \left(1 - \frac{1}{D} \sum_{\mu} \cos p_{\mu} \right) & \beta \geq \beta_c \end{cases} \quad (3.20)$$

and

$$\mathcal{U}_2 = \begin{cases} \frac{2}{\delta D} - \frac{1}{\beta D} & \beta \leq \beta_c \\ 2 - \frac{1}{\beta D} & \beta \geq \beta_c \end{cases} \quad (3.21)$$

where in this case

$$\beta_c = \beta_L = \frac{1}{2D} f \quad (3.22)$$

At such point there is no discontinuity in the internal energy but the correlation length, Eq. (2.38), diverges. The order of the transition depends on D , yielding the classical (i.e., Gaussian) critical exponents for $D > 4$. The curves described by Eqs. (3.20) and (3.21) are plotted in Fig. 3.2, for $D = 4$. As we pointed out previously, when $D \rightarrow 2$, $f \rightarrow \infty$ and $D = 2$ is exactly the physically interesting case when the intermediate phase extends up to $\beta = \infty$ and asymptotic freedom is present in the continuum limit. This case will be discussed in Section 4. Further modifications in behaviour occur when we consider $D = 1$. In this case the integrations over momenta can be performed exactly and the resulting expressions are

$$F_1 = \frac{1-\alpha\beta}{2} + \ln \alpha\beta + \ln \frac{1}{2} \left(1 + \sqrt{1 - \frac{2(\alpha\beta-1)}{\alpha^2\beta}} \right) \quad (3.23)$$

and

$$F_2 = 1 - \alpha\beta + \ln \alpha\beta + \ln \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{\alpha^2}} \right) \quad (3.24)$$

and the corresponding saddle point equations reduce to

$$\alpha^2 \beta^2 = 2\beta(\alpha\beta - 1) + 1 \quad (3.25)$$

and

$$\beta = \frac{1}{\sqrt{\alpha^2 - 4}} \quad (3.26)$$

respectively.

S_1 therefore corresponds to a two-phase system; the weak coupling solution is

$$\alpha = 2 - \frac{1}{\beta} \quad (3.27)$$

and the thermodynamical quantities are

$$F_1 = \begin{cases} 0 & \beta \leq 1 \\ 1 - \beta + \ln \beta & \beta \geq 1 \end{cases} \quad (3.28)$$

$$U_1 = \begin{cases} 0 & \beta \leq 1 \\ 1 - \frac{1}{\beta} & \beta \geq 1 \end{cases} \quad (3.29)$$

A second order phase transition occurs at $\beta_c = 1$.

The action S_2 leads to a simpler system: there is only one phase corresponding to the solution of Eq. (3.26):

$$\alpha = \sqrt{4 + \frac{1}{\beta^2}} \quad (3.30)$$

It is immediate to establish that

$$F_2 = 1 - \sqrt{1 + 4\beta^2} + \ln \frac{1 + \sqrt{1 + 4\beta^2}}{2} \quad (3.31)$$

$$U_2 = \sqrt{4 + \frac{1}{\beta^2}} - \frac{1}{\beta} \quad (3.32)$$

The one-dimensional results are best understood by observing that the one-dimensional L link system factorizes¹⁷⁾:

$$Z^{(L)} \equiv \int \prod_{n=1}^{L+1} \frac{dz_n d\bar{z}_n}{(2\pi i)^N} \Gamma(N) \delta(\bar{z}_n z_{n-1}) e^{-S} = \left(Z^{(1)} \right)^L \quad (3.33)$$

where the one-link, two-site model is described by the actions:

$$S_1 = -N\beta (\bar{z}_2 \cdot z_1)(\bar{z}_1 \cdot z_2) \quad (3.34)$$

$$S_2 = -N\beta (\bar{z}_2 \cdot z_1 \lambda + \bar{z}_1 \cdot z_2 \bar{\lambda}) \quad (3.35)$$

Therefore we must only compute $Z^{(1)}$, that, in turn, can be calculated exactly for arbitrary N , by exploiting the fact that the z_2 integration can be eliminated by an appropriate global gauge fixing, thus reducing the problem to a single-link external field integral. External field integrals are discussed in Appendix B. The gauge $z_2^i = \delta_{1i}$ leads to the effective sources

$$\alpha_{ij} = \beta \delta_{1i} \delta_{1j} \quad (3.36)$$

and

$$h_i = \beta \delta_{1i} e^{i\theta} \quad (3.37)$$

respectively. The results presented in Appendix B allow us to write down immediately the expressions for the partition function of the single-link models

$$Z_1^{(1)} = \Gamma(N) \left[\frac{e^{N\beta}}{(N\beta)^{N-1}} - \sum_{m=0}^{N-2} \frac{(N\beta)^{m-N+1}}{m!} \right] = \sum_{m=0}^{\infty} \frac{(N-1)!}{(m+N-1)!} (N\beta)^m \quad (3.38)$$

$$Z_2^{(1)} = \Gamma(N) \frac{I_{N-1}(2N\beta)}{(N\beta)^{N-1}} = \sum_{m=0}^{\infty} \frac{(N\beta)^{2m}}{m!} \frac{(N-1)!}{(N-1+m)!} \quad (3.39)$$

The large N limits of these expressions are also presented in Appendix B. It is immediate to realize that they lead to Eqs. (3.28), (3.29) and to Eqs. (3.31) and (3.32) respectively. It is also straightforward to apply the method presented in Section 2 directly to the two-link model, with the introduction of appropriate Lagrange multipliers. The intermediate steps of this approach are slightly different from the $D = 1$ case, but the final result turns out to be coincident.

4. - THE TWO-DIMENSIONAL CASE

Physically, the most interesting case is the two-dimensional one. In this case, the free energy per degree of freedom (at $N = \infty$) is for the action S_1

$$\begin{aligned} F &= \ln(\alpha\beta) - \frac{\alpha\beta}{2} + \frac{1}{2} + \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \ln(1 - \gamma \sum_{\mu} \cos p_{\mu}) \\ &= \ln(\alpha\beta) - \frac{\alpha\beta}{2} + \frac{1}{2} + \frac{2}{\pi} \int_0^{\pi/2} d\theta \ln \left[\frac{1 + \sqrt{1 - 4\gamma^2 \sin^2 \theta}}{2} \right] \end{aligned} \quad (4.1)$$

while the equations of motion can now be written as:

$$\alpha\beta = \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{1}{1 - \gamma \sum_{\mu} \cos p_{\mu}} = \frac{2}{\pi} K(2\gamma) \quad (4.2)$$

$K(2\gamma)$ being the complete elliptic integral of first kind, and:

$$\alpha - 4\lambda^2 = \beta^{-1} \quad (4.3)$$

As usual, there are two solutions to these equations. One is given by

$$\gamma = 0 \quad ; \quad \alpha = \beta^{-1} \quad (4.4)$$

and dominates at high temperature.

In this phase, which is defined up to $\beta = 1$, one has $F = 0$. The weak coupling phase, which extends up to $\beta = \infty$ and corresponds to what we called the intermediate region, is given by solution with $\lambda \neq 0$. Really, this saddle point is twice degenerate, since Eqs. (4.2) and (4.3) are invariant for $\lambda \rightarrow -\lambda$. However, such a degeneracy is immaterial when dealing with quantities like the free energy per link or per degree of freedom

From Eqs. (4.2) and (4.3) one obtains

$$\beta = \frac{4\gamma^2 K^2(2\gamma)}{2\pi K(2\gamma) - \pi^2} \quad (4.5)$$

The plot of such a function in terms of 2γ is given in Fig. 4.1. We see that when $\beta_{\min} \leq \beta \leq 1$ (when $\beta_{\min} \approx 0.943$) for each value of β there are two possible values of γ , and correspondingly two branches for the free energy F

which, for the relevant range in β , is plotted in Fig. 4.2. One of the branches always stays above the strong coupling value ($F = 0$) coinciding with the latter at $\beta = 1$ and $\gamma = 0$. The other branch instead crosses the value $F = 0$ at the critical value $\beta = \beta_c \approx 0.956$ yielding a first-order phase transition.

In the weak coupling region ($\beta \rightarrow \infty$), one approaches the singularity of the elliptic integral, i.e., $\gamma \rightarrow 1/2$. Moreover, from the equations of motion one also finds:

$$\alpha = \frac{2 K(2\gamma) - \pi}{2 \gamma^2 K(2\gamma)} \quad (4.6)$$

which yields $\alpha \rightarrow 4$ as $\gamma \rightarrow 1/2$. Therefore, the weak coupling limit corresponds to: $\gamma \rightarrow 1/2$; $\alpha \rightarrow 4$; $\lambda \rightarrow 1$. From the asymptotic behaviour¹⁸⁾

$$K(2\gamma) \underset{\gamma \rightarrow 1/2}{\sim} \frac{1}{2} \ln \frac{16}{1-4\gamma^2} + O((1-2\gamma) \ln(1-2\gamma)) \quad (4.7)$$

and from the relation which follows from the saddle point equations:

$$K(2\gamma) = \frac{\pi \beta}{4 \gamma^2} \left(1 + \sqrt{1 - \frac{4 \gamma^2}{\beta}} \right) \quad (4.8)$$

one finds

$$\gamma^2 \underset{\beta \rightarrow \infty}{\sim} \frac{1}{4} - 4 e^{-2\pi \beta \left(1 + \sqrt{1 - \frac{1}{\beta}} \right)} \quad (4.9)$$

We see that γ has no perturbative part in weak coupling, i.e., it has no expansion in powers of $1/\beta$.

This will correspond to the fact that physical quantities with no perturbative tail will be expressed only in terms of γ . On the other hand, from Eq. (4.3), it is easy to find that

$$\alpha \underset{\beta \rightarrow \infty}{\sim} 2 \left(1 + \sqrt{1 - \frac{1}{\beta}} \right) + 64 e^{-2\pi \left(1 + \sqrt{1 - \frac{1}{\beta}} \right)} \quad (4.10)$$

so that α has also a perturbative part

$$\alpha_{\text{pert.}} = 2 \left(1 + \sqrt{1 - \frac{1}{\beta}} \right)$$

It is interesting to notice that this expression is defined only for $\beta \geq 1$, so that from the perturbative behaviour, one expects a phase transition at $\beta = 1$. The non-perturbative effects move the critical value at $\beta_c \approx 0.956$.

To conclude this section, we consider the two-dimensional case for the quadratic action. The free energy per degree of freedom is then (at $N = \infty$):

$$F_2 = \beta(\alpha - 4) - \int \frac{d^2 p}{(2\pi)^2} \ln [\alpha - 2 \sum_{\mu} \cos p_{\mu}] \quad (4.11)$$

and we have just one equation of motion (2.24) which can be written as

$$\alpha \beta = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{1 - \frac{2}{\alpha} \sum_{\mu} \cos p_{\mu}} = \frac{2}{\pi} K\left(\frac{2}{\alpha}\right) \quad (4.12)$$

In the strong coupling region, we have $\alpha \sim 1/\beta$, while in weak coupling

$$\alpha \sim 4 \left(1 + 8 e^{-4\pi\beta}\right) \quad (4.13)$$

We see that in this case there is only one phase. However, it is interesting to notice that the strong coupling expansion obtained from Eq. (4.11) does not coincide with the large N limit of the strong coupling series evaluated in Ref. 11), which is not defined as $N \rightarrow \infty$, while it reproduces the weak coupling i.e., for this action the $N \rightarrow \infty$ and the strong coupling limits do not commute^{*)}. On the other hand, the two phases of the model obtained using the quartic action S_1 do reproduce respectively the large N limit of the strong coupling and weak coupling series respectively.

