SPINOR EQUATIONS FOR BINARY ATOMS

H. Pilkuhn^{*)}

CERN - Geneva

A B S T R A C T

New relativistic equations are presented for atoms composed of two particles of spins < 1 and arbitrary mass ratios. The equation for one spin- $\frac{1}{2}$ particle has exact solutions for the point Coulomb potential. The equation for two spin- $\frac{1}{2}$ particles contains no tensor force and predicts a new value for the hyperfine quadrupole moment.

*)

On leave of absence from University of Karlsruhe, Federal Republic of Germany.

1. - INTRODUCTION

Binary atoms consist of two particles a and b of arbitrary masses m_a , m_b and spins S_a and S_b . This paper contains two new relativistic equations for such atoms, one for $S_a = \frac{1}{2}$, $S_b = 0$, the other for $S_a = S_b = \frac{1}{2}$. The case $S_a = S_b = 0$ has already been treated by Todorov¹⁾⁻³⁾, but has not yet found any applications.

Like several other equations, these equations reproduce the atomic energy levels to order α^4 (with the exception of 3S_1 positronium, where the annihilation graph is omitted). But other equations are either incomplete to order α^4 or cannot provide relativistic wave functions. Usually they contain the radial operator $V'/r = \alpha/r^3$ in the hyperfine structure. This operator is so singular for $r \rightarrow 0$ that it can be trusted only in first-order perturbation theory. The oldest and best-known example is the Breit equation⁴⁾, where the Breit operator makes little sense beyond first-order perturbation theory⁵⁾.

The new equation (Eq. (2.5) below) for $S_a = \frac{1}{2}$, $S_b = 0$ contains no such operator and has exact solutions. The equation (3.1) for $S_a = S_b = \frac{1}{2}$ still contains V'/r in the "recoil spin-orbit" term $\vec{L}(\vec{\sigma}_a + \vec{\sigma}_b)V'/rE$ (Eq. (3.9)), where $E \approx m_a + m_b$ is the total energy of the system. The resulting wave functions are still unreliable, except of course in places where the singular operator does not appear. One example is the hyperfine quadrupole moment of 3S_1 states (Eq. (5.14)), which differs from that of both the Breit⁶⁾ and Dirac⁷⁾ equations. The most interesting result for $S_a = S_b = \frac{1}{2}$ is the following, however: in a relativistic two-body description, vector-meson exchange produces the well-known tensor potential (3.10) only in connection with anomalous magnetic moment couplings (Sections 4 and 5). Thus, muonium has no tensor potential. The effects which are normally attributed to this potential arise from the combination of the (vectorial) Dirac operators of the two particles. The same is true for positronium, charmonium, etc., when the annihilation graph does not contribute, and also for pp scattering (see the discussion in Section 6).

In principle, the relativistic two-body problem is solved by the Bethe-Salpeter equation. But practical progress has been slow, particularly for comparable masses m_a and m_b . One needs a "start equation", upon which one can build a perturbation theory. Usually the start equation is a one-body

equation (Schrödinger, Klein-Gordon or Dirac) with exact solutions, but with different meanings of its parameters and eigenvalues: mass, charge and energy are replaced by reduced mass, effective charge and effective energy. These parameters are determined from a comparison with QED, either via the Bethe-Salpeter equation, or simply via the Born series of the scattering amplitude: the Born series of the effective particle scattering amplitude must agree with that of the covariant two-body scattering amplitude⁸⁾⁻¹⁰⁾. To order α^4 in the binding energy and for a point Coulomb potential, only the first Born approximation and gauge invariance are needed. When the effective potential V includes vacuum polarization or nuclear size, the second Born approximation produces small "recoil corrections" to V^2 . Such effects are neglected in the following; only the first Born approximation is used.

As an illustration of the method, we now rederive Todorov's equation³⁾ for $S_a = S_b = 0$ from the KG-equation and the Born approximation. We use units $\hbar = c = 1$, $e^2/4\pi = \alpha = 1/137$. The KG-equation for a particle of mass m_a and charge $-e$ in an external four-potential A^μ can be written as:

$$K_a \Psi = 0, \quad K_a = \pi_a^0{}^2 - \vec{\pi}_a^2 - m_a^2, \quad (1.1)$$

$$\pi_a^0 = i\partial_t - V, \quad \vec{\pi}_a = -i\nabla + e\vec{A}, \quad V = -eA^0. \quad (1.2)$$

We specialize A^μ to the static potential $\vec{A} = 0$, $A^0 = A^0(r)$, $\partial_t = E_a$, and replace m_a and E_a by unknown parameters μ_E and ε :

$$K\Psi = 0, \quad K = (\varepsilon - V)^2 + \nabla^2 - \mu_E^2. \quad (1.3)$$

The point Coulomb potential corresponds to

$$V_c = \hat{\alpha}/r, \quad \hat{\alpha} = -Ze\alpha, \quad (1.4)$$

where $-Ze^2 = Z_a Z_b e^2$ is the product of charges of particles a and b (we thus refrain from using $\hat{\alpha}$ as an unknown parameter^{8),9)}, for a reason which will become clear below). We also define

$$\varepsilon^2 - \mu_E^2 = k^2, \quad (1.5)$$

such that the Klein-Gordonian K becomes

$$K = k^2 - 2\varepsilon V + V^2 + \nabla^2. \quad (1.6)$$

For $V = 0$, it has plane-wave solutions $\exp(i\varepsilon t - i\vec{k}\vec{r})$. The connection between this formalism and relativistic two-body scattering is given in the centre-of-mass (Fig. 1). We assume $V(r \rightarrow \infty) = 0$, such that the asymptotic two-particle states have $\vec{k}_a = -\vec{k}_b = \vec{k}$ before scattering and $\vec{k}'_a = -\vec{k}'_b = \vec{k}'$ afterwards. Thus k^2 plays a central rôle in the equation. It is the eigenvalue of the operator $-\nabla^2 + 2\varepsilon V - V^2$; bound states have negative k^2 .

The total energy E of the system appears less directly:

$$E = E_a + E_b = (m_a^2 + k^2)^{1/2} + (m_b^2 + k^2)^{1/2}. \quad (1.7)$$

Note that V^2 in (1.6) follows from the gauge invariance of the original KG-equation (1.1). Corrections to V^2 for $V \neq V_c$ have been derived in Ref. 10). If one regards $\hat{\alpha}$ as a free parameter, then its variation must be cancelled by another parameter in front of V^2 .

Once we have settled on V^2 , we merely have to compute the one-foton exchange matrix element $T_{ab}^{(1)}$ for elastic scattering of particles a and b in the centre-of-mass:

$$T_{ab}^{(1)} = -Ze^2 \mathcal{J}_a \cdot \mathcal{J}_b / q^2, \quad \vec{q} = \vec{k} - \vec{k}', \quad (1.8)$$

where, for spinless particles, we have

$$\mathcal{J}_a^\mu = p_a^\mu + p_a'^\mu = (2E_a, \vec{k} + \vec{k}'), \quad \mathcal{J}_b^\mu = (2E_b, -\vec{k} - \vec{k}'), \quad (1.9)$$

$$\mathcal{J}_a \cdot \mathcal{J}_b = \mathcal{J}_a^\mu \mathcal{J}_{b\mu} = 4(E_a E_b + k^2) - q^2. \quad (1.10)$$

The differential centre-of-mass cross-section is

$$d\sigma_{ab}/d\Omega = |T_{ab}/8\pi E|^2, \quad (1.11)$$

which must be reproduced by the scattering amplitude F of the effective particle on the fictitious external potential A^0 . As the cross-section is $d\sigma/d\Omega = |F|^2$ in this case, we must have $F = T/8\pi E$, or in the one-photon-exchange approximation

$$F^{(1)} = T_{ab}^{(1)}/8\pi E = \hat{\alpha} \mathbf{j}_a \cdot \mathbf{j}_b / 2q^2 E \quad (1.12)$$

Turning now to Todorov's equation (1.3), we find of course that F is given by the zero-component of the effective current operator,

$$F^{(1)} = \hat{\alpha} \mathcal{J}^0 / q^2, \quad \mathcal{J}^0 = \varepsilon + \varepsilon' = 2\varepsilon, \quad (1.13)$$

because of energy conservation in scattering on a static field, $\varepsilon' = \varepsilon$. Insertion of (1.10) into (1.12) gives

$$\frac{1}{2} \mathcal{J}^0 = \frac{1}{4E} \mathbf{j}_a \cdot \mathbf{j}_b = (E_a E_b + k^2)/E = \varepsilon, \quad (1.14)$$

plus a correction $-q^2/4E$ which accounts for the term $-q^2$ in Eq. (1.10). The Fourier transform of $-q^2/4Eq^2 = -1/4E$ is a $\delta(\vec{r})/4E$ which should be added as an extra operator to (1.3)^{1), 2)}. We omit this term, as it does not appear in the spin case.