Physical Quantities

We now want to evaluate some relevant physical quantities in the two-dimensional case and discuss their relations with the corresponding continuum quantities.

a) First we look at the spontaneously generated mass, i.e., the inverse correlation length. Equation (2.38) yields:

$$am = \ln \frac{\gamma}{1 - \gamma - \sqrt{1 - 2\gamma}} \quad (4.14)$$

^{*)} See Ref. 13

The large β behaviours of γ for the two actions, using (4.9) and (4.13), are

$$\gamma_1 \underset{\beta \rightarrow \infty}{\sim} \frac{1}{2} - 4 e^{-2\pi\beta(1 + \sqrt{1 - \frac{1}{\beta}})} \quad (4.15)$$

and

$$\gamma_2 = \frac{2}{\alpha} \underset{\beta \rightarrow \infty}{\sim} \frac{1}{2} - 4 e^{-4\pi\beta} \quad (4.16)$$

It is easy, then, to find the weak coupling behaviour for the masses:

$$am \underset{\beta \rightarrow \infty}{\sim} 4\sqrt{2} e^{\pi/2} e^{-2\pi\beta} \quad (4.17)$$

for the action S_1 , and:

$$am \underset{\beta \rightarrow \infty}{\sim} 4\sqrt{2} e^{-2\pi\beta} \quad (4.18)$$

for the action S_2 .

As we see both reproduce the right exponential fall-off predicted by the renormalization group^{*)}; however, the coefficients are different due to the different values of Λ_{lattice} for the two models. Indeed since

$$\Lambda_L = a^{-1} e^{-2\pi\beta} \quad (4.19)$$

we may write for the two cases

$$m_1 = 4\sqrt{2} e^{\pi/2} \Lambda_{1L}$$

and

$$m_2 = 4\sqrt{2} \Lambda_{2L} \quad (4.20)$$

^{*)} For the beta function and for a short review of the large N expansion results in the continuum, see Appendix A.1.

On the continuum, on the other hand, the two-dimensional saddle point equation, (Pauli-Villars regularized) is:

$$\beta = \frac{1}{4\pi} \ln \frac{\Lambda_c^2}{m^2} \quad (4.21)$$

therefore, the requirement that the physical mass be independent of the renormalization procedures, (i.e., $m_1 = m_2 = m$), gives the following values for the Λ 's ratios:

$$\frac{\Lambda_{1L}}{\Lambda_c} = \frac{1}{\sqrt{32}} e^{-\pi/2}; \quad \frac{\Lambda_{2L}}{\Lambda_c} = \frac{1}{\sqrt{32}} \quad (4.22)$$

coinciding with the direct computation.

b) More involved is the discussion of the internal energy per link as defined in (2.29). For the action S_1 one has, in two dimensions:

$$\mathcal{U}_1 = \frac{1}{4} \left(\alpha - \frac{1}{\beta} \right)$$

In order to look at its weak coupling behaviour, we may use (4.10) which yields:

$$\mathcal{U}_1 \underset{\beta \rightarrow \infty}{\sim} \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{\beta}} - \frac{1}{4\beta} + 16 e^{\pi} e^{-4\pi\beta} \quad (4.23)$$

These relations deserve some comments. The predicted exponential fall-off is explicitly recovered but we found also a perturbative part (i.e., powers of $1/\beta$) that, already at a small value of β , obscures the renormalization group behaviour. However, since we know the expression for the perturbative tail we may subtract it to $\mathcal{U}$ in order to define a quantity which has the right continuum limit.

The expansion of the "exact" perturbative part in (4.23) reproduces the series of the internal energy obtained by a weak coupling calculation at finite N , as shown in Ref. 11). We will see in the following that also the mean field approach, at $N = \infty$, reproduces the perturbative tail, while the non-perturbative behaviour is completely lost by this method.

In an analogous way, we may look at the internal energy as defined for the action S_2 :

$$\mathcal{U}_2 = \frac{1}{2} \left(\alpha - \frac{1}{\beta} \right)$$

using for α (4.13), we have then in weak coupling:

$$\mathcal{U}_2 \underset{\beta \rightarrow \infty}{\sim} 2 - \frac{1}{2\beta} + 16 e^{-4\pi\beta} \quad (4.24)$$

The same considerations as before can be made: we recover the right exponential behaviour and moreover the factor in front is changed according to the Λ 's ratios (4.22). We found again a perturbative tail which is now different but reproduces the result obtained in Ref. 11) for the corresponding quantity.

In conclusion, in both models the explicit solution at $N = \infty$ allows us to subtract the complete perturbative tail of the internal energy leaving us with a quantity which scales as predicted by the renormalization group.

For the purpose of comparison with the other results of Ref. 11), we want briefly to analyze the strong coupling region both for U_1 and U_2 . With the action S_1 the region $\beta \rightarrow 0$ is dominated by the saddle point $\gamma = 0$ and $\alpha = 1/\beta$, therefore

$$\mathcal{U}_1 = 0 \quad (4.25)$$

For the action S_2 we have to look at the asymptotic behaviour of $K(4/\alpha)$ in the region where $\alpha \gg 4$; it can be found to be:

$$\alpha\beta = \frac{2}{\pi} K\left(\frac{4}{\alpha}\right) = 1 + \frac{4}{\alpha^2} + \frac{36}{\alpha^4} + O(1/\alpha^6) \quad (4.26)$$

from which we found the expansion:

$$\alpha = \frac{1}{\beta} + 4\beta + 4\beta^3 + O(\beta^5) \quad (4.27)$$

Equation (4.27) then yields:

$$\mathcal{U}_2 = 2\beta + 2\beta^3 + O(\beta^5) \quad (4.28)$$

In Ref. 11) both U_1 and U_2 were obtained by a strong coupling technique at finite N . Now while Eq. (4.25) for the action S_1 is recovered sending $N \rightarrow \infty$, for the action S_2 this limit yields definitely a different result. As we already noticed the strong coupling and the large N do not commute for the model given by S_2 .

c) We want finally to evaluate the magnetic susceptibility defined as

$$\begin{aligned}\chi &\equiv \sum_m G(n, m) = \\ &= \sum_m \sum_i \langle \bar{z}_m \lambda^i z_m \bar{z}_m \lambda^i z_m \rangle\end{aligned}\quad (4.29)$$

Using (2.35) one finds

$$\begin{aligned}\chi &= \frac{2}{\beta^2} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{[D(p)]^2} \\ &= - \frac{2}{\beta^2} \frac{\partial}{\partial \alpha} \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{1}{D(p)}\end{aligned}\quad (4.30)$$

The integral is related to the complete elliptic integral of first kind, whose asymptotic behaviour is given by Eq. (4.7) for the action S_1 . Therefore, the weak coupling expansions may be found to be:

$$\chi_1 \underset{\beta \rightarrow \infty}{\sim} \frac{1}{4\pi\beta^2} \frac{e^{4\pi\beta}}{32 e^\pi} \quad (4.31)$$

and

$$\chi_2 \underset{\beta \rightarrow \infty}{\sim} \frac{1}{4\pi\beta^2} \frac{e^{4\pi\beta}}{32} \quad (4.32)$$

Both have the behaviour predicted by the renormalization group and the coefficients agree with the correct A_L 's ratio. Moreover, we may notice that using the expressions (4.17) and (4.18) for the masses, for both cases it is:

$$\chi = \frac{1}{4\pi\beta^2} \frac{1}{a^2 m^2} \quad (4.33)$$

and this relation holds also on the continuum [see A.11)].

5. - 1/N CORRECTIONS AND TOPOLOGICAL CHARGE

In Section 2, we have performed the functional integration over the fundamental fields z_n^α and we are left with the effective Lagrangians (2.11) and (2.12) that contain only the auxiliary fields α_n and $\lambda_{n,\mu}$. We have then expanded the effective actions around the large N saddle point. The leading term in N is given by (2.18) and (2.19). Here we want to compute the $1/N$ corrections, that come from the terms of the effective actions quadratic in the fluctuations α_n and $\delta\lambda_{n,\mu}$.

Let us start considering the action S_1 . At large N it is convenient to parametrize the fluctuations as follows

$$\delta\alpha_n = \gamma_n \quad \delta\lambda_{n\mu} = \rho_{n\mu} + i\lambda\theta_{n\mu} \quad (5.1)$$

where λ is the saddle point value determined by Eqs. (2.22) and (2.23).

The terms quadratic in the fluctuations (5.1) can easily be extracted from (2.11) and they are given by:

$$-(S_1^{\text{eff}})^{(2)} = N \left\{ -\beta \sum_{n\mu} [\rho_{n\mu}^2 + \lambda^2 \theta_{n\mu}^2] + \frac{1}{2} \text{Tr}[(B_0^{-1}S)^2] + \frac{1}{2} \text{Tr}[(B_0^{-1}A)^2] \right\} \quad (5.2)$$

where B_0^{-1} is given in (2.25) and

$$S_{mm} = -i\gamma_m \delta_{mm} - \sum_{\mu} (\rho_{m\mu} \delta_{m+m\mu} + \rho_{m\mu} \delta_{m+\mu m}) \quad (5.3)$$

$$A_{mm} = -i\lambda \sum_{\mu} (\theta_{m\mu} \delta_{m+m\mu} - \theta_{m\mu} \delta_{m+\mu m}) \quad (5.4)$$

Using (2.25) we can transform (5.2) as follows:

$$\begin{aligned} -(S^{\text{eff}})^{(2)} = & \frac{N}{2} \sum_{n,m} \int \frac{d^2k}{(2\pi)^2} e^{i(m-n) \cdot k} \left\{ i\gamma_m i\gamma_n \Gamma(k) + \right. \\ & + i \sum_{\mu} [\gamma_m \rho_{m\mu} + \gamma_n \rho_{n\mu}] \Gamma_1(k) + \sum_{\mu,\nu} \rho_{m\mu} \rho_{n\nu} [\Gamma_{2\mu\nu} - 2\beta \delta_{\mu\nu}] + \\ & + \sum_{\mu,\nu} 2\lambda^2 \theta_{m\mu} \theta_{n\nu} \tilde{\delta}_{\mu\nu} A(k) \left. \right\} \quad (5.5) \end{aligned}$$

where

$$\Gamma(k) = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{D(p) D(p+k)} \quad (5.6)$$

$$\Gamma_{1\mu}(k) = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{-ip_\mu} + e^{i(p_\mu + k_\mu)}}{D(p) D(p+k)} \quad (5.7)$$

$$\Gamma_{2\mu\nu}(k) = \int \frac{d^2 p}{(2\pi)^2} \frac{[e^{-ip_\mu} + e^{i(p_\mu + k_\mu)}][e^{-ip_\nu} + e^{i(p_\nu + k_\nu)}]}{D(p) D(p+k)} \quad (5.8)$$

and

$$\tilde{\delta}_{\mu\nu} = \delta_{\mu\nu} - \frac{(e^{ik_\mu} - 1)(e^{-ik_\nu} - 1)}{\sum_s (e^{ik_s} - 1)(e^{-ik_s} - 1)} \quad (5.9)$$

$$A(k) = \int \frac{d^2 p}{(2\pi)^2} \frac{2 \sum_\mu \sin^2(p_\mu + \frac{k_\mu}{2})}{D(p) D(p+k)} - 2\beta \quad (5.10)$$

Notice that the presence of the transversal expression $\tilde{\delta}_{\mu\nu}$ in the last term of (5.5) is a consequence of the U(1) gauge invariance that is explicitly preserved by the lattice regularization.

Using $(S_{\text{eff}})^{(2)}$ one can compute the $1/N$ corrections to the free and internal energy or one can also compute other physical quantities as the topological susceptibility that are vanishing at the leading order in N . In the following, we will limit ourselves to evaluate the last quantity on the lattice and we will show that in the limit $a \rightarrow 0$ and $\beta \rightarrow \infty$ the result of the continuum theory is recovered. The topological charge density for the CP^{N-1} model on the lattice has been originally constructed by Berg and Lüscher¹²⁾, who wrote for it the following expression:

$$Q_n = \frac{1}{2\pi} \text{Im} \left\{ \ln [\text{Tr} (P_{n+1+2} P_{n+1} P_n)] + \ln [\text{Tr} (P_{n+2} P_{n+1+2} P_n)] \right\} \quad (5.11)$$

where n denotes the lower left corner of a plaquette and the imaginary part of the logarithm is to be taken in the interval $(-\pi, \pi)$. P_n is a $n \times n$ projection matrix given

$$P_n = z_n^\alpha \bar{z}_n^\beta \quad (5.12)$$

Introducing the following quantity:

$$e^{i\theta_{n\mu}} = \frac{z_{n+\mu}^\alpha \bar{z}_n^{-\alpha}}{|\bar{z}_{n+\mu}^\alpha \cdot z_n^\alpha|} \quad (5.13)$$

It is possible to rewrite (5.11) in the form introduced by Polonyi¹⁹⁾:

$$Q_n = \frac{1}{4\pi} \epsilon_{\mu\nu} (\theta_{n\mu} + \theta_{n+\mu\nu} - \theta_{n+\mu} - \theta_{n\nu}) \quad (5.14)$$

At large N the variable $\theta_{n,\mu}$ appearing in (5.14) coincides with the variable $\theta_{n,\mu}$ in the fluctuation (5.1). By means of the effective action (5.5) we can compute the θ propagator. We get

$$\langle \theta_{m\mu} \theta_{n\nu} \rangle = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{2\lambda^2 N} \frac{\tilde{\delta}_{\mu\nu}}{A(p)} e^{ip \cdot (m-n)} \quad (5.15)$$

Using this propagator we can then easily compute the topological susceptibility obtaining the following expression:

$$\langle Q_m \sum_n Q_n \rangle = \frac{4}{2\lambda^2 N} \frac{1}{(4\pi)^2} \lim_{p \rightarrow 0} \left(\frac{p^2}{A(p)} \right) \quad (5.16)$$

In order to get a finite value in the right-hand side of (5.16), one needs that $A(0) = 0$. This can be easily shown by starting from the saddle point condition for the λ field, which after a partial integration, can be rewritten as follows:

$$2\beta\lambda = \int \frac{d^2 q}{(2\pi)^2} \frac{2\lambda \sum_r \sin^2 q_r}{[D(q)]^2} \quad (5.17)$$

This relation implies of course that $A(0) = 0$ if $\lambda \neq 0$. Therefore, we can expand $A(p)$ around $p = 0$ as follows:

$$A(p) = \frac{1}{2} p_\mu p_\nu \left(\frac{\partial^2}{\partial p_\mu \partial p_\nu} A(p) \right) \Big|_{p=0} + O(p^4) \quad (5.18)$$

where

$$\begin{aligned} \frac{\partial A(p)}{\partial p_\mu \partial p_\nu} \Big|_{p=0} = & \delta_{\mu\nu} \int \frac{d^2 q}{(2\pi)^2} \left\{ -\frac{\cos 2q_\mu}{[D(q)]^2} - (4\lambda)^2 \frac{\sin^2 q_\mu \sum_s \sin^2 q_s}{[D(q)]^4} \right. \\ & \left. + 4\lambda \frac{\cos q_\mu [2\sin^2 q_\mu + \sum_s \sin^2 q_s]}{[D(q)]^3} \right\} \end{aligned} \quad (5.19)$$

$A(p)$ is, of course, a function of β because it contains the variables α and λ , whose dependence on β is fixed by the saddle point conditions (2.22) and (2.23). In the continuum limit ($\beta \rightarrow \infty$) after some calculation we get

$$\frac{\partial A(p)}{\partial p_\mu \partial p_\nu} \Big|_{p=0} \underset{\beta \rightarrow \infty}{\sim} \frac{\delta_{\mu\nu}}{3\pi\alpha^2} \frac{1}{1-K} + O(\ln(1-K)) \quad (5.20)$$

where $K = 4\lambda/\alpha$.

Using the asymptotic expressions (4.9) and (4.10), we get the following asymptotic limit

$$\begin{aligned} \frac{\partial A(p)}{\partial p_\mu \partial p_\nu} \Big|_{p=0} \underset{\beta \rightarrow \infty}{\sim} & \frac{\delta_{\mu\nu}}{3\pi\alpha^2} \frac{e^{2\pi\beta [1 + \sqrt{1-1/\beta}]} }{8} \\ & \underset{\beta \rightarrow \infty}{\sim} \frac{e^{4\pi\beta} \delta_{\mu\nu}}{8 \cdot 3\pi e^{\pi} \alpha^2} (1 + O(1/\beta)) \end{aligned} \quad (5.21)$$

Inserting this asymptotic expression in (5.16), we get finally:

$$\langle Q_m \sum_m Q_m \rangle \underset{\beta \rightarrow \infty}{\sim} \frac{3}{\pi N} \left[32 e^{\pi} \alpha^{-2} e^{-4\pi\beta} \right] (1 + O(1/\beta)) \quad (5.22)$$

where we have explicitly written the lattice constant a in order to have the right dimensions. In the continuum limit, the expression inside the bracket in (5.22) is kept fixed and it coincides with the generated mass m^2 (4.17).

The final result (5.22) reproduces exactly the result of the continuum theory. We confirm therefore the result obtained by Lüscher in Ref. 7). Notice that the quantity (5.22) has no perturbative tail, but there are corrections $1/\beta$, that may obscure the pure exponential behaviour when one tries to extract it at finite β . This fact does not happen if we use the action S_2 . In the last part of this section, we will compute the topological susceptibility with the action S_2 and we will show that the continuum limit is recovered without the occurrence of power corrections as in (5.22).

In order to compute $1/N$ corrections for the action S_2 , it is convenient first to exponentiate the constraint $\lambda_{n,\mu} \bar{\lambda}_{n,\mu} = 1$ obtaining, instead of (2.12), the following effective Lagrangian

$$-S_2^{\text{eff}} = N \left\{ -\beta \sum_n i\alpha_n + \frac{V(N-1)}{N} \ln \frac{1}{\beta N} + \frac{V}{N} \ln \Gamma(N) - \text{Tr} \ln B \right\} - i \sum_n \ell_{n\mu} (\lambda_{n\mu} \bar{\lambda}_{n\mu} - 1) \quad (5.23)$$

The large N expansion is then performed by writing

$$\alpha_n = i\alpha + \delta\alpha_n ; \quad \lambda_{n\mu} = \lambda_\mu + \delta\lambda_{n\mu} ; \quad \ell_{n\mu} = iN\ell_\mu + \delta\ell_{n\mu} \quad (5.24)$$

where α , λ_μ and ℓ_μ are determined by minimizing the leading term of the action:

$$\left(-\frac{S_2^{\text{eff}}}{NV} \right)^{(0)} = \beta\alpha - \ln \beta - \int \frac{d^2 p}{(2\pi)^2} \ln \left[\alpha - \sum_\mu (\lambda_\mu e^{ip_\mu} + \bar{\lambda}_\mu e^{-ip_\mu}) \right] + \sum_\mu \ell_\mu (\bar{\lambda}_\mu \lambda_\mu - 1) - 1 \quad (5.25)$$

The saddle point equations can be written as follows

$$\beta = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\alpha - 2 \sum_\mu \cos p_\mu} \quad (5.26)$$

$$\bar{\lambda}_\mu \lambda_\mu = 1 \quad (5.27)$$

$$\ell_\mu = - \int \frac{d^2 p}{(2\pi)^2} \frac{\cos p_\mu}{\alpha - 2 \sum_\nu \cos p_\nu} \quad (5.28)$$

They determine α and ℓ_μ as functions of β . The terms quadratic in the fluctuations (5.24) are given by

$$\begin{aligned}
 -\left(S_2^{\text{eff}}\right)^{(2)} = N \left\{ \frac{1}{2} \text{Tr} \left((B_0^{-1} S)^2 \right) + \frac{1}{2} \text{Tr} \left((B_0^{-1} A)^2 \right) + \sum_{\mu} \ell_{\mu} \delta \lambda_{n\mu} \delta \bar{\lambda}_{n\mu} \right. \\
 \left. + \frac{1}{N} \sum_{n,\mu} \delta \ell_{n\mu} \left[\lambda_{\mu} \delta \bar{\lambda}_{n\mu} + \bar{\lambda}_{\mu} \delta \lambda_{n\mu} \right] \right\}. \quad (5.29)
 \end{aligned}$$

where S and A are given by (5.3) and (5.4).

Assuming, for simplicity, that $\lambda_{\mu} = 1$ and parametrizing the fluctuation of $\lambda_{n,\mu}$ as follows

$$\delta \lambda_{n\mu} = \rho_{n\mu} + i \theta_{n\mu} \quad (5.30)$$

we get the following contribution to $(-S_2^{\text{eff}})^{(2)}$ of the terms quadratic in $\theta_{n,\mu}$:

$$-\left(S_2^{\text{eff}}\right)_{\theta\theta}^{(2)} = -N \sum_{\substack{m,n \\ \mu,\nu}} \theta_{m\mu} \theta_{n\nu} \int \frac{d^2 k}{(2\pi)^2} e^{ik(n-m)\tilde{\mu}} \delta_{\mu\nu} F(k) \quad (5.31)$$

where

$$F(k) = \int \frac{d^2 q}{(2\pi)^2} \frac{\sum_{\mu} \cos q_{\mu}}{D(q)} - \int \frac{d^2 q}{(2\pi)^2} \frac{2 \sum_{\mu} \sin^2 \left(q_{\mu} + \frac{k_{\mu}}{2} \right)}{D(q) D(q+k)} \quad (5.32)$$

and $D(q) = \alpha - 2 \sum_{\mu} \cos q_{\mu}$.