We see that both ε and E are relatively complicated functions of k^2 . It comes as a surprise that μ_E according to (1.5) is simply related to E :

$$\mu_E = (\varepsilon^2 - k^2)^{1/2} = m_a m_b / E = \mu m / E, \quad (1.15)$$

$$\mu \equiv m_a m_b / m, \quad m \equiv m_a + m_b. \quad (1.16)$$

μ is the non-relativistic reduced mass, μ_E is its relativistic generalization.

For $V = \hat{\alpha}/r$, the solution of (1.3) is trivial: $V^2 = \hat{\alpha}^2/r^2$ is combined with the centrifugal potential $\ell(\ell+1)/r^2$ into an effective centrifugal potential $\ell'(\ell'+1)/r^2$ for the radial wave function R :

$$\left[k^2 - 2\varepsilon V + r^{-1} \partial_r^2 r - \ell'(\ell'+1)/r^2 \right] R(r) = 0. \quad (1.17)$$

This is now an effective Schrödinger equation. The binding energies are computed in Appendix A.

2. - THE "KLEIN-DIRAC" EQUATION FOR $S_a = \frac{1}{2}$, $S_b = 0$

Here we need an equation which interpolates between the KG-equation for $m_a/m_b = \infty$ and the Dirac equation for $m_a/m_b = 0$, without introducing singular operators. We first take the Dirac equation for particle a in an external field

$$\pi_a^\mu \gamma_\mu \Psi_D = m_a \Psi_D \quad (2.1)$$

and put it into a form resembling (1.1). We use the "high-energy" representation of ϕ_D for this purpose:

$$\begin{aligned} (\pi_a^0 - \vec{\pi}_a \vec{\sigma}) \Psi_r &= m_a \Psi_\ell \\ (\pi_a^0 + \vec{\pi}_a \vec{\sigma}) \Psi_\ell &= m_a \Psi_r \end{aligned} \quad (2.2)$$

where $\vec{\sigma}$ are the Pauli matrices, ϕ_r is a right-handed or "undotted" spinor, and ϕ_ℓ is a left-handed or "dotted" spinor. Multiplying the first of these coupled equations by $\pi_a^0 + \vec{\pi}_a \vec{\sigma}$ and observing the second, one obtains an equation for ϕ_r alone:

$$(\pi_a^0 + \vec{\pi}_a \vec{\sigma}) (\pi_a^0 - \vec{\pi}_a \vec{\sigma}) \Psi_r = m_a^2 \Psi_r. \quad (2.3)$$

Insertion of π_a^μ (1.2) and use of the Pauli commutation relations leads to

$$\left[K_a - e \vec{\sigma} (\vec{B} - i \vec{E}) \right] \Psi_r = 0, \quad (2.4)$$

with K_a = Klein-Gordonian (1.1) and $\vec{B} = \text{rot} \vec{A}$, $\vec{E} = -\nabla A^0 - \partial_t \vec{A}$. This is the quadratic form of the Dirac equation. It has been used by Kramers¹¹⁾ for a late derivation of the Dirac equation; following a discussion of Brown¹²⁾, we shall call it the Kramers equation. Replacing e by ec with $0 \leq c \leq 1$, we

have an equation which interpolates between KG and Dirac, but which does not allow field quantization^{13),12)}. On the other hand, the modified equation can be used to calculate the motion of an effective particle in a static potential. Setting $\vec{A} = 0$ as before and $e\vec{E} = \nabla V = \hat{r}V'(r)$ with $\hat{r} = \vec{r}/r$, we arrive at the desired "Klein-Dirac" equation

$$(K + ic\sigma_r V') \Psi_r = 0, \quad \sigma_r = \vec{\sigma} \hat{r}, \quad (2.5)$$

with K given by (1.3) or (1.6). It remains to determine its two free parameters ε and c from $T_{ab}^{(1)}$, using

$$J_a^\mu = \bar{u}_a' \gamma^\mu u_a, \quad J_b^\mu = p_b^\mu + p_b'^\mu. \quad (2.6)$$

We use the "high-energy" representation of the plane-wave Dirac spinor u :

$$\begin{aligned} u_r(\vec{k}) &= (E_a + m_a + \vec{k} \vec{\sigma}) [2m_a(E_a + m_a)]^{-1/2} \chi_a \\ &= m_a^{-1/2} (E_a + \vec{k} \vec{\sigma})^{1/2} \chi_a \equiv A_a(\vec{k}) \chi_a, \end{aligned} \quad (2.7)$$

where χ_a is a Pauli spinor for particle a at rest and $A_a(\vec{k})$ is a "boost" to momentum $\vec{k}$. Below, we shall need an effective boost A for our KD-equation.

We note already now that any boost must obey

$$A(-\vec{k}) A(\vec{k}) = A(\vec{k}) A(-\vec{k}) = 1, \quad (2.8)$$

i.e., $A(-\vec{k})$ is the inverse of $A(\vec{k})$. The left-handed spinor u_λ is the parity transform of u_r :

$$u_\ell(\vec{k}) = A_a(-\vec{k}) \chi_a = A_a^T(-\vec{k}) u_r(\vec{k}) \quad (2.9)$$

which allows one to express the entire current $\bar{u}_a' \gamma^\mu u_a$ as follows:

$$J_a^\mu = \chi_a'^\dagger A_a(-\vec{k}') \hat{J}_a^\mu A_a(\vec{k}) \chi_a \quad (2.10)$$

$$\hat{J}_a^0 = E_a + E_a' - \vec{\sigma} \vec{q} \quad (2.11)$$

$$\hat{\vec{J}}_a = \vec{k} + \vec{k}' + i \vec{q} \times \vec{\sigma} - \vec{\sigma} (E_a - E_a') \quad (2.12)$$

A part of this transformation is the "Gordon transformation", which exhibits the matrix elements $p_a^\mu + p_a'^\mu$ of the spinless current. In Section 4, a straightforward derivation of these Feynman rules will be given, starting from the Kramers equation. The product of currents is now in the centre-of-mass system:

$$\hat{J}_a \cdot J_b = 2E_b (2E_a - \vec{q} \cdot \vec{\sigma}) + (\vec{k} + \vec{k}') \left[\vec{k} + \vec{k}' + i \vec{q} \times \vec{\sigma} \right] \quad (2.13)$$

The last term can be rewritten as $2i\vec{p} \times \vec{q} \cdot \vec{\sigma}$, with $\vec{p} = -i\nabla$. Using $\vec{r} \times \vec{p} = \vec{L}$, it gives rise to the usual $\vec{L} \cdot \vec{\sigma} / r^3$ operator for the binary atom (for example, $\mu^- \pi^+$ or $\pi^- \mu^+$ atoms). However, this operator can be eliminated by an appropriate choice of the boost A for the effective particle. We rewrite

$$i(\vec{k} + \vec{k}') \vec{q} \times \vec{\sigma} = 2\vec{k}' \cdot \vec{\sigma} \vec{k} \cdot \vec{\sigma} - 2\vec{k}' \cdot \vec{k}, \quad (2.14)$$

$$\frac{1}{2} \hat{J}_a \cdot J_b = E_b (2E_a - \vec{\sigma} \cdot \vec{q}) + k^2 + \vec{k}' \cdot \vec{\sigma} \vec{k} \cdot \vec{\sigma} \quad (2.15)$$

The calculation is easiest for helicity states $\chi_a(\vec{k}, \lambda)$, which satisfy

$$\vec{k} \cdot \vec{\sigma} \chi_a = 2k\lambda \chi_a, \quad \chi_a'^+ \vec{\sigma} \cdot \vec{k}' = 2k\lambda' \chi_a'^+, \quad (2.16)$$

with $\lambda = \pm \frac{1}{2}$, $\lambda' = \pm \frac{1}{2}$ and are eigenstates of the boost:

$$A_a(\vec{k}) \chi_a = m_a^{-1/2} (E_a + 2k\lambda)^{1/2} \chi_a \quad (2.17)$$

In this way we find

$$J_a \cdot J_b / 2E = 2(\varepsilon \delta_{\lambda\lambda'} + m_a E_b / E \delta_{\lambda, -\lambda'}) \chi_a'^+ \chi_a. \quad (2.18)$$

In the effective boost A, we replace $E_a \rightarrow \varepsilon$, $\vec{\sigma} \rightarrow c\vec{\sigma}$ and $m_a \rightarrow w$ = "boost-mass". The delicate point here is that the boostmass cannot be μ_E , because the boost must obey (2.8)

$$A(\vec{k}) = w^{-1/2} (\varepsilon + c \vec{k} \cdot \vec{\sigma})^{1/2}, \quad (2.19)$$

$$W = (\varepsilon^2 - c^2 k^2)^{1/2}. \quad (2.20)$$

Next, we have to compute the Born approximation of the effective particle scattering from (2.5). This will be done as a by-product in Section 4, but the result is obvious without calculation: only the zero-component of the effective current J^μ appears, and J^0 is obtained from (2.11) by replacing $E_a = E'_a \rightarrow \varepsilon$, $\vec{\sigma} \rightarrow c\vec{\sigma}$:

$$\hat{J}^0 = 2\varepsilon - c\vec{\sigma}\vec{q}, \quad (2.21)$$

$$J^0 = \chi_a'^+ A(-\vec{k}') \hat{J}^0 A(\vec{k}) \chi_a = 2(\varepsilon \delta_{\lambda\lambda'} + W \delta_{\lambda, -\lambda'}) \chi_a'^+ \chi_a. \quad (2.22)$$

It is not even necessary to calculate (2.22) from (2.21) and (2.19), because it follows also from (2.18) in the static limit $E_b/E = 1$ by replacing $m_a \rightarrow w$. From (2.22) it follows that the full Born amplitude (2.18) is reproduced for

$$W = m_a E_b / E. \quad (2.23)$$

The new parameter c is now fixed by (2.20) as

$$c = k^{-1} (\varepsilon^2 - W^2)^{1/2} = (1 - m_a^2 / E^2)^{1/2}. \quad (2.24)$$

For electronic and muonic atoms, c is close to 1. For $\pi^- p$ and $K^- p$ atoms, on the other hand, particle a is the proton (its anomalous magnetic moment is included in Section 5 below), and then c approaches zero. Particle a becomes non-relativistic in this limit, and E can be approximated by $m_a + E_b$:

$$c (m_a > m_b) \approx (2E_b/m - E_b^2/m^2)^{1/2}. \quad (2.25)$$

The effective boost A approaches the spinless particle boost $A(\vec{k}) \rightarrow A_0(\vec{k}) = 1$ in this limit, as expected. The spin reduction by a factor c was already discovered by Grotch and Yennie⁸⁾ (they treated the case $S_a = S_b = \frac{1}{2}$ but neglected the hyperfine interaction at this point). However, these authors used the Dirac equation as their start equation and had to describe the effect as well as possible by the parameters ε , μ_E and $\hat{\alpha}$. The resulting

inconsistencies to order $\hat{\alpha}^4$ appear in the spin-independent terms, particularly in the Lamb shift (see Appendix A). The Klein-Dirac equation (2.5) has an extra parameter (namely c) and is not equivalent to any Dirac equation.

Before leaving the subject, we show how (2.5) is reduced to the effective Schrödinger equation (1.17) for $V = V_c = \hat{\alpha}/r$: one has to diagonalize the operator $L^2 + ic\hat{\alpha}\sigma_r$. Coupling $\vec{L} + \vec{\sigma}/2 = \vec{J}$ as usual, and denoting the eigenstates of $\vec{J}^2$ and J_z by $\chi_\lambda^{jm_j}$ with $\lambda = j \pm \frac{1}{2}$, one has

$$\sigma_r \chi_{j \pm 1/2}^{jm_j} = - \chi_{j \mp 1/2}^{jm_j}. \quad (2.26)$$

In the space of these two states (j and m_j fixed), one has

$$L_{lj}^2 \equiv L^2 + ic\hat{\alpha}\sigma_r = \begin{pmatrix} (j+\frac{1}{2})(j+\frac{3}{2}) & -ic\hat{\alpha} \\ -ic\hat{\alpha} & j^2 - \frac{1}{4} \end{pmatrix} \quad (2.27)$$

The eigenvalues λ of this matrix are

$$\lambda = (j+\frac{1}{2})^2 \pm \gamma_c, \quad \gamma_c = [(j+\frac{1}{2})^2 - c^2 \hat{\alpha}^2]^{1/2}. \quad (2.28)$$

Denoting the total strength of r^{-2} by $\ell'(\ell'+1)$ as before, one finds

$$\ell'(\ell'+1) = \lambda - \hat{\alpha}^2 = \gamma^2 \pm \gamma_c, \quad (2.29)$$

$$\gamma = [(j+\frac{1}{2})^2 - \hat{\alpha}^2]^{1/2} \quad (2.30)$$

The wave functions are given in Eq. (C.1) below.

3. - THE BISPINOR EQUATION FOR $S_a = S_b = \frac{1}{2}$

In this case, the effective particle carries two spins, with Pauli spin operators $\vec{\sigma}_a$ and $\vec{\sigma}_b$. In order to reproduce the respective Dirac equations in the two static limits $m_a/m_b = 0$ and $m_a/m_b = \infty$, the start equation must have the form

$$[K + i (\sigma_{ar} c_a - \sigma_{br} c_b) V' - X] \Psi = 0, \quad (3.1)$$

where X is an operator that vanishes at both static limits. Ideally, X would again be proportional to V', in which case the spin-orbit operator in front of it must be a pseudoscalar, for example,

$$\sigma_{xr} = \vec{\sigma}_x \cdot \hat{r}, \quad \vec{\sigma}_x = \vec{\sigma}_a \times \vec{\sigma}_b. \quad (3.2)$$

Unfortunately, it seems that any combination of σ_{ar} , σ_{br} and σ_{xr} is insufficient. While I have found a matrix representation of the desired operator, I have been unable to express it in terms of $\vec{\sigma}_a$ and $\vec{\sigma}_b$.

Let us therefore immediately discuss the product $J_a \cdot J_b$, with $J_a^\mu = \bar{u}_a' \gamma^\mu u_a$ and $J_b^\mu = \bar{u}_b' \gamma^\mu u_b$. Using the representation (2.10) for J_a^μ in the centre-of-mass system and corresponding expressions for J_b^μ with $\vec{k}_b = -\vec{k}$,

$$\hat{J}_b^0 = 2E_b + \vec{\sigma}_b \cdot \vec{q}, \quad \hat{\vec{J}}_b = -\vec{k} - \vec{k}' - i \vec{q} \times \vec{\sigma}_b, \quad (3.3)$$

we obtain

$$\begin{aligned} \hat{J}_a \cdot \hat{J}_b &= (2E_a - \vec{\sigma}_a \cdot \vec{q})(2E_b + \vec{\sigma}_b \cdot \vec{q}) + (\vec{k} + \vec{k}')^2 + 2i(\vec{q} \times \vec{\sigma}_a) \cdot \vec{k} \\ &\quad + 2i(\vec{q} \times \vec{\sigma}_b) \cdot \vec{k} - (\vec{q} \times \vec{\sigma}_a) \cdot (\vec{q} \times \vec{\sigma}_b) \\ &= 4E_a E_b - 2E_b \vec{\sigma}_a \cdot \vec{q} + 2E_a \vec{\sigma}_b \cdot \vec{q} + 2i\vec{q} \cdot (\vec{k} \times \vec{q}) - q^2(1 + \vec{\sigma}_a \cdot \vec{\sigma}_b) \end{aligned} \quad (3.4)$$

$$\vec{\sigma} \equiv \vec{\sigma}_a + \vec{\sigma}_b. \quad (3.5)$$

In the last form of (3.4), we have used

$$(\vec{q} \times \vec{\sigma}_a) \cdot (\vec{q} \times \vec{\sigma}_b) + \vec{\sigma}_a \cdot \vec{q} \vec{\sigma}_b \cdot \vec{q} = q^2 \vec{\sigma}_a \cdot \vec{\sigma}_b \quad (3.6)$$

Taking now the effective particle boost as the product of the original single-particle boosts

$$A = A_a(\vec{k}) A_b(-\vec{k}) = (m_a m_b)^{-\frac{1}{2}} (E_a + \vec{k} \cdot \vec{\sigma}_a)^{\frac{1}{2}} (E_b - \vec{k} \cdot \vec{\sigma}_b)^{\frac{1}{2}}, \quad (3.7)$$

we immediately arrive at Eq. (3.1) with

$$c_a = E_b/E, \quad c_b = E_a/E, \quad (3.8)$$

$$\chi = \vec{L} \vec{\sigma} V'/Er - (1 + \vec{\sigma}_a \vec{\sigma}_b) \Delta V / 2E. \quad (3.9)$$

In the usual relativistic two-body equations, the term $(\vec{q} \times \vec{\sigma}_a)(\vec{q} \times \vec{\sigma}_b)$ leads to the well-known tensor potential

$$V_T = (3 \sigma_{ar} \sigma_{br} - \vec{\sigma}_a \vec{\sigma}_b) V' / 4 m_a m_b r. \quad (3.10)$$

This potential is absent here, due to the cancellation (3.6). The expert reader will of course wonder how (3.1) can reproduce the known hyperfine structure without V_T . The answer is that the diagonalization of the matrix $L_{\lambda j f}^2$ (Eq. (3.21) below) mixes matrix elements of σ_{ar} and σ_{br} , such that tensorial effects arise from the interplay of the two fundamental operators $\sigma_{ar} = \vec{\sigma}_a \hat{r}$ and $\sigma_{br} = \vec{\sigma}_b \hat{r}$.