Using (5.31) we can now compute the topological susceptibility and we get:

$$\langle Q_m \sum_m Q_m \rangle = \frac{4}{2Na^2} \frac{1}{(4\pi)^2} \lim_{p \rightarrow 0} \frac{p^2}{F(p)} \quad (5.33)$$

The function $F(p)$ has the following properties

$$F(0) = 0 \quad (5.34)$$

$$\left. \frac{\partial^2 F(p)}{\partial p_\mu \partial p_\nu} \right|_{p=0} \underset{\beta \rightarrow \infty}{\sim} \delta_{\mu\nu} \frac{K^2}{48\pi} \frac{1}{1-K} \quad (5.35)$$

with

$$\frac{4}{a^2} = K \underset{\beta \rightarrow \infty}{\sim} 1 - 8 e^{-4\pi\beta} \quad (5.36)$$

Inserting (5.34), (5.35) and (5.36) in (5.33) we get finally

$$\langle Q_m [Q_m] \rangle \underset{\beta \rightarrow \infty}{\sim} \frac{3}{N\pi} \left[32 e^{-4\pi\beta} \frac{1}{a^2} \right] \quad (5.37)$$

The term inside the bracket is the spontaneously generated mass and therefore the result of the continuum theory is recovered. Notice that, since the expression in (5.36) for large β is correct apart from exponentially small corrections, the topological susceptibility computed with the action S_2 has no power correction in $1/\beta$ as in (5.22).

In conclusion, the result of the continuum theory can be recovered with both actions S_1 and S_2 , but the action S_2 has only exponentially small corrections in β , while the action S_1 has also power corrections. In both cases, however, there is no perturbative tail as happened for the average energy computed in Section 4.

6. - O(N) MODELS IN ARBITRARY DIMENSIONS

For the sake of comparison, we review in this section the main properties of O(N) lattice spin models in D dimensions. O(N) models are constructed by introducing N real fields $S_\alpha(x)$ transforming according to the fundamental (vector) representation of O(N) and satisfying the constraint

$$\sum_\alpha S^\alpha(x) S^\alpha(x) = 1 \quad (6.1)$$

The action is invariant under global O(N) transformations and in the continuum it is given by

$$S = \frac{1}{g} \int d^D x \sum_\alpha \partial_\mu S^\alpha \partial_\mu S^\alpha \quad (6.2)$$

In complete analogy with the CP^{N-1} case, we shall consider two lattice versions of this model, described by the actions^{6),20)}

$$S_1 = \frac{1}{2g} \sum_{\mu} (\vec{S}_{n+\mu} \cdot \vec{S}_n)^2 \quad (6.3)$$

$$S_2 = \frac{1}{g} \sum_{\mu} \vec{S}_{n+\mu} \cdot \vec{S}_n \quad (6.4)$$

In both cases the constraint

$$\vec{S}_n \cdot \vec{S}_n = 1 \quad (6.5)$$

is imposed, and the lattice functional measure

$$Ds = \prod_{n,\alpha} \frac{ds_n^\alpha}{\pi^{1/2}} \prod_n \Gamma\left(\frac{N}{2}\right) \delta(\vec{S}_n \cdot \vec{S}_n - 1) \quad (6.6)$$

is employed. Again, we introduce the lattice coupling $\beta = 1/gN$.

Standard Lagrange multiplier techniques are used in order to turn the actions Eqs. (6.3) and (6.4) into the effective actions:

$$S_1^{\text{eff}} = \frac{N\beta}{2} \left[\sum_n 2i\alpha_n + \sum_{\mu} \lambda_{n\mu} \lambda_{n\mu} \right] + \frac{N}{2} \text{Tr} \ln B + V \left[\left(\frac{N}{2} - 1 \right) \ln \frac{N\beta}{2} - \ln \Gamma\left(\frac{N}{2}\right) \right] \quad (6.7)$$

$$S_2^{\text{eff}} = \frac{N\beta}{2} \left[\sum_n 2i\alpha_n \right] + \frac{N}{2} \text{Tr} \ln B' + V \left[\left(\frac{N}{2} - 1 \right) \ln \frac{N\beta}{2} - \ln \Gamma\left(\frac{N}{2}\right) \right] \quad (6.8)$$

where

$$B_{nm} = -2i\alpha_n \delta_{nm} - \sum_{\mu} \left[\lambda_{n\mu} \delta_{m+\mu, n} + \lambda_{n\mu} \delta_{m, n+\mu} \right] \quad (6.9)$$

and B' is obtained from B through the replacement $\lambda_{n\mu} \rightarrow 1$. In the large N limit, the standard method of Fourier transforming the variables and choosing translation invariant solutions reduces the previous expressions to:

$$\frac{S_1^{\text{eff}}}{NV} = \frac{1-2\alpha\beta + \beta \sum_{\mu} \lambda_{\mu}^2}{2} + \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln [2\alpha - \sum_{\mu} 2\lambda_{\mu} \cos p_{\mu}] + \frac{1}{2} \ln \beta \quad (6.10)$$

$$\frac{S_2^{\text{eff}}}{NV} = \frac{1-2\alpha\beta}{2} + \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln [2\alpha - \sum_{\mu} 2 \cos p_{\mu}] + \frac{1}{2} \ln \beta \quad (6.11)$$

It is easy to recognize that the mapping

$$2\alpha \rightarrow \alpha \quad ; \quad \frac{N}{2} \rightarrow N \quad (6.12)$$

turns Eqs. (6.10) and (6.11) into the corresponding expressions for the CP^{N-1} models (with the request that λ_u be real). The discussion of solutions for arbitrary D is therefore the same as that presented in Section 3. We shall not repeat the discussion but only mention a few results for further reference. Because of the local symmetry

$$\vec{Z}_m \rightarrow -\vec{Z}_m \quad (6.13)$$

the action S_1 leads to a system admitting an (unbroken) strong coupling phase, while no such phase appears in the system described by S_2 . In more than two dimensions, the thermodynamical quantities show the following behaviour:

$$F_1 = \begin{cases} 0 & \beta \leq \beta_c \\ \frac{1}{2} \left[\frac{1}{2} - \frac{\beta D}{2} \left(1 + \sqrt{1 - \frac{2}{\beta D}} \right) + \ln \beta D \left(1 + \sqrt{1 - \frac{2}{\beta D}} \right) + \int \frac{d^D p}{(2\pi)^D} \ln \left(1 - \frac{1}{D} \sum_r \cos p_r \right) \right] & \beta \geq \beta_c \end{cases} \quad (6.14)$$

$$u_1 = \begin{cases} 0 & \beta \leq \beta_c \\ \frac{1}{2} \left[\frac{1}{2} \sqrt{1 - \frac{2}{\beta D}} + \frac{1}{2} - \frac{1}{2\beta D} \right] & \beta \geq \beta_c \end{cases} \quad (6.15)$$

and there is a first-order phase transition at $\beta = \beta_c$. The action S_2 leads to

$$F_2 = \begin{cases} \frac{1}{2} - \frac{\beta}{\gamma} + \frac{1}{2} \ln \frac{2\beta}{\gamma} + \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln \left(1 - \gamma \sum_r \cos p_r \right) & \beta \leq \beta_c \\ \frac{1}{2} - \beta D + \frac{1}{2} \ln 2\beta D + \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln \left(1 - \frac{1}{D} \sum_r \cos p_r \right) & \beta \geq \beta_c \end{cases} \quad (6.16)$$

$$u_2 = \begin{cases} \frac{1}{\gamma D} - \frac{1}{2\beta D} & \beta \leq \beta_c \\ 1 - \frac{1}{2\beta D} & \beta \geq \beta_c \end{cases} \quad (6.17)$$

where $\gamma = 1/\alpha$ is a solution of

$$2\beta = \gamma f_D(\gamma) \quad (6.18)$$

As it is well known, the action S_2 yields the phase structure of the spherical model²⁰⁾, and the same comments apply as for the corresponding CP^{N-1} case. When $D = 2$, asymptotic freedom and the phenomenology described in Section 4 and in Ref. 6) are recovered: the $O(N)$ models keep behaving (in the large N limit) as the corresponding $CP^{N/2-1}$.

The same can be said when we consider the one-dimensional $O(N)$ models. Again the reduction to single-link models can be performed^{20),17)} and a solution can be found for arbitrary values of N . We refer to Appendix B for the relevant external field integrals.

7. - THE MEAN FIELD APPROACH IN THE $1/N$ EXPANSION

It is instructive to compare the results we have obtained so far with what would have been predicted by the mean field approximation.

In the case of CP^{N-1} models with action S_1 , we first rapidly sketch a "naïve" deduction of the mean field results, then give a formal derivation within the scheme usually employed in order to turn the mean field approximation into a loop expansion. The naïve mean field equation takes the form:

$$\langle z_i \bar{z}_j \rangle = \frac{\int d\mu(z) z_i \bar{z}_j \exp \{2N\beta d z_i \bar{z}_j \langle z_j \bar{z}_i \rangle\}}{\int d\mu(z) \exp \{2N\beta d z_i \bar{z}_j \langle z_j \bar{z}_i \rangle\}} \quad (7.1)$$

with the condition that

$$\langle z_i \bar{z}_i \rangle = 1 \quad (7.2)$$

We can try and solve it by the ansatz²¹⁾

$$\langle z_i \bar{z}_j \rangle = A P_{ij} + B \delta_{ij} \quad (7.3)$$

when P_{ij} is a projection operator. Two conditions then follow from Eqs. (7.1) and (7.2):

$$A P_{ij} + B \delta_{ij} = \frac{\int d\mu(z) z_i \bar{z}_j \exp \{2N\beta d A P_{ij} z_i \bar{z}_j\}}{\int d\mu(z) \exp \{2N\beta d A P_{ij} z_i \bar{z}_j\}} \quad (7.4a)$$

$$A + NB = 1 \quad (7.4b)$$

By defining

$$A = \frac{x}{2\beta d} \quad B = \frac{1}{N} \left(1 - \frac{x}{2\beta d}\right) \quad (7.5)$$

$$\mathcal{E}(x) = \frac{1}{N} \ln \int d\mu(z) \exp N x P_{ij} z_i \bar{z}_j \quad (7.6)$$

The mean field equations reduce to

$$x = \frac{1}{N-1} \left[N \frac{\partial \mathcal{E}}{\partial x} - 1 \right] 2\beta d \quad (7.7)$$

$\partial \mathcal{E} / \partial x$ can be computed and found to be, to all orders in $1/N$ perturbation theory (see Appendix A)

$$\frac{\partial \mathcal{E}}{\partial x} = 1 - \frac{N-1}{N} \frac{1}{x} \quad (7.8)$$

Therefore,

$$x = 2\beta d \left(1 - \frac{1}{x}\right) \quad \bar{x} = \beta d \left(1 + \sqrt{1 - \frac{2}{\beta d}}\right) \quad (7.9)$$

In the large N limit, this simple result is found to be exact to all perturbative orders in a $1/N$ (weak coupling) expansion. Let us now sketch its formal derivation and compute the first correction around the naïve mean field result. The reformulation of the theory in terms of unconstrained variables is based on a standard trick²²⁾:

$$1 = \int dV_{ij} \delta(z_i \bar{z}_j - V_{ij}) = N \int dV_{ij} \frac{d\alpha_{ji}}{2\pi i} \exp N \alpha_{ji} (z_i \bar{z}_j - V_{ij}) \quad (7.10)$$

$$\begin{aligned} Z_N &= \int d\mu(\bar{z}_i) \exp N \beta \sum_{i,j} (z_i^n \bar{z}_j^n) (z_j^{n+\mu} \bar{z}_i^{n+\mu}) = \\ &= \int d\mu(\bar{z}_i^n) N \frac{d\alpha_{ji}^{ij}}{2\pi i} dV_{ij} \exp N \left[\sum_{j,i} \alpha_{ji}^{ij} z_i^n \bar{z}_j^n - \sum_{j,i} \alpha_{ji}^{ij} V_{ij}^{nn} + \beta \sum_{i,j} V_{ij}^{nn} V_{ji}^{n+\mu, n+\mu} \right] \end{aligned} \quad (7.11)$$

We can define the one-link free energy

$$\exp N w_n(\alpha_{ji}^{ij}) = \int d\mu(\bar{z}_i^n) \exp N \alpha_{ji}^{ij} z_i^n \bar{z}_j^n \quad (7.12)$$

and rewrite the free energy for the CP^{N-1} models in the form

$$F_N = \frac{1}{N} \ln \int \prod_{ij} dV_{ij} \exp N \left[\sum_i w_i(\alpha_{ij}) - \sum_{ij} \alpha_{ij}^n V_{ij}^n + \beta \sum_{\mu} V_{ij}^n V_{ji}^{n+\mu} \right] \quad (7.13)$$

The mean field equations are equivalent to the saddle point equations resulting from Eq. (7.13)

$$V_{ij}^n = \frac{\partial w_i}{\partial \alpha_{ij}^n} \quad (7.14a)$$

$$\alpha_{ji}^n = \beta \sum_{\mu} (V_{ji}^{n+\mu} + V_{ji}^{n-\mu}) \quad (7.14b)$$

that is, assuming translation invariance for the solutions:

$$\alpha_{ji} = 2\beta d \frac{\partial w}{\partial \alpha_{ij}} \quad (7.15)$$

Further progress requires explicit knowledge of $w(a)$. In the Appendix B, we show that, to all order in the $1/N$ expression [that is removing only terms that are $O(e^{-N})$] one obtains w as a function of the eigenvalues of α_{ij} :

$$w(\alpha_{ij}) = \alpha_0 - \frac{1}{N} \sum_{i=1}^{N-1} \ln(\alpha_0 - \alpha_i) - 1 \quad (7.16)$$

and Eq. (7.15) is rephrased in terms of the eigenvalues:

$$\alpha_i = 2\beta d \frac{\partial w}{\partial \alpha_i} \quad (7.17)$$

Equation (7.17) implies that all α_k ($k = 1, \dots, N-1$) take the same value and

$$\alpha_0 = \frac{2\beta d}{N} \left[N + \frac{1-N}{\alpha_0 - \alpha_k} \right] \quad (7.18a)$$

$$\alpha_k = \frac{2\beta d}{N} \frac{1}{\alpha_0 - \alpha_k} \quad (7.18b)$$

solved by

$$\alpha_0 = 2\beta d \left(1 - \frac{N-1}{N\bar{x}} \right) \quad (7.19a)$$

$$\alpha_k = \frac{2\beta d}{N\bar{x}} \quad (7.19b)$$

where $\bar{x}$ is defined by Eq. (7.9)

It is immediate to recover from Eqs. (7.19) the perturbative results presented in Section 3. In particular, we observe that from Eq. (7.13) and the equations of motion we may obtain expressions for the free energy and the internal energy (per unit volume and per unit link)

$$\frac{F_N}{V} = w(\alpha_{ij}) - \frac{1}{4\beta d} \alpha_{ji} \alpha_{ij} = \frac{N-1}{N} \left[\frac{\bar{\alpha}-1}{\bar{\alpha}} - \ln \bar{\alpha} \right] + \frac{\beta d-1}{N} \quad (7.20)$$

$$U = \frac{1}{d} \frac{\partial}{\partial \beta} \left(\frac{F_N}{V} \right) = \frac{\alpha_{ji} \alpha_{ij}}{(2\beta d)^2} = \frac{1}{N} + \frac{N-1}{N} \frac{\bar{\alpha}-1}{2\beta d} \quad (7.21)$$

The large N limit of the (mean field) internal energy agrees with the corresponding exact perturbative value, while there is a discrepancy in the expression for the free energy, whose exact (perturbative) value is

$$F_{\text{pert}}(N=\infty) = F_{\text{mf}}(N=\infty) - \int \prod_{k=1}^d \frac{d\theta_k}{2\pi} \ln \left(1 - \frac{1}{d} \sum_k \cos \theta_k \right) \quad (7.22)$$

This constant is exactly reproduced by the one-loop correction to the mean field result (and it is actually zero in the $d \rightarrow \infty$ limit). The one-loop correction is computed by considering the quadratic expansion around the effective action:

$$\alpha_{ij} = \alpha_{ij}^0 + \gamma_{ij} N^{-\frac{1}{2}} \quad (7.23a)$$

$$V_{ij} = V_{ij}^0 + y_{ij} N^{-\frac{1}{2}} \quad (7.23b)$$

$$Z = \exp N F_0 \int \frac{d\gamma_n^{ij}}{2\pi i} dy_n^{ij} \exp \frac{1}{2} \left[\sum_n \gamma_n^{ij} \frac{\partial^2 W}{\partial \alpha_{ij} \partial \alpha_{lm}} \gamma_n^{lm} - 2 \sum_n \gamma_n^{ij} y_n^{ji} + 2\beta \sum_{n\mu} y_n^{ij} y_{n+\mu}^{ji} \right] \quad (7.24)$$

and after diagonalization of the saddle point solutions and some algebra, one finds:

$$Z = \exp N F_0 \int \frac{d\tilde{\gamma}_n^{kk'}}{2\pi i} d\tilde{y}_n^{kk'} \exp \frac{1}{2} \left[\sum_n \sum_{\substack{k,k' \\ k' \neq k}} \tilde{\gamma}_n^{k'k} \left(\frac{\partial W}{\partial \alpha_k} - \frac{\partial W}{\partial \alpha_{k'}} \right) \tilde{\gamma}_n^{kk'} \right. \\ \left. + \sum_n \sum_{kk'} \tilde{\gamma}_n^{kk} \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_{k'}} \tilde{\gamma}_n^{k'k'} - 2 \sum_n \sum_{kk'} \tilde{\gamma}_n^{k'k} \tilde{y}_n^{kk'} + 2\beta \sum_{n\mu} \tilde{y}_n^{kk'} \tilde{y}_{n+\mu}^{k'k} \right] \quad (7.25)$$

We can perform the Gaussian integrations and obtain:

$$F = F_0 - \frac{1}{2N} \sum_{\mathbf{k}, \mathbf{k}'} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{\frac{\partial W}{\partial a_{\mathbf{k}}} - \frac{\partial W}{\partial a_{\mathbf{k}'}}}{a_{\mathbf{k}} - a_{\mathbf{k}'}} 2\beta \sum_m \cos p_m \right) - \frac{1}{2N} \int \frac{d^d p}{(2\pi)^d} \text{Tr} \ln \left(\delta_{\mathbf{k}\mathbf{k}'} - \frac{\partial^2 W}{\partial a_{\mathbf{k}} \partial a_{\mathbf{k}'}} 2\beta \sum_m \cos p_m \right) \quad (7.26)$$

We can now use a few explicit properties of the solutions:

$$\frac{1}{a_{\mathbf{k}} - a_{\mathbf{k}'}} \left(\frac{\partial W}{\partial a_{\mathbf{k}}} - \frac{\partial W}{\partial a_{\mathbf{k}'}} \right) = \begin{cases} \frac{1}{2\beta d} & \text{when } \mathbf{k} \text{ or } \mathbf{k}' = 0 \\ \frac{1}{N\bar{\alpha}^2} & \text{when } \mathbf{k}, \mathbf{k}' \neq 0 \end{cases} \quad (7.27)$$

$$\frac{\partial^2 W}{\partial a_0 \partial a_0} = \frac{N-1}{N\bar{\alpha}^2} \quad \frac{\partial^2 W}{\partial a_0 \partial a_{\mathbf{k}}} = -\frac{1}{N\bar{\alpha}^2} \quad \frac{\partial^2 W}{\partial a_{\mathbf{k}} \partial a_{\mathbf{k}'}} = \frac{1}{N\bar{\alpha}^2} \delta_{\mathbf{k}\mathbf{k}'} \quad (7.28)$$

and finally obtain

$$F = F_0 - \frac{N-1}{N} \int \ln \left(1 - \frac{1}{d} \sum_m \cos p_m \right) - \frac{N-2}{N} \int \ln \left(1 - \frac{2\beta}{N\bar{\alpha}^2} \sum_m \cos p_m \right) - \frac{1}{2N} \int \ln \left(1 - \frac{2\beta}{\bar{\alpha}^2} \sum_m \cos p_m \right) \quad (7.29)$$

In the large N limit Eq. (7.22) is reproduced, while for $N = 1$ all corrections vanish, as expected. A more complete derivation of these results is presented in Appendix C. A completely analogous scheme can be followed in the case of the $O(N)$ models with action S_1 , with the exception that there are no $1/N$ corrections to all orders for $w(\alpha)$. For a discussion, see Appendix B. When we choose the quadratic action S_2 , the one-link integral (with the first $1/N$ correction) turns out to be essentially the same for $\mathbb{CP}^{N-1}$ and $O(2N)$ models. For simplicity, we shall focus on the $O(N)$ case, where

$$w(h) = \frac{1}{N} \ln \int d\mu(\varphi_i) \exp N h_i \varphi_i = \frac{1}{2} \sqrt{1+4h^2} - \frac{1}{2} - \frac{1}{2} \ln \left(\frac{1}{2} \sqrt{1+4h^2} + \frac{1}{2} \right) - \frac{1}{N} \left[\frac{1}{2} \ln \sqrt{1+4h^2} + \ln \frac{\sqrt{1+4h^2} - 1}{2h^2} \right] \quad (7.30)$$

and there are $1/N$ corrections to all orders. The mean field equation takes the form

$$h = 2\beta d \frac{\partial w}{\partial h} \quad (7.31)$$

and is solved (in the large N limit) by

$$h = \sqrt{2\beta d(2\beta d - 1)} \quad (7.32)$$

The resulting mean field values for the free and internal energies are

$$F_{mf} = (\beta d - \frac{1}{2}) - \frac{1}{2} \ln 2\beta d \quad (7.33a)$$

$$\frac{1}{d} \frac{\partial F}{\partial \beta} = 1 - \frac{1}{2\beta d} \quad (7.33b)$$

The same behaviour is shown as that resulting from perturbative weak coupling analysis. The one-loop correction to the mean field is

$$-\frac{1}{2N} \int \frac{d^d p}{(2\pi)^d} \text{Tr} \ln \left[\delta_{ij} - 2\beta \frac{\partial^2 w}{\partial h_i \partial h_j} \sum_k \cos p_k \right] \quad (7.34)$$

and since the large N limit saddle point gives

$$\frac{\partial^2 w}{\partial h_i \partial h_j} = \delta_{ij} \frac{1}{2\beta d} \quad (7.35)$$

the standard constant perturbative term is reproduced

$$F = F_0 - \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln \left(1 - \frac{1}{d} \sum_k \cos p_k \right) \quad (7.36)$$

In the conclusion of this section, a few general remarks are due. First of all, we have always discussed the mean field approximation in the weak coupling region. As one can easily see from Eqs. (7.9) and (7.32), a phase transition occurs when $\beta d = 2$ and $1/2$ respectively and it is first order for the quartic actions and second order for the quadratic ones. The mean field strong coupling phase in the large N limit is a trivial one:

$$w = 0 \quad F = 0 \quad U = 0 \quad (7.37)$$

consistently with our previous strong coupling analysis.