Before solving the four-component equation (3.1), we must choose a basis for the four components. By rotational invariance, the total angular momentum

$$\vec{F} = \vec{L} + \frac{1}{2} \vec{\sigma} \quad (3.11)$$

is conserved, and all four components may be taken as eigenstates of F^2 and F_z , with fixed eigenvalues $f(f+1)$ and m_f as usual. They may also be taken as eigenstates of L^2 , with eigenvalues $\lambda(\lambda+1)$, $\lambda = f-1, f, f+1$. The states $\chi_{\lambda}^{f m_f}$ with $\lambda = f \pm 1$ are unique. For $m_f = f$, they are explicitly

$$\begin{aligned} \chi_{f-1}^{ff} &= Y_{f-1}^{f-1}(\hat{r}) \chi_{a+} \chi_{b+} \\ \chi_{f+1}^{ff} &= C_- Y_{f-1}^{f+1} \chi_{a+} \chi_{b+} + C_0 Y_f^{f+1} (\chi_{a+} \chi_{b-} + \chi_{a-} \chi_{b+}) \\ &\quad + C_+ Y_{f+1}^{f+1} \chi_{a-} \chi_{b-}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} C_- &= [(f+1)(2f+3)]^{-1/2}, \quad C_0 = -(f+\frac{1}{2})^{1/2} [(f+1)(2f+3)]^{-1/2}, \\ C_+ &= (2f+1)^{1/2} (2f+3)^{-1/2}. \end{aligned} \quad (3.13)$$

The value $\lambda = f$ occurs twice. An extra quantum number is needed here, which may be taken either as the total spin s (states $\chi_{\lambda s}^{fm_f}$), or the total angular momentum j of particle a (states $\chi_{\lambda, j}^{fm_f}$). The transformation from one basis to the other is

$$\chi_{f, f+1/2}^{fm_f} = \left(\frac{f}{2f+1}\right)^{1/2} \chi_{f, 1}^{fm_f} + \left(\frac{f+1}{2f+1}\right)^{1/2} \chi_{f, 0}^{fm_f} \quad (3.14)$$

$$\chi_{f, f-1/2}^{fm_f} = \left(\frac{f+1}{2f+1}\right)^{1/2} \chi_{f, 1}^{fm_f} - \left(\frac{f}{2f+1}\right)^{1/2} \chi_{f, 0}^{fm_f}. \quad (3.15)$$

All this is of course known from non-relativistic angular momentum addition. The only relativistic aspect is that the required equation mixes all four states. On the basis $\chi_{f, j}^{fm_f}$, the operators σ_{ar} and σ_{br} are

$$\sigma_{ar} = \begin{pmatrix} -\hat{\sigma} & 0 \\ 0 & -\hat{\sigma} \end{pmatrix}, \quad \sigma_{br} = \frac{1}{2f+1} \begin{pmatrix} \hat{\sigma} & -2F\sigma_0 \\ -2F\sigma_0 & -\hat{\sigma} \end{pmatrix} \quad (3.16)$$

$$\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = [f(f+1)]^{1/2}, \quad (3.17)$$

where the $\chi_{f, j}^{fm_f}$ are taken in the order $\chi_{f+1}^{fm_f}$, $\chi_{f, f+1/2}^{fm_f}$, $\chi_{f, f-1/2}^{fm_f}$, $\chi_{f-1}^{fm_f}$. For the point Coulomb potential, Eq. (3.1) can be solved to all orders in $\sigma_{ar} V'$ and $\sigma_{br} V'$, and to first order in the remaining operators. We do this by re-writing¹⁴⁾

$$\begin{aligned} \vec{L} \vec{\sigma} / r^3 &= -\hat{\alpha} \vec{L} \vec{\sigma} \mu / r^2 L^2 + \vec{L} \vec{\sigma} \gamma_r, \\ \gamma_r &= r^{-3} + \hat{\alpha} \mu / L^2 r^2, \end{aligned} \quad (3.18)$$

because the non-relativistic expectation values of γ_r vanish (remember that $\hat{\alpha}$ is negative). Normally, the requirement $\lambda > 0$ is added, but one may use (3.18) also for $\lambda = 0$, if one simultaneously omits the operator ΔV in (3.9). For the calculation of the hyperfine energies to order $\hat{\alpha}^4$, one may neglect γ_r altogether. We thus approximate (3.9) by

$$X = \hat{\alpha}^2 \frac{\hbar}{m} \vec{L} \vec{\sigma} (r^2 L^2)^{-1}. \quad (3.19)$$

For the point Coulomb potential, (3.1) is now of the form

$$[k^2 - 2\varepsilon V + r^{-1} \partial_r^2 r - r^{-2} (L_{eff}^2 - \hat{\alpha}^2)] \psi = 0 \quad (3.20)$$

$$L_{eff}^2 = L^2 + i\hat{\alpha} (c_a \sigma_{ar} - c_b \sigma_{br}) + \hat{\alpha}^2 \frac{\hbar}{m} \vec{L} \vec{\sigma} / L^2 \quad (3.21)$$

After diagonalization of L_{eff}^2 , the equation is again of the Schrödinger type (1.17), and the complete hyperfine structure follows, including a number of recoil effects which are absent in the Dirac equation with hyperfine interaction but present in the Breit equation (see Appendix A). As the form (3.19) of X is much nicer than (3.9), one could take the point of view that it is closer to "the correct" form. This may be so for $V = \hat{\alpha}/r$ but makes little sense for $V \neq \hat{\alpha}/r$. The only possible generalization of $\hat{\alpha}^2/r^2$ in (3.19) is V^2 , and this operator is excluded from a one-photon exchange amplitude. The correct procedure would be to rewrite the troublesome term $\vec{\sigma}(\vec{k} \times \vec{q})$ in a way analogous to (2.14) and to send the unwanted terms into a cleverly chosen boost $A(\vec{k})$.

4. - PARTICLE WITH ANOMALOUS MAGNETIC MOMENT IN EXTERNAL FIELD

Protons and heavier spin- $\frac{1}{2}$ nuclei have large anomalous magnetic moments κ , which are frequently expressed in terms of the nuclear g-factor

$$g = m^{-1} 2m_p Z (1 + \kappa) \quad (4.1)$$

m being the particle mass and m_p the proton mass. With this definition, protons and antiprotons have $\kappa = 1.79$. For ${}^3\text{He}$, the magnetic moment of -2.1276 nuclear magnetons implies $1 + \kappa = -3.185$, i.e.,

$$\kappa ({}^3\text{He}) = -4.185 \quad (4.2)$$

Before admitting such moments in binary atoms, we have to study the motion of the particle in an external field. We therefore refer to it as particle a for the time being. The relevant Dirac equation is

$$(\pi_a^\mu \gamma_\mu + \pi_a^\mu \pi_a^\nu \sigma_{\mu\nu} \kappa / 2m_a - m_a) \psi_D = 0 \quad (4.3)$$

$$2\sigma^{\mu\nu} = \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \quad (4.4)$$

with $\pi_a^\mu \pi_a^\nu \sigma_{\mu\nu} = \frac{1}{2} i F^{\mu\nu} \sigma_{\mu\nu}$ for $Z_a = -1$ and $F^{\mu\nu}$ = electromagnetic field. In the "high-energy" representation (2.2), the new operator is diagonal:

$$\pi_a^\mu \pi_a^\nu \sigma_{\mu\nu} = -e \vec{\sigma} \begin{pmatrix} \vec{B}^- & 0 \\ 0 & \vec{B}^+ \end{pmatrix}, \quad \vec{B}^\pm = \vec{B} \pm i \vec{E} \quad (4.5)$$

It is again possible to eliminate ϕ_ℓ , but the resulting equation for ϕ_r involves the ugly operator

$$\left(1 + \vec{\sigma} \vec{B}^+ \frac{\kappa e}{2m_a^2}\right)^{-1} = \left(1 - \vec{\sigma} \vec{B}^+ \frac{\kappa e}{2m_a^2}\right) \left(1 - \vec{B}^{+2} \frac{\kappa^2 e^2}{4m_a^4}\right)^{-1} \quad (4.6)$$

In view of the applications to binary atoms, it is appropriate to describe protons and antiprotons in external fields not by the generalized Dirac equation (4.3), but by a generalized Kramers equation. To find such an equation, we rewrite (2.3) in covariant form³⁾:

$$\pi_a^\mu \sigma_{\ell\mu} \pi_a^\nu \sigma_\nu \psi_r = m_a^2 \psi_r \quad (4.7)$$

$$\sigma^\nu = (1, \vec{\sigma}), \quad \sigma_\ell^\mu = (1, -\vec{\sigma}). \quad (4.8)$$

As $\sigma_\ell^\mu \sigma^\nu$ is obviously a tensor, we have two covariant operators at hand which are linear in $F^{\mu\nu}$. The first of these is

$$\sigma_\ell^\mu F_{\mu\nu} \sigma^\nu = 2i \vec{\sigma} \vec{B}^-, \quad (4.9)$$

which already occurs in the Kramers equation (2.4). A reduction of its strength leads to the Klein-Dirac equation and not to an anomalous magnetic moment. The other operator is

$$\sigma^\mu F_{\mu\nu} \sigma_\nu^\nu = 2i\vec{\sigma} \vec{B}^+. \quad (4.10)$$