If we compare the mean field results to the exact results when $N \rightarrow \infty$, we notice the following features. When $d > 2$ the mean field (one-loop) contribution reproduces exactly the weak coupling (and strong coupling) phase. When $d = 2$ the mean field reproduces the perturbative tail in the weak coupling phase exactly. However, it says nothing about the non-perturbative part of the solution. The situation when $d = 1$ is peculiar, because the $d = 1$ solution [both for CP^{N-1} and for $O(N)$ models] turns out to coincide, for arbitrary N , with the single link solution^{17),20)}, such that the identity

$$F_{d=1}(\beta) = F^{(1)}(\beta) = w(\beta) \quad (7.38)$$

holds. This result is not reproduced by the mean field result. It has, however, been observed recently that not even standard perturbation theory (in weak coupling) can reproduce the exact $d = 1$ result, and it rather converges towards the mean field prediction.

The agreement between mean field and perturbation theory, and their disagreement with $d = 1$ exact results, are explained by their common nature of analytic continuations to low d of results that are defined around $d = \infty$.

We do not understand why the mean field perturbation series truncates after one loop in the large N limit. It is, however, apparent that further contributions would appear as perturbative terms in the β^{-1} series, and therefore they must vanish because the solution is already (perturbatively) exact. It is worth noticing that even for finite N , Eq. (7.21) already reproduces all known terms in the perturbative expansion ($O(\beta^{-3})$). We, therefore, expect the $O(1/N)$ mean field corrections to be automatically of higher order also in the β^{-1} series.

In conclusion, it appears as if the random field transform employed to define the mean field expansion managed to remove the non-perturbative contributions to the free energy (at least in the large N limit). If this phenomenon is confirmed in gauge theories, the mean field approach might become an efficient way of isolating (and subtracting) the perturbative tails in the physical quantities for asymptotically free theories.

8. - LATTICE ACTIONS FOR THE SUPERSYMMETRIC CP^{N-1} MODELS

In this section we want to discuss briefly the possibility of considering supersymmetric extensions of the CP^{N-1} models on the lattice, in analogy with the discussion given in Ref. 6) for the $O(N)$ models.

Let us remind the continuum version of the CP^{N-1} models with supersymmetry:

$$\begin{aligned} \mathcal{L} = & \overline{D}_\mu \bar{z} D_\mu z - i \bar{\psi} \not{\partial} \psi + \frac{1}{4} [(\bar{\psi} \psi)^2 + (\bar{\psi} \gamma_5 \psi)^2 - (\bar{\psi} \gamma_\mu \psi)^2] = \\ & = \partial_\mu \bar{z} \partial_\mu z - i \bar{\psi} \not{\partial} \psi - \left[\frac{1}{2} \bar{\psi} \gamma_\mu \psi + \frac{\bar{z} \partial_\mu z - z \partial_\mu \bar{z}}{2i} \right]^2 + \frac{1}{4} [\bar{\psi} \psi - i \bar{\psi} \gamma_5 \psi]^2 \end{aligned} \quad (8.1)$$

supplemented with the constraints

$$\bar{z} z = 1 \quad \bar{z} \psi = \bar{\psi} z = 0 \quad (8.2)$$

As usual, there is a wide choice of lattice actions that correspond to the previous expression in the naive continuum limit, but we would like to consider those preserving most properties of the continuum theory. Using the large N limit as a test of our choice, we shall consider the possibility of preserving the $U(1)$ gauge invariance of the model and of realizing explicit supersymmetry properties on the saddle point solutions.

The request of gauge invariance puts very strong constraints on the form of the lattice action. Actually, the only solution to these constraints that we have found admitting the correct naive continuum limit and involving only z and ψ fields, is

$$\begin{aligned} S = N\beta \left\{ \sum_{n,\mu} \left[(\bar{z}_n z_{n+\mu} + \frac{i}{2} \bar{\psi}_n \gamma_\mu \psi_{n+\mu}) (\bar{z}_{n+\mu} z_n - \frac{i}{2} \bar{\psi}_{n+\mu} \gamma_\mu \psi_n) - 1 \right] \right. \\ \left. - \frac{1}{4} \sum_n [(\bar{\psi}_n \psi_n)^2 + (\bar{\psi}_n \gamma_5 \psi_n)^2] \right\} \end{aligned} \quad (8.3)$$

In order to consider the large N limit of this model, we introduce Lagrange multipliers

$$\begin{aligned} S = N \left\{ \sum_{n,\mu} \left[\bar{\lambda}_{n\mu} (\bar{z}_{n+\mu} z_n - \frac{i}{2} \bar{\psi}_{n+\mu} \gamma_\mu \psi_n) + \lambda_{n\mu} (\bar{z}_n z_{n+\mu} + \frac{i}{2} \bar{\psi}_n \gamma_\mu \psi_{n+\mu}) - \lambda_{n\mu} \bar{\lambda}_{n\mu} \right] \right. \\ + \sum_n \left[\frac{\phi_n}{2} (\bar{\psi}_n \psi_n + i \bar{\psi}_n \gamma_5 \psi_n) + \frac{\bar{\phi}_n}{2} (\bar{\psi}_n \psi_n - i \bar{\psi}_n \gamma_5 \psi_n) \right. \\ \left. \left. - \beta \bar{\phi}_n \phi_n + i\beta \sum_n a_n (\bar{z}_n z_n - 1) - \beta \right] + i\beta \sum_n (\bar{z}_n \psi_n \bar{z}_n + \psi_n \bar{\psi}_n z_n) \right\} \end{aligned} \quad (8.4)$$

and integrate over z_n and ψ_n with the standard large N translationally invariant ansatz, thus obtaining the effective action

$$S_{\text{eff}} = N \left\{ \beta(\alpha - \phi^2) - \text{Tr} \ln [\alpha - 2\lambda_\mu \cos p_\mu + \lambda_\mu \lambda_\mu + 2] \right. \\ \left. + \text{Tr} \ln [\varphi + \lambda_\mu \lambda_\mu \sin p_\mu] \right\} \quad (8.5)$$

The equations of motion are

$$\beta = \int \frac{1}{\alpha - 2\lambda_\mu \sum_\mu \cos p_\mu + 2 \lambda_\mu \lambda_\mu + 2} \quad (8.6)$$

$$\beta \phi = \int \frac{\phi}{\phi^2 + \lambda_\mu^2 \sum_\mu \sin^2 p_\mu} \quad (8.7)$$

$$\int \frac{2 \sum_\mu \cos p_\mu - 4\lambda_\mu}{\alpha + 2 + 2\lambda_\mu^2 - 2\lambda_\mu \sum_\mu \cos p_\mu} + \int \frac{2\lambda_\mu \sum_\mu \sin^2 p_\mu}{\phi^2 + \lambda_\mu^2 \sum_\mu \sin^2 p_\mu} = 0 \quad (8.8)$$

There are two different phases: the strong coupling solution is

$$\lambda = 0 \quad \alpha = \frac{1}{\beta} - 2 \quad \phi = \frac{1}{\beta} \frac{1}{2} \quad (8.9)$$

and the free energy takes the value

$$F = - \frac{S}{N} = 2\beta \quad (8.10)$$

In the weak coupling regime, it is possible to derive the equation

$$\alpha + 2 - 2\lambda^2 + \beta - 2\phi^2 = 0 \quad (8.11)$$

and after substitution the equations reduce to

$$\beta = \int \frac{1}{\phi^2 + \lambda_\mu^2 \sum_\mu \sin^2 p_\mu} \quad (8.12)$$

$$\beta = \int \frac{1}{\lambda_\mu^2 - \lambda_\mu + \frac{\phi^2}{2} - \frac{1}{4\beta} + \lambda_\mu \sum_\mu \sin^2 p_\mu} \quad (8.13)$$

The perturbative solution is

$$\varphi = 0 \quad \lambda = \frac{1 + \sqrt{1 + \frac{1}{\beta}}}{2} \quad \alpha = \sqrt{1 + \frac{1}{\beta}} - 1 - \frac{1}{2\beta} \quad (8.14)$$

and the free energy does not vanish.

This last feature is unpleasant in comparison with what happened for the $O(N)$ models: we do not find in this case explicit restoration of supersymmetry in the large N limit. This fact motivated the search for a different action fulfilling the previously described property, even if this forced us to relax the condition of explicit gauge invariance. We finally obtained the following (unpleasant-looking) action:

$$S' = N\beta \sum_{n\mu} \left[\frac{\bar{z}_{n+\mu} z_{n-\mu} + \bar{z}_{n-\mu} z_{n+\mu} - 2}{4} \right] - i\bar{\psi}_n \gamma^\mu \frac{(\psi_{n+\mu} - \psi_{n-\mu})}{2} \\ - \frac{1}{4} \left[\bar{z}_n z_{n+\mu} - z_n \bar{z}_{n+\mu} + i\bar{\psi}_n \gamma^\mu \psi_n \right]^2 - \frac{1}{4} \left[(\bar{\psi}_n \psi_n)^2 + (\bar{\psi}_n \gamma_5 \psi_n)^2 \right] \quad (8.15)$$

By rescaling the fields and introducing Lagrange multipliers, it turns into

$$S' = \frac{N}{4} \left\{ \sum_{n\mu} \left[(\bar{z}_{n+\mu} z_{n-\mu} + \bar{z}_{n-\mu} z_{n+\mu} - 2) - 2i\bar{\psi}_n \gamma^\mu (\psi_{n+\mu} - \psi_{n-\mu}) \right. \right. \\ \left. \left. + 2i\lambda_{n\mu} (\bar{z}_n z_{n+\mu} - z_n \bar{z}_{n+\mu} + i\bar{\psi}_n \gamma^\mu \psi_n) - \lambda_{n\mu} \lambda_{n\mu} + \phi_n (\bar{\psi}_n \psi_n + i\bar{\psi}_n \gamma_5 \psi_n) \right. \right. \\ \left. \left. + \bar{\phi}_n (\bar{\psi}_n \psi_n - i\bar{\psi}_n \gamma_5 \psi_n) \right] + \sum_n \left[i\beta \frac{q_n}{4} (\bar{z}_n z_n - 1) + iC_n \psi_n \bar{z}_n + i\bar{C}_n z_n \bar{\psi}_n - \frac{\beta}{4} \phi_n \bar{\phi}_n \right] \right\} \quad (8.16)$$

By introducing the large N saddle point ansatz

$$-i\alpha_n = \alpha \quad \lambda_{n\mu} = \lambda_\mu \quad C_n = \bar{C}_n = 0 \quad \phi_n = \bar{\phi}_n = \phi \quad (8.17)$$

We obtained the following effective action (in momentum representation)

$$S'_{\text{eff}} = \frac{\beta}{4} (\alpha - \phi^2) - \text{Tr} \ln [\alpha - 4\lambda_\mu \sin p_\mu - \cos 2p_\mu + \lambda_\mu \lambda_\mu + 2] \\ + \text{Tr} \ln [2\gamma^\mu \sin p_\mu + \phi - \gamma_\mu \lambda_\mu] \quad (8.18)$$

and after a rescaling of the momentum integration $P_\mu \rightarrow P_\mu/2$

$$S'_{\text{eff}} = \frac{1}{4} \left[(\alpha - \phi^2) \beta - \text{Tr} \ln \left[\alpha + \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)^2 \right] + \text{Tr} \ln \left[\phi^2 + \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)^2 \right] \right] \quad (8.19)$$

The resulting equations of motion

$$\beta = \int \frac{1}{\alpha + \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)^2} \quad (8.20)$$

$$\beta \phi = \int \frac{\phi}{\phi^2 + \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)^2}$$

$$\int \frac{2 \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)}{\alpha + \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)^2} = \int \frac{2 \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)}{\phi^2 + \sum_\mu (2 \sin \frac{P_\mu}{2} - \lambda_\mu)^2} \quad (8.21)$$

have an obvious solution:

$$\alpha = \phi^2 \quad \lambda_\mu \text{ arbitrary} \quad (8.22)$$

So that to our surprise not only the free energy turns out to be identically zero (and supersymmetry is restored on the lattice on the large N limit) but also gauge invariance, that was not present in our action, appears in the solution, manifesting itself in the arbitrariness of λ_μ .