It appears in the left-handed Kramers equation (the one for ψ_λ), and is not covariant when it operates directly on ψ_r . On the other hand, the operator $\pi_a^\nu \sigma_\nu$ transforms ψ_r into something left-handed (compare the first line of Eq. (2.2)), and therefore the following equation is Lorentz-invariant:

$$\left[\pi_a^\mu \sigma_{\ell\mu} (1 - e\kappa \vec{\sigma} \vec{B}^+ / 2m_a^2) \pi_a^\nu \sigma_\nu - e\kappa \vec{\sigma} \vec{B}^- / 2 - m_a^2 \right] \psi_r = 0 \quad (4.11)$$

This is no longer equivalent to any Dirac equation, the difference being precisely the denominator of (4.6). From the physical point of view, the difference is immaterial, as it is quadratic in $F^{\mu\nu}$. The anomalous magnetic moment is defined as the linear response to $F^{\mu\nu}$; quadratic terms go into the magnetic polarizability. Thus the value of the polarizability is different in different equations. From the mathematical point of view, (4.11) is simpler than (4.3). This becomes evident when one reduces the four coupled equations (4.3) to two coupled equations. Of course, neither (4.3) nor (4.11) is good enough for second quantization. The anomalous magnetic moment is accompanied by a form factor which reaches at least to distances m_a^{-1} , and which, for $r \rightarrow 0$, turns $\kappa \rightarrow 0$.

We now derive the Born approximation for antiproton scattering on a given field. We introduce the free particle Klein-Gordonian

$$K_a^{(0)} = -\partial_t^2 + \nabla^2 - m_a^2, \quad (4.12)$$

and the normal and anomalous interactions:

$$N = A_\mu \partial^\mu + \partial_\mu A^\mu - ieA^2 + i\vec{\sigma} \vec{B}^- \quad (4.13)$$

$$A = i\kappa (\pi_a^\mu \sigma_{\ell\mu} \vec{\sigma} \vec{B}^+ \pi_a^\nu \sigma_\nu + m_a^2 \vec{\sigma} \vec{B}^-) / 2m_a^2 \quad (4.14)$$

in which case (4.11) assumes the form

$$[K_a^{(0)} + ie (N + A)] \Psi_r = 0 \quad (4.15)$$

for a particle of charge $-e$ (the equation applies also to neutrons, with $N = 0$). Introducing the Green's function $\Delta(x)$ via $K_a^{(0)} \Delta = \delta_4(x)$, we rewrite (4.15) as an integral equation:

$$\Psi_r(x) = \Psi_0 + ie \int d^4y \Delta(x-y) (N+A) \Psi_r(y). \quad (4.16)$$

For $x_0 > y_0$, $\Delta(x-y)$ may be expanded in terms of a complete set of positive-energy states:

$$\Delta(x-y) = \int d^3p e^{-iP \cdot x} A_a(\vec{p}) \sum_{M=\pm 1/2} \lambda_a(M) \lambda_a^\dagger(M) A_a^{-1}(\vec{p}) e^{iP \cdot y} \quad (4.17)$$

We drop all terms which are quadratic in e , replace $\phi_r(y)$ by a plane wave spinor $A(\vec{k}) \chi_a(\lambda, \hat{k}) \exp i(-E_a t + \vec{k} \cdot \vec{y}) = \phi_{\vec{k}\lambda}(y)$ and project out of $\phi_r(x)$ a corresponding spinor $\phi_{\vec{k}'\lambda'}(x)$. The coefficient of this spinor is essentially the matrix element J_a^μ of the current operator. The various operators in N and A act as follows

$$\pi_a^\nu \sigma_\nu \rightarrow E_a - \vec{k} \cdot \vec{\sigma}, \quad \pi_a^\dagger \sigma_{\ell r} \rightarrow E'_a + \vec{k}' \cdot \vec{\sigma}, \quad (4.18)$$

$$\vec{B}^\pm \rightarrow -i\vec{q} \times \vec{A} \mp \vec{q} A^0 \mp \vec{A} (E_a - E'_a) \quad (4.19)$$

Integration over the time components gives $E'_a = P_0 = E_a$ for static fields, leading to

$$\hat{J}_a^0 = 2E_a - \vec{\sigma} \cdot \vec{q} + \frac{1}{2} \kappa (\vec{\sigma}_\kappa - \vec{\sigma}) \cdot \vec{q}, \quad (4.20)$$

$$\hat{J}_a^i = \vec{k} + \vec{k}' + i\vec{q} \times \vec{\sigma} + \frac{1}{2} i\kappa \vec{q} \times (\vec{\sigma}_\kappa + \vec{\sigma}), \quad (4.21)$$

$$\vec{\sigma}_\kappa = m_a^{-2} (E_a + \vec{k}' \cdot \vec{\sigma}) \vec{\sigma} (E_a - \vec{k} \cdot \vec{\sigma}). \quad (4.22)$$

It is useful to expand $\vec{\sigma}_k$ as follows:

$$\vec{\sigma}_k = \vec{\sigma} + m_a^{-2} E_a \left[i \vec{\sigma} \times (\vec{k} + \vec{k}') - \vec{q} \right] - m_a^{-2} \left[(\vec{k}' \cdot \vec{\sigma}) \vec{\sigma} (\vec{k} \cdot \vec{\sigma}) - k^2 \vec{\sigma} \right] \quad (4.23)$$

Near the non-relativistic limit, one may neglect the last square bracket of $\vec{\sigma}_k$ in $\hat{J}_a^0$:

$$\hat{J}_a^0 \approx 2 E_a - \vec{\sigma} \cdot \vec{q} - m_a^{-1} \kappa \left[i \vec{\sigma} \cdot (\vec{q} \times \vec{k}) + \frac{1}{2} q^2 \right] \quad (4.24)$$

and both square brackets in $\hat{J}_a$:

$$\hat{J}_a \approx \vec{k} + \vec{k}' + i \vec{q} \times \vec{\sigma} (1 + \kappa) \quad (4.25)$$

We shall also need $\vec{\sigma}_k$ in the helicity representation:

$$\begin{aligned} \vec{\sigma}_k &= m_a^{-2} (E_a + 2\lambda' k)(E_a - 2\lambda k) \vec{\sigma} \\ &= \vec{\sigma} \left[1 - 4 m_a^{-2} \lambda k (E_a - 2\lambda k) \delta_{\lambda, -\lambda'} \right] \end{aligned} \quad (4.26)$$

5. - ANOMALOUS MAGNETIC MOMENTS IN BINARY EQUATIONS.

ATOMIC QUADRUPOLE MOMENT

We begin with the case $S_a = S_b = \frac{1}{2}$, where the binary boost A (3.7) is simply the product of the usual single-particle boosts. We content ourselves with the non-relativistic reductions (4.24) and (4.25) for $\hat{J}_a^0$ and $\hat{J}_a$ and the corresponding expressions for $\hat{J}_b^0$ and $\hat{J}_b$:

$$\begin{aligned}
 \hat{J}_a \cdot \hat{J}_b &= \hat{J}_a^0 \hat{J}_b^0 - \vec{J}_a \cdot \vec{J}_b \\
 &\approx 4E_a E_b - 2E_b \vec{\sigma}_a \vec{q} + 2E_a \vec{\sigma}_b \vec{q} - \vec{\sigma}_a \vec{q} \cdot \vec{\sigma}_b \vec{q} \\
 &\quad - 2E_b m_a^{-1} \kappa_a (i \vec{\sigma}_a \vec{q} \times \vec{k} + \frac{1}{2} q^2) - 2E_a m_b^{-1} \kappa_b (i \vec{\sigma}_b \vec{q} \times \vec{k} + \frac{1}{2} q^2) \\
 &\quad + (\vec{k} + \vec{k}')^2 - 2(\vec{q} \times \vec{k}) (\vec{\sigma} + \kappa_a \vec{\sigma}_a + \kappa_b \vec{\sigma}_b) \\
 &\quad - (1 + \kappa_a)(1 + \kappa_b) (\vec{q} \times \vec{\sigma}_a) (\vec{q} \times \vec{\sigma}_b)
 \end{aligned} \tag{5.1}$$

The bispinor equation (3.1) now becomes

$$\left\{ K + i(E_b \sigma_{ar} - E_a \sigma_{br}) V'/m - X - 2\mu V_K \right\} \Psi = 0 \tag{5.2}$$

$$V_K = \kappa_a V_{LSa} + \kappa_b V_{LSb} + (\kappa_a + \kappa_b + \kappa_a \kappa_b) (V_T + \vec{\sigma}_a \vec{\sigma}_b \Delta V / 6 m_a m_b) \tag{5.3}$$

$$V_{LSa} = (r^{-1} V' \vec{L} \vec{\sigma}_a + \Delta V m_b / 2m) / 2\mu m_a \tag{5.4}$$

$$V_{LSb} = (r^{-1} V' \vec{L} \vec{\sigma}_b + \Delta V m_a / 2m) / 2\mu m_b \tag{5.5}$$

These are the usual spin-orbit and tensor potentials of the non-relativistic reduction of the Breit equation.

The Klein-Dirac equation is more complicated. The change to the boost (2.19) requires a change from $\vec{\sigma}_K$ to an effective operator

$$\begin{aligned}
 \vec{\sigma}_{Ke} &= w^{-2} (\varepsilon + c \vec{k}' \cdot \vec{\sigma}) \vec{\sigma} (\varepsilon - c \vec{k} \cdot \vec{\sigma}) \\
 &= \vec{\sigma} - w^{-2} c \varepsilon [i \vec{\sigma} \times (\vec{k} + \vec{k}') - \vec{q}] - w^{-2} c^2 [(\vec{k}' \cdot \vec{\sigma}) \vec{\sigma} (\vec{k} \cdot \vec{\sigma}) - k^2 \vec{\sigma}]
 \end{aligned} \tag{5.6}$$

and the effective $\hat{J}^0$ of (2.21) is replaced by

$$\begin{aligned}\hat{j}^0 &= 2\varepsilon - c \vec{\sigma} \vec{q} + i \frac{\kappa}{4\varepsilon} (\vec{k} + \vec{k}') \vec{q} \times (\vec{\sigma}_{ke} + \vec{\sigma}) + \kappa E_b \frac{w}{cm_a} E \vec{q} (\vec{\sigma}_{ke} - \vec{\sigma}) \\ &\approx 2\varepsilon - c \vec{\sigma} \vec{q} - \kappa [i \vec{\sigma} (\vec{q} \times \vec{k}) + q^2 m_b / 2m] / m_a\end{aligned}\quad (5.7)$$

The resulting additional spin-orbit potential is again $\kappa_a V_{LS_a}$.

Returning to $S_a = S_b = \frac{1}{2}$, we observe that a state with $f = \lambda + 1$, which non-relativistically has an angular dependence Y_m^λ (compare (3.12)), also has a component (3.13) with angular dependence $Y_m^{\lambda+2}$. For $\lambda = 0$ and $\kappa_a = \kappa_b = 0$, for example, the total spin-angular function χ is an eigenfunction of the operator L_{ojf}^2 (3.21) and has the form

$$\chi^{11} \approx \hat{\alpha}^2 a_2 \chi_2^{11} + i \hat{\alpha} a_{3/2} \chi_{3/2}^{11} + i \hat{\alpha} a_{1/2} \chi_{1/2}^{11} + \chi_0^{11} \quad (5.8)$$

where $\chi_{3/2}^{11}$ and $\chi_{1/2}^{11}$ are given in (3.14) and (3.15).

$\hat{\alpha}^2 a_2$ is the D state admixture to the S-state. If we are not interested in $a_{3/2}$ and $a_{1/2}$, we can omit the operator $\vec{L} \vec{\sigma} / L^2$ from the eigenvalue equation $L_{ojf}^2 \chi = 0(\hat{\alpha}^2) \chi$ and obtain

$$a_2 = 2^{-\frac{1}{2}} \mu / 9m \quad (5.9)$$

The D-state admixture is believed to cause an atomic quadrupole moment Q in an external electric field, which is taken as the expectation value of the operator $\hat{Q}$:

$$\hat{Q} = r^2 (3 \cos^2 \theta - 1) \quad (5.10)$$

$$\begin{aligned}Q &= \langle \chi | \hat{Q} | \chi \rangle = \langle r^2 \rangle 2a_2 \hat{\alpha}^2 \langle \chi_{f-1}^{ff} | 2P_2(\cos \theta) | \chi_{f+1}^{ff} \rangle \\ &= \langle r^2 \rangle 4a_2 \hat{\alpha}^2 C_- \int Y_{f-1}^{f-1} * P_2 Y_{f+1}^{f+1} d\Omega\end{aligned}\quad (5.11)$$

with C_- given in (3.13). The final result for 1S-states is

$$Q(\kappa=0) = 4(15 m_a m_b)^{-1} \quad (5.12)$$

This would be the hyperfine quadrupole moment of 1S muonium, say. For normal electronic atoms, the anomalous magnetic moment κ_b introduces the operator V_T of (3.10), which also admixes a D-state (that calculation is restricted to first-order perturbation theory, as long as κ is taken as a constant). In this case, the admixture has a different radial dependence from the unperturbed S-state, and the result for the 1S-state is ^{7),8)}

$$Q_\kappa = \kappa(3 m_a m_b)^{-1} \quad (5.13)$$

The total result for electronic atoms is thus

$$Q = (3 m_a m_b)^{-1} (0.8 - \kappa_b) \quad (5.14)$$

and differs drastically from the usual result where the entire effect is attributed to V_T . In that case, $0.8 - \kappa_b$ is obviously replaced by $-1 - \kappa_b$. Clearly, operators which produce identical energy levels to order $\hat{\alpha}^4$ can produce different wave functions. Our derivation of the 0.8 in (5.14) involves only the operators σ_{ar} and σ_{br} of the bispinor equation (3.1), which are known to be correct in the static limits. The usual Dirac equation with hyperfine interaction applies only to one static limit, and the usual Breit operators (5.3) are correct only for the anomalous magnetic moment.

However, a correct calculation of the electric quadrupole moment Q should perhaps not take Eq. (5.11) for granted, but start from the interaction of the composite particle with an external field. The quadrupole moment of the deuteron does come out of (5.11), but it is caused by pion exchange and undoubtedly due to V_T .

It appears that the atomic hyperfine quadrupole moment has not yet been measured. It could be measurable for example in Rydberg states of ^3He in sufficiently large ℓ -states of the outer electron. Note that $0.8 - \kappa_b = 4.985$ is large and sufficiently different from $-1 - \kappa_b = 3.185$ in this case.

6. - DISCUSSION OF RESULTS AND OUTLOOK

The new spinor equations are based on two-dimensional representations of the proper Lorentz group. They lack a linear parity transformation but are nevertheless parity-invariant. Due to their simple completeness relation,

$$\sum_{\lambda = \pm 1/2} \chi_{\lambda} \chi_{\lambda}^{\dagger} = 1, \quad (6.1)$$

their propagator (4.17) is that of a spinless particle. A clever choice of the associated boost $A(\vec{k})$ allows one to get rid of some unwanted interactions. For $S_a = \frac{1}{2}$, $S_b = 0$, the resulting Klein-Dirac equation (2.5) is unique and has exact solutions. The binding energies (A.4) are generally correct to order $\hat{\alpha}^4$. The spin structure (fine and hyperfine) sets in at order $\hat{\alpha}^4$. A good wave function should then permit a calculation of $\hat{\alpha}^6$ -corrections to the spin structure. This is done in Eq. (A.11) for the Lamb shift. The calculation of recoil corrections to the $\hat{\alpha}^6$ -hyperfine structure is in progress. Another test is provided by the vacuum polarization in exotic atoms, which can be incorporated in the potential V (Uehling potential). In this case, the spin structure tests the wave equation already to order $\hat{\alpha}^4$. Examples would be $\mu^{-}\pi^{+}$ and $\bar{p}^4\text{He}$ -atoms.

The case $S_a = S_b = \frac{1}{2}$ includes muonium, positronium and charmonium and is therefore more interesting. Equations (A.17) and (A.18) below include recoil corrections to the muonium hyperfine structure

$$\frac{2\mu/m}{f+1/2} \left[1 - 3 \frac{f(f+1)}{(f+1/2)^2} \frac{\mu}{m} \delta_{f,e} \right] \quad (6.2)$$

but it must be remembered that such expressions follow already from the generalized Breit equations¹⁵⁾.

The most important property of the bispinor equation (3.1) is its lack of tensor operators. It is easily checked that the matrix elements of the usual Dirac current for particle a can be put into the form

$$J_a^{\mu} = \bar{u}_a' \gamma^{\mu} u_a = u_{a\ell}'^{\dagger} \hat{J}_a^{\mu} u_{ar}, \quad (6.3)$$

with u_{ar} and $u_{a\ell}$ referring to the "high-energy" representation (2.7), (2.8) and $\hat{J}_a^{\mu}$ given by (2.11) and (2.12). Doing the same thing for particle b one

finds that $\hat{J}_a^0 \hat{J}_b^0$ and $\hat{J}_a^\mu \hat{J}_b^\mu$ contain the same tensor matrix element, such that it disappears from the scalar product $\hat{J}_a^\mu \hat{J}_b^\mu$. Why has this gone unnoticed for so many years? One reason is the popularity of the "low-energy representation" (B.2), which mixes the two inequivalent spinor representations.

The most striking consequence of the missing tensor operator is a new D-state admixture (5.9) to the ground state, which should imply a new quadrupole moment (5.13). Similar effects occur for ω - and ρ -exchange in nuclear bound states (deuteron), but the changes for ρ -exchange are less pronounced since the ρ -meson couples predominantly via the anomalous magnetic moment potential (5.3).