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APPENDIX A

In this appendix, we briefly perform the large N expansion and we compute some physical quantities in the continuum theory.

The Lagrangian of the CP^{N-1} model is given by

$$\mathcal{L} = \frac{1}{g} \overline{D_\mu z} D_\mu z \quad (A.1)$$

where

$$D_\mu = \partial_\mu + iA_\mu \quad (A.2)$$

and the field z satisfies the following constraint:

$$\bar{z} \cdot z = 1 \quad (A.3)$$

The generating functional of the Euclidean Green function is given by:

$$Z(J, \bar{J}) = \int D z D \bar{z} D A_\mu D \alpha e^{-\frac{1}{g} \int d^d x \overline{D_\mu z} D_\mu z + \int d^d x i \alpha (\bar{z} z - 1) + \int d^d x [J \bar{z} + \bar{J} z]} \quad (A.4)$$

Rescaling the fields in the following way:

$$z \rightarrow g^{1/2} z \quad \alpha \rightarrow \frac{\alpha}{g} \quad (A.5)$$

and performing the integration over the z fields, we can rewrite (A.4) as follows:

$$Z(J, \bar{J}) = \int D A_\mu D \alpha e^{-N [\text{Tr} \ln \Delta + \int d^d x i \alpha(x) \beta]} e^{g \int d^d x \int d^d y \bar{J}(x) \Delta^{-1}(x, y) J(y)} \quad (A.6)$$

where

$$\Delta = -D^\mu D_\mu + i \alpha + m^2 \quad (A.7)$$

At large N we get the following saddle point condition:

$$\frac{1}{gN} \equiv \beta = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} = \frac{\Gamma(1 - \frac{D}{2})}{(4\pi)^{\frac{D}{2}} (m^2)^{1 - \frac{D}{2}}} \quad (A.8)$$

where the integral in (A.8) has been dimensionally regularized.

We can compute Green functions involving the fields z and $\bar{z}$ by differentiating (A.6) with respect to the sources J and $\bar{J}$. At large N we get for instance:

$$\langle \bar{z}^\alpha(x) z^\beta(y) \rangle = g \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(x-y)}}{p^2 + m^2} \delta^{\alpha\beta} \quad (A.9)$$

$$\begin{aligned} \langle \bar{z}^\alpha(x) z^\gamma(u) z^\beta(y) z^\delta(v) \rangle &= g^2 \delta^{\alpha\beta} \delta^{\gamma\delta} \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(v-u)}}{p^2 + m^2} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq(y-x)}}{q^2 + m^2} + \\ &+ g^2 \delta^{\alpha\gamma} \delta^{\beta\delta} \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(v-x)}}{p^2 + m^2} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq(y-u)}}{q^2 + m^2} \end{aligned} \quad (A.10)$$

From (A.10), we can easily compute the magnetic susceptibility at large N , that is given by:

$$\chi = \langle \bar{z}(x) \lambda^i z(x) \int d^D y \bar{z}(y) \lambda^j z(y) \rangle = g^2 \text{Tr}(\lambda^i \lambda^j) \frac{1}{4\pi m^2} \quad (A.11)$$

where λ^i are the generators of $SU(N)$ and the vacuum expectation value of the energy momentum tensor

$$\begin{aligned} \langle \Theta_{\mu\nu} \rangle &= \frac{1}{g} \langle -\overline{D_\mu z^a} D_\nu z^a + \frac{1}{2} \delta_{\mu\nu} \overline{D^2 z^a} D_\rho z^a \rangle = \\ &= -N \int \frac{d^D p}{(2\pi)^D} \frac{(p_\mu p_\nu - \frac{1}{2} p^2 \delta_{\mu\nu})}{p^2 + m^2} \end{aligned} \quad (A.12)$$

The previous integral can be easily computed and we get:

$$\langle \Theta_{\mu\nu} \rangle = N \frac{\delta_{\mu\nu}}{8\pi} m^2 \quad (A.13)$$

In particular the trace of the energy momentum tensor

$$\langle \Theta_\mu^\mu \rangle = \frac{N}{4\pi} m^2 \quad (\text{A.14})$$

is not vanishing, as a consequence of the fact that the conformal invariance of the classical theory is broken by the quantum corrections.

We can also compute at large N the analogous in the CP^{N-1} model of the gluon condensate and we get:

$$\langle \overline{D_\mu z^a} D_\mu z^a \rangle = gN \int \frac{d^D p}{(2\pi)^D} \frac{p^2}{p^2 + m^2} = -m^2 \quad (\text{A.15})$$

where we used for gN the saddle point condition (A.8) and we made at the end of the calculation the limit $D \rightarrow 2$. Comparing (A.14) and (A.15), we get in the leading order in N that:

$$\Theta_\mu^\mu = \frac{\beta(g)}{g} \overline{D_\mu z} D_\mu z \quad (\text{A.16})$$

being the one loop β function given by

$$\beta(g) = -\frac{N}{4\pi} g^2 \quad (\text{A.17})$$

APPENDIX B

B.1 Useful integrals

$$\int_0^{2\pi} \frac{d^D p}{(2\pi)^D} \ln [1 - \gamma \sum_{\mu} \cos p_{\mu}] = \int_0^{\infty} \frac{dt}{t} e^{-t} [1 - I_0(\gamma t)^D] \quad (B.1)$$

$$\int_0^{2\pi} \frac{d^D p}{(2\pi)^D} \frac{1}{1 - \gamma \sum_{\mu} \cos p_{\mu}} = \int_0^{\infty} dt e^{-t} [I_0(\gamma t)]^D \quad (B.2)$$

$$\int_0^{\infty} \frac{dt}{t} e^{-t} (1 - I_0(\gamma t)) = \ln \frac{1 + \sqrt{1 - \gamma^2}}{2} \quad (B.3)$$

$$\int_0^{\infty} dt e^{-t} I_0(\gamma t) = \frac{1}{\sqrt{1 - \gamma^2}} \quad (B.4)$$

$$\int_0^{\infty} \frac{dt}{t} e^{-t} (1 - I_0(\gamma t)^2) = \int_0^{2\pi} \frac{dp}{2\pi} \ln \frac{1 + \sqrt{1 - 4\gamma^2 \sin^2 p}}{2} \quad (B.5)$$

$$\int_0^{\infty} dt e^{-t} I_0(\gamma t)^2 = \int_0^{2\pi} \frac{dp}{2\pi} \frac{1}{\sqrt{1 - 4\gamma^2 \sin^2 p}} = \frac{2}{\pi} K(2\gamma) \quad (B.6)$$

$$\int_0^{2\pi} \frac{dp}{2\pi} \frac{e^{imp}}{1 - \gamma \cos p} = \frac{1}{\sqrt{1 - \gamma^2}} \left[\frac{\sqrt{1 - \gamma^2} - 1}{\gamma} \right]^m$$

B.2 One-link integrals

Quartic interaction (S_1)

$$\begin{aligned} w(\alpha_{ji}) &= \frac{1}{N} \ln \int \frac{dz d\bar{z}}{(2\pi i)^N} \Gamma(N) \delta(z\bar{z} - 1) \exp N \alpha_{ji} z_i \bar{z}_j = \\ &= \frac{1}{N} \ln \frac{\Gamma(N)}{N^{N-1}} \int \frac{d\alpha}{2\pi} \exp \{ -iN\alpha - \sum_i \ln(-i\alpha - \alpha_i) \} \end{aligned} \quad (B.7)$$

where we have introduced a Lagrange multiplier α for the constraint, we have integrated over the z fields and expressed the result in terms of the eigenvalues α_i of the matrix α_{ji} . Therefore

$$w(\alpha_{ji}) = \frac{1}{N} \ln \frac{\Gamma(N)}{N^{N-1}} \sum_{i=0}^{N-1} \frac{e^{N\alpha_i}}{\prod_{j \neq i} (\alpha_i - \alpha_j)} \quad (B.8)$$

In the $1/N$ expansion, one might at first conjecture that all the eigenvalues α_i take the values that are $O(1)$ at the mean field saddle point. It is, however, easy to show that the effective action reaches its absolute minimum when only one eigenvalue is $O(1)$ and all the others are $O(1/N)$: this phenomenon is related to the fact that $z_i \hat{z}_j$ is itself a projection operator and the same is expected to happen (in leading order) for its expectation value [in this context, see also Ref. 21)]. Under this assumption, Eq. (B.8) is $1/N$ expandable and the result is:

$$W(\alpha_{ji}) = \alpha_0 - \frac{1}{N} \sum_{i=1}^{N-1} \ln(\alpha_0 - \alpha_i) - 1 + O(e^{-N}) \quad (B.9)$$

In the special case when α_{ij} is proportional to a projection operator:

$$\alpha_{ji} = x P_{ji} \quad (B.10)$$

$$W(\alpha_{ji}) \rightarrow \mathcal{E}_N(x) = \frac{1}{N} \ln \Gamma(N) \left[\frac{\exp Nx}{(Nx)^{N-1}} - \sum_{n=0}^{N-2} \frac{(Nx)^{n-N+1}}{n!} \right] \quad (B.11)$$

and in the large N limit

$$\mathcal{E}(x) = x - \frac{N-1}{N} \ln x - 1 + \frac{1}{2N} \ln 2\pi N + O(e^{-N}) \quad (B.12)$$

$$\frac{\partial \mathcal{E}}{\partial x} = 1 - \frac{N-1}{N} \frac{1}{x} + O(e^{-N}) \quad (B.13)$$

and when $x = \beta$ we obtain from Eq. (B.12) the $1/N$ expansion of the one-dimensional free energy,

$$F = 1 - \beta + \frac{N-1}{N} \ln \beta - \frac{1}{2N} \ln 2\pi N + O(e^{-N}) \quad (B.14)$$

to be compared with Eq. (3.28). When we restrict the integration in (B.7) to real variables only, that is when we consider

$$\begin{aligned} W(\alpha_{ji}) &= \frac{1}{N} \ln \int d\vec{s} \delta(\vec{s}^2 - 1) \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} \exp \frac{N}{2} \alpha_{ji} s_i s_j = \\ &= \frac{1}{N} \ln \frac{\Gamma(\frac{N}{2})}{(\frac{N}{2})^{\frac{N}{2}-1}} \int \frac{d\alpha}{2\pi} \exp -i\frac{N}{2} \alpha - \frac{1}{2} \sum_i \ln(-i\alpha - \alpha_i) \end{aligned} \quad (B.15)$$

we are no longer led to an explicitly integrable expression. However, in the special case, when α_{ij} is proportional to a projector, the asymptotic form of Eq. (B.15) can be found to be

$$w(x) = \frac{x}{2} - \frac{1}{2} - \frac{1}{2} \ln x \quad (B.16)$$

The large N limit is therefore the same as that of Eq. (B.12) and the one-dimensional free energy is correctly reproduced. However, in this case $1/N$ corrections appear to all orders in the expansion of $w(\alpha)$.