Other applications can be foreseen in positronium and charmonium. In such cases, the singlet-triplet basis $\chi_{f,0}, \chi_{f,1}$ is more appropriate than the basis $\chi_{f,j}$ with $j = f \pm \frac{1}{2}$ (3.14), (3.15). Taking the four states in order $\chi_{f,0}, \chi_{f+1}, \chi_{f,1}, \chi_{f-1}$, we have the following matrix representations of the operators entering $L_{\lambda f}^2$ of (3.21):

$$\frac{1}{m} (m_b \sigma_{ar} - m_a \sigma_{br}) = \frac{1}{(2f+1)^{1/2}} \begin{pmatrix} 0 & -(f+1)^{1/2} & 0 & f^{1/2} \\ -(f+1)^{1/2} & 0 & f^{1/2} \Delta \hat{m} & 0 \\ 0 & f^{1/2} \Delta \hat{m} & 0 & (f+1)^{1/2} \Delta \hat{m} \\ f^{1/2} & 0 & (f+1)^{1/2} \Delta \hat{m} & 0 \end{pmatrix}$$

$$\Delta \hat{m} \equiv (m_b - m_a)/m \quad (6.4)$$

$$L^2 + \hat{\alpha}^2 \frac{\mu}{m} \frac{\vec{L} \cdot \vec{\sigma}}{L^2} = \begin{pmatrix} f(f+1) & & & 0 \\ (f+1)(f+2) - \hat{\alpha}^2 \frac{\mu}{m} \frac{2}{f+1} & & & \\ & f(f+1) - \hat{\alpha}^2 \frac{\mu}{m} \frac{2}{f(f+1)} & & \\ 0 & & f^2 - f + \hat{\alpha}^2 \frac{\mu}{m} \frac{2}{f} & \end{pmatrix} \quad (6.5)$$

Notice that $\vec{L} \cdot \vec{\sigma}/L^2$ is well-defined even for $\lambda = 0$. The operator (6.5) may yield a good wave function only for $V = V_c = \hat{\alpha}/r$, but not, e.g., for vacuum polarization. Clearly, it would be best to have a wave equation of the type

(3.1) where X contains no bad operator at all.

ACKNOWLEDGEMENTS

The author would like to thank R. Glauber, M. Kolsrud, P. Lepage, J. Sapirstein, T. Fulton and D. Yennie for helpful comments.

This work has been completed during the author's sabbatical term at Harvard and CERN (earlier parts have appeared as Karlsruhe and Harvard preprints). It has been supported by the US Department of Energy under Contract No. AC 02-76 ERO 3064, by CERN and by a travel grant from the Deutsche Forschungsgemeinschaft.

APPENDIX A - LAMB SHIFT AND HYPERFINE STRUCTURE

The radial eigenvalue equation is always of the form (1.17) with non-integer orbital angular momentum ℓ' . It has the following regular solutions:

$$R = x^{\ell'} e^{-x/2} {}_1F_1(-n_r, 2\ell'+2, x), \quad x = -2ikr \quad (\text{A.1})$$

$$-n_r = \ell' + 1 + i\hat{\alpha}^2 \epsilon/k \quad (\text{A.2})$$

For bound states, n_r is a non-negative integer. The binding energy $E_B = E - m$ is given by ($n = n_r + \ell + 1 = \text{integer}$), $\Delta\ell = \ell' - \ell$:

$$E_B (1 + E_B/2m) + \mu = \mu \left[1 + \frac{\hat{\alpha}^2}{(n + \Delta\ell)^2} \right]^{-1/2} \equiv E_\mu \quad (\text{A.3})$$

The term $E_B^2/2m$ is the Bechert-Meixner recoil correction ¹⁶⁾ to order $\hat{\alpha}^4$. For our purposes, it will be sufficient to expand (A.3) to order $(\Delta\ell)^2$:

$$E_\mu - \mu = \frac{-\hat{\alpha}^2 \mu}{2(n + \Delta\ell)^2} \left[1 - \frac{3}{4} \left(\frac{\hat{\alpha}^2}{n + \Delta\ell} \right)^2 \right] = -\frac{\hat{\alpha}^2 \mu}{2n^2} \left(1 - \frac{3}{4} \frac{\hat{\alpha}^2}{n^2} \right) + E_{\ell'} \quad (\text{A.4})$$

$$E_{\ell'} = \mu \hat{\alpha}^2 n^{-3} \Delta\ell \left[1 - \frac{3}{2} \left(\frac{\hat{\alpha}^2}{n} \right)^2 - \frac{3}{2} \frac{\Delta\ell}{n} \right]. \quad (\text{A.5})$$

The only relevant energy in the following will be $E_{\ell'}$. We can therefore drop the distinction between μ and μ_E . Next comes the specification of ℓ' . Here we choose a notation which is appropriate to electronic and muonic atoms:

$$\ell'(\ell'+1) = \gamma(\gamma \pm 1) - \hat{\alpha}^2 x/\gamma \quad (\text{A.6})$$

with γ given by (2.30). For equal masses, one may simply use $\gamma = j + \frac{1}{2}$. For the Klein-Dirac equation, expansion of γ_c in (2.29) to first order in $1 - c^2 = m_a^2/E^2 \approx m_a^2/m^2$ gives

$$x = (\ell - j) m_a^2 / m^2 \quad (\text{A.7})$$

Using

$$\ell' = \gamma - \frac{1}{2} \pm \frac{1}{2} + \hat{\alpha}^2 x / \gamma (2\gamma \pm 1), \quad (\text{A.8})$$

we can separate $E_{\ell'}$ into two pieces :

$$E_{\ell'} = E_{\ell'pD} + E_{\ell'jx}$$

$$E_{\ell'pD} = -\hat{\alpha}^4 \frac{\mu}{2j+1} n^{-3} \left[1 - \frac{3}{2} \hat{\alpha}^2 \left(\frac{1}{2j+1} - \frac{1}{n} + \frac{1}{n^2} \right) \right] \quad (\text{A.9})$$

$$E_{\ell'jx} = \hat{\alpha}^4 \frac{\mu x / \gamma}{2\gamma \pm 1} n^{-3} \left[1 + \frac{3}{2n} \hat{\alpha}^2 \left(\frac{2}{2j+1} - \frac{1}{n} \right) \right] \quad (\text{A.10})$$

The first part, $E_{\ell'pD}$ is the usual point Dirac fine structure energy, and ℓ is the value of ℓ' for $\hat{\alpha} = 0$. $E_{\ell'pD}$ is independent of ℓ ("parity degeneracy"), but the correction $E_{\ell'jx}$ does depend on ℓ . The case at hand has

$$E_{\ell'jx} = \hat{\alpha}^4 \frac{\mu(\ell-j)m_a^2/m^2}{(j+\frac{1}{2})(2\ell+1)} n^{-3} \left[1 + \frac{3}{2n} \hat{\alpha}^2 \left(\frac{1}{j+\frac{1}{2}} - \frac{1}{n} \right) + 2\hat{\alpha}^2 \frac{1 + (\ell-j)/(2j+1)}{(j+\frac{1}{2})(2\ell+1)} \right] \quad (\text{A.11})$$

This is the recoil contribution of a spinless (or spin-averaged, at least for $\ell > 0$) nucleus to the Lamb shift to order m_a^2/m^2 and $\hat{\alpha}^6$.

In the bispinor case, we have to find the eigenvalues λ of the matrix $L_{\ell j}^2$ (3.21). The matrix $L_{\ell j f}^2 - \lambda$ has the following structure in the basis $(\chi_{f+1}, \chi_{f+\frac{1}{2}}, \chi_{f-\frac{1}{2}}, \chi_{f-1})$:

$$L_{\ell j f}^2 - \lambda = \begin{pmatrix} d_{++} - i\hat{\alpha}d_+ & i\hat{\alpha}e^{1/2} & 0 & 0 \\ -i\hat{\alpha}d_+ & d_{+-} & -\hat{\alpha}^2 d/F & i\hat{\alpha}e^{1/2} \\ i\hat{\alpha}e^{1/2} & -\hat{\alpha}^2 d/F & d_{-+} & -i\hat{\alpha}d_- \\ 0 & i\hat{\alpha}e^{1/2} & -i\hat{\alpha}d_- & d_{--} \end{pmatrix} \quad (\text{A.12})$$

$$\begin{aligned}
 d_{\pm} &= m_b/m \pm m_a/m(2f+1), \quad d = 2\mu/m(2f+1), \\
 e^{1/2} &= 2Fm_a/m(2f+1) \\
 d_{++} &= (f+1)(f+2) - 2\hat{\alpha}^2\mu/m(f+1) - \lambda \\
 d_{--} &= f(f-1) + 2\hat{\alpha}^2\mu/mf - \lambda \\
 d_{\pm\mp} &= F^2 - d / (f + \frac{1}{2} \pm \frac{f}{2}) - \lambda
 \end{aligned} \tag{A.13}$$

In evaluating the determinant of (A.12), it is advisable to keep the product of determinants D_+D_- of the 2×2 diagonal submatrices intact:

$$D_+ = d_{++}d_{+-} + \hat{\alpha}^2 d_+^2, \quad D_- = d_{--}d_{-+} + \hat{\alpha}^2 d_-^2 \tag{A.14}$$

and to pull $\hat{\alpha}^2 e$ into D_+ and D_- . In this way one finds

$$\begin{aligned}
 & \left[d_{++}d_{+-} + \hat{\alpha}^2 \left(1 - 4 \frac{\mu}{m} \frac{f}{2f+1} \right) \right] \left[d_{--}d_{-+} + \hat{\alpha}^2 \left(1 - 4 \frac{\mu}{m} \frac{f+1}{2f+1} \right) \right] \\
 & - \left[\hat{\alpha}^2 4F \frac{\mu}{m} \frac{1}{2f+1} \right]^2 + \dots = 0,
 \end{aligned} \tag{A.15}$$

where the remaining terms are of order $\hat{\alpha}^4 \mu^2/m^2$ for λ near $f \pm 1$ and of order $\hat{\alpha}^6 \mu^2/m^2$ for λ near f . The round brackets multiplying $\hat{\alpha}^2$ are $d_+^2 + e$ and $d_-^2 + e$, respectively. The equation contains m_a and m_b only in the combination $m_a m_b / m^2 = \mu/m$. It is easily solved to first order in μ/m and all orders in $\hat{\alpha}^2$. Setting $f = j - \frac{1}{2}$ in the first square bracket and $f = j + \frac{1}{2}$ in the other, one finds that the vanishing of either square bracket amounts to

$$0 = \left[(j + \frac{1}{2})(j + \frac{3}{2}) - \lambda \right] \left[j^2 - \frac{1}{4} - \lambda \right] + \hat{\alpha}^2 \left[1 - 2(-1)^{j-f} \frac{2j+1}{2f+1} \frac{\mu}{m} \right] \tag{A.16}$$

The effective angular momentum ℓ' and energy $E_{\ell j x}$ follow again from (A.8) and (A.10), with x given by

$$x = 2(f-j) \frac{\mu}{m} \frac{2j+1}{2f+1} \tag{A.17}$$

and $(2j+1)/\gamma = 2$ to order $\hat{\alpha}^4$. Thus the usual Dirac hyperfine structure emerges to lowest order in μ/m , at least to order $\hat{\alpha}^4$. For λ near $(f+1)(f+2)$ or

f^2-f , Eq. (A.16) is also true to order $\hat{\alpha}^4$ and all orders in μ/m , because the last term in (A.15) is of order $\hat{\alpha}^4$ and therefore negligible (except for λ near F^2 , where the other square brackets are both of order $\hat{\alpha}^2$). For λ near F^2 , on the other hand, inclusion of the last term leads to

$$\begin{aligned}\lambda &= F^2 - \left(\frac{\hat{\alpha}}{2F}\right)^2 \left\{ 1 \pm \left[1 + 4F^2 \left(1 - 4\frac{\mu}{m}\right) \right]^{1/2} \right\} \\ &= F^2 - \left(\frac{\hat{\alpha}}{2F}\right)^2 \left\{ 1 \pm (2f+1) \left[1 - 4\frac{\mu}{m} \left(\frac{F}{f+1/2}\right)^2 \right]^{1/2} \right\} \quad (\text{A.18})\end{aligned}$$

For small μ/m , expansion of the square root gives

$$\lambda \approx \ell(\ell+1) - \hat{\alpha}^2(f-j) \left[\frac{1}{j+1/2} - \frac{2\mu/m}{f+1/2} + 6 \frac{(F\mu/m)^2}{(f+1/2)^3} \dots \right], \quad (\text{A.19})$$

which agrees with (A.17) in its mass dependence only for $f = 0$.

APPENDIX B - PERTURBATION THEORY

The Dirac equation for a particle of mass μ may be written in Hamiltonian form, with $\vec{\alpha} = \gamma^0 \vec{\gamma}$:

$$H_{tot} \Psi_{Dtot} = \varepsilon_{tot} \Psi_{Dtot}, \quad H_{tot} = \mu \gamma^0 + \vec{\pi}_{tot} \vec{\alpha} + V_{tot} \quad (B.1)$$

where Ψ_D is a four-component spinor, normally used in its low-energy representation:

$$\Psi_D = \begin{pmatrix} \psi_g \\ \psi_k \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad (B.2)$$

and ε_{tot} , $\vec{\pi}_{tot}$ and V_{tot} are expanded around unperturbed quantities ε , $\vec{\pi}$, V :

$$\varepsilon_{tot} = \varepsilon + \delta\varepsilon, \quad \vec{\pi}_{tot} = \vec{\pi} + e \delta \vec{A}, \quad V_{tot} = V + \delta V. \quad (B.3)$$

Then the first-order change $\delta\varepsilon$ in the energy ε may be shown to be

$$\delta\varepsilon = \int d^3r \Psi_D^\dagger (\delta V + e \vec{\alpha} \delta \vec{A}) \Psi_D \quad (B.4)$$

where Ψ_D is a solution of the unperturbed equation

$$H \Psi_D = \varepsilon \Psi_D, \quad H = \mu \gamma^0 + \vec{\pi} \vec{\alpha} + V, \quad (B.5)$$

precisely as in non-relativistic quantum mechanics. However, this situation is not typical for perturbation theory of relativistic single-particle equations. The Klein-Gordon equation is quadratic in ε and $\vec{\pi}$ and does not lead to (B.4). It can be linearized in ε by a doubling of components¹⁷⁾, but terms of the form $\vec{\pi} \delta \vec{A}$ remain in the perturbation "Hamiltonian" δH . The Kramers equation (2.3) is also quadratic in ε , but the operator on its left-hand side factorizes neatly and makes it equivalent to the Dirac equation. After inclusion of the anomalous magnetic moment (4.11), however, it does not factorize, and neither does any of the spinor equations for binary atoms presented in this paper. A more general relativistic perturbation theory is therefore needed. The first order is in fact quite easy, as we shall see (for the spinless equation, a second-order perturbation theory has been given¹⁰⁾, but only to first order in relativistic effects).

Let the equation to be solved perturbatively be of the form

$$\left[(\varepsilon_{tot} - V_{tot})^2 - \mu_E^2 - \vec{\pi}^2 - ec \vec{\sigma} \cdot \vec{B}^- + \delta D \right] \Psi = 0 \quad (B.6)$$

with $\vec{B}^- = \vec{B} - i\vec{E}$ and $\vec{\pi}$ at their unperturbed values, i.e., the differences between $\vec{\pi}_{tot}^2$ and $\vec{\pi}^2$ and between $\vec{\sigma} \cdot \vec{B}_{tot}^-$ and $\vec{\sigma} \cdot \vec{B}^-$ are included in the operator δD , which may also contain hyperfine interactions. The constant c is arbitrary real, but to avoid unnecessary complications we take it energy-independent. We also introduce the fixed-mass energy E_μ and approximate ε :

$$E_\mu^2 \equiv \mu^2 + k^2, \quad \varepsilon \approx E_\mu - k^2 \mu / 2m E_\mu \quad (B.7)$$

which means that μ_E is treated to first order in relativity only (the approximation $c = \text{const}$ is better, it includes for example the Klein-Gordon ($c = 0$) and Dirac ($c = 1$) equations). Setting finally

$$\Psi = \Psi_r + \delta \Psi, \quad (B.8)$$

the unperturbed equation is

$$\left[k^2 + V^2 - 2V \left(E_\mu - \frac{k^2 \mu}{2m E_\mu} \right) - \vec{\pi}^2 - ec \vec{\sigma} \cdot \vec{B}^- \right] \Psi_r = 0 \quad (B.9)$$

and $\delta \Psi$ satisfies the inhomogeneous equation

$$\begin{aligned} & \left[k^2 + V^2 - 2V \left(E_\mu - \frac{k^2 \mu}{2m E_\mu} \right) - \vec{\pi}^2 - ec \vec{\sigma} \cdot \vec{B}^- \right] \delta \Psi \\ &= -2 \left[(E_\mu - V) (\delta E_\mu - \delta V) + \frac{\mu}{m} V \delta E_\mu + \frac{1}{2} \delta D \right] \Psi_r \end{aligned} \quad (B.10)$$

As a preparation to perturbation theory, we derive the orthogonality relations of the unperturbed functions ϕ_r from (B.9). The unusual feature here is that the operator $\vec{\sigma} \cdot \vec{B}^-$ is not Hermitian: its Hermitian conjugate is $\vec{\sigma} \cdot \vec{B}^+ = \vec{\sigma} \cdot (\vec{B} + i\vec{E})$. Consequently, we must also consider "left-handed" spinors ϕ_λ which are solutions of the unperturbed equation

$$\left[k^2 + V^2 - 2V \left(E_\mu - k^2 \mu / 2m E_\mu \right) - \vec{\pi}^2 - ec \vec{\sigma} \cdot \vec{B}^+ \right] \Psi_\ell = 0 \quad (B.11)$$

We shall need the Hermitian adjoint of (B.11) at