Quadratic interaction (S_2)

Let us focus first on the integral over real variables:

$$w(h_i) = \frac{1}{N} \ln \int \frac{ds_i}{\pi^{N/2}} \Gamma\left(\frac{N}{2}\right) \exp N h_i s_i \delta(s_i; s_i - 1) \quad (B.17)$$

This integral is most easily evaluated by changing variables to polar co-ordinates and fixing $\ln = h \delta_{ii}$

$$w(h) = \frac{1}{N} \ln \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})} \int \sin^{N-2} \theta e^{N h \cos \theta} d\theta = \frac{1}{N} \ln \Gamma\left(\frac{N}{2}\right) \frac{I_{\frac{N-1}{2}}(N h)}{\left(\frac{N h}{2}\right)^{\frac{N-1}{2}}} \quad (B.18)$$

In order to perform a $1/N$ expansion, we use the representation

$$w(h) = \frac{1}{N} \ln \int \frac{d\alpha}{\pi^{N/2}} \exp \left\{ -i N \alpha - \frac{N}{2} \ln -2i\alpha + \frac{i N h^2}{4\alpha} - \frac{N}{2} \right\} \quad (B.19)$$

Determination of the saddle point and evaluation of the quadratic fluctuations lead to:

$$w(h) = \frac{1}{2} \sqrt{1+4h^2} - \frac{1}{2} - \frac{1}{2} \ln \left(\frac{1}{2} \sqrt{1+4h^2} + \frac{1}{2} \right) - \frac{1}{N} \left[\frac{1}{2} \ln \sqrt{1+4h^2} + \ln \frac{\sqrt{1+4h^2}-1}{2h^2} \right] + O\left(\frac{1}{N^2}\right) \quad (B.20)$$

The choice $h = \beta$ leads to a determination of the one dimensional free energy. The extension to complex variables is trivial, and we only quote the results:

$$\begin{aligned} w(h, \bar{h}) &= \frac{1}{N} \ln \int \frac{dz d\bar{z}}{(2\pi i)^N} \Gamma(N) \exp N (z\bar{h} + h\bar{z}) \delta(z\bar{z}-1) = \\ &= \frac{1}{N} \ln \Gamma(N) \frac{I_{N-1}(2Nh)}{(Nh)^{N-1}} \end{aligned} \quad (B.21)$$

and in the large N limit²³⁾

$$w(k) = \sqrt{1+4k^2} - 1 + \ln \left(\frac{1}{2} \sqrt{1+4k^2} + \frac{1}{2} \right) + O\left(\frac{1}{N}\right) \quad (\text{B.22})$$

reproducing the result for the one-dimensional free energy Eq. (3.31).

APPENDIX C

COMPUTATION OF THE QUADRATIC FLUCTUATIONS AROUND THE MEAN FIELD

Equation (7.24) leads to the Gaussian integral

$$\int \frac{d\gamma_n^{ij}}{2\pi i} d\gamma_n^{ij} \exp \frac{1}{2} \left[\sum_n \gamma_n^{ij} \frac{\partial^2 W}{\partial \alpha_{ij} \partial \alpha_{lm}} \gamma_n^{lm} - 2 \sum_n \gamma_n^{ij} \gamma_n^{ji} + 2\beta \sum_{n\mu} \gamma_n^{ij} \gamma_{n+\mu}^{ji} \right] \quad (C.1)$$

We introduce the unitary matrices R_{ij} that diagonalize the saddle point solutions α_{ij} :

$$\alpha_{ij}^0 = \sum_k R_{ik} R_{jk}^* \alpha_k \quad (C.2)$$

$$\frac{\partial W}{\partial \alpha_{ij}} = \sum_k R_{jk} R_{ik}^* \frac{\partial W}{\partial \alpha_k} \quad (C.3)$$

$$\frac{\partial^2 W}{\partial \alpha_{ij} \partial \alpha_{lm}} = \sum_{k, k'} \frac{R_{jk} R_{lk}^* R_{mk} R_{l'k'}^*}{\alpha_k - \alpha_{k'}} \left(\frac{\partial W}{\partial \alpha_k} - \frac{\partial W}{\partial \alpha_{k'}} \right) + \sum_{k, k'} R_{jk} R_{lk}^* R_{mk} R_{l'k'}^* \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_{k'}} \quad (C.4)$$

It is convenient to define the new integration variables

$$\tilde{\gamma}_n^{pq} = R_{ip}^* R_{jq} \gamma_n^{ij} \quad \tilde{\gamma}_n^{qp} = R_{ip} R_{jq}^* \gamma_n^{ji} \quad (C.5)$$

such that the quadratic action now reads:

$$\begin{aligned} & \frac{1}{2} \left[\sum_n \sum_{k, k'} \tilde{\gamma}_n^{k'k} \left(\frac{\partial W}{\partial \alpha_k} - \frac{\partial W}{\partial \alpha_{k'}} \right) \tilde{\gamma}_n^{kk'} + \sum_n \sum_{k, k'} \tilde{\gamma}_n^{kk} \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_{k'}} \tilde{\gamma}_n^{k'k'} \right. \\ & \left. - 2 \sum_n \sum_{k, k'} \tilde{\gamma}_n^{k'k} \tilde{\gamma}_n^{kk'} + 2\beta \sum_{n\mu} \sum_{ij} \tilde{\gamma}_n^{ij} \tilde{\gamma}_{n+\mu}^{ji} \right] \end{aligned} \quad (C.6)$$

We can now decompose the integration in the form:

$$\begin{aligned} & \int \prod_{k \neq k'} \frac{d\tilde{\gamma}_n^{kk'}}{2\pi i} d\tilde{\gamma}_n^{kk'} \exp \frac{1}{2} \left[\sum_n \tilde{\gamma}_n^{k'k} \frac{\partial W}{\partial \alpha_k} - \frac{\partial W}{\partial \alpha_{k'}} \tilde{\gamma}_n^{kk'} - 2 \tilde{\gamma}_n^{k'k} \tilde{\gamma}_n^{kk'} + 2\beta \sum_{\mu} \tilde{\gamma}_n^{kk'} \tilde{\gamma}_{n+\mu}^{k'k} \right] \\ & \int \prod_k \frac{d\tilde{\gamma}_n^{kk}}{2\pi i} d\tilde{\gamma}_n^{kk} \exp \frac{1}{2} \left[\sum_n \tilde{\gamma}_n^{kk} \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_{k'}} \tilde{\gamma}_n^{k'k'} - 2 \tilde{\gamma}_n^{kk} \tilde{\gamma}_n^{kk} + 2\beta \sum_{\mu} \tilde{\gamma}_n^{kk} \tilde{\gamma}_{n+\mu}^{kk} \right] \end{aligned} \quad (C.7)$$

and by standard techniques, we now obtain

$$\exp \left\{ -\frac{1}{2} \sum_{k,k'} \int \frac{d^0 p}{(2\pi)^0} \ln \left(1 - \frac{\frac{\partial w}{\partial a_k} - \frac{\partial w}{\partial a_{k'}}}{a_k - a_{k'}} 2\beta \sum_m \cos p_m \right) \right. \\ \left. - \frac{1}{2} \int \frac{d^0 p}{(2\pi)^0} \text{Tr} \ln \left(\delta_{kk'} - \frac{\frac{\partial^2 w}{\partial a_k \partial a_{k'}}}{a_k - a_{k'}} 2\beta \sum_m \cos p_m \right) \right\} \quad (C.8)$$

Now notice that

$$\frac{1}{a_k - a_{k'}} \left(\frac{\partial w}{\partial a_k} - \frac{\partial w}{\partial a_{k'}} \right) = \begin{cases} \frac{1}{2\beta d} & \text{when } k \text{ or } k' = 0 \\ \frac{1}{N\bar{x}^2} & \text{when } k, k' \neq 0 \end{cases} \quad (C.9)$$

$$\frac{\partial^2 w}{\partial a_0 \partial a_0} = \frac{N-1}{N\bar{x}^2} \quad \frac{\partial^2 w}{\partial a_0 \partial a_k} = -\frac{1}{N\bar{x}^2} \quad \frac{\partial^2 w}{\partial a_k \partial a_{k'}} = \frac{1}{N\bar{x}^2} \delta_{kk'} \quad (C.10)$$

and obtain

$$\exp \left\{ -(N-1) \int \frac{d^0 p}{(2\pi)^0} \ln \left(1 - \frac{1}{d} \sum_\mu \cos p_\mu \right) - \frac{(N-1)(N-2)}{2} \int \frac{d^0 p}{(2\pi)^0} \ln \left(1 - \frac{2\beta}{N\bar{x}^2} \sum_\mu \cos p_\mu \right) \right. \\ \left. - \frac{(N-2)}{2} \int \frac{d^0 p}{(2\pi)^0} \ln \left(1 - \frac{2\beta}{N\bar{x}^2} \sum_\mu \cos p_\mu \right) - \frac{1}{2} \int \frac{d^0 p}{(2\pi)^0} \ln \left(1 - \frac{2\beta}{\bar{x}^2} \sum_\mu \cos p_\mu \right) \right\} \quad (C.11)$$

where we have used the fact that

$$\det \left[\mathbb{1} - A \begin{pmatrix} N-1 & -1 \\ -1 & \mathbb{1} \end{pmatrix} \right] = (1-NA)(1-A)^{N-2} \quad (C.12)$$

Therefore, Eq. (7.29) is proved.

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FIGURE CAPTIONS

Fig. 3.1 : For $D = 4$ in the case of the action S_1 :

a) the free energy $F_1(\beta)$, Eq. (3.18);

b) the internal energy, Eq. (3.19), showing the first order phase transition.

The dashed line is the continuation of the analytic weak coupling solution in the unphysical region.

Fig. 3.2 : For $D = 4$ in the case of the action S_2 :

a) the free energy $F_2(\beta)$, Eq. (3.20);

b) the internal energy U_2 , Eq. (3.21).

Fig. 4.1 : Plot of β as function of $4\gamma^2$, Eq. (4.5).

Fig. 4.2 : The free energy $F(\beta)$ in the $D = 2$ CP^{N-1} model with quartic action, Eq. (4.1).

Fig. 3.1

Fig. 3.2

Fig. 4.1

Fig. 4.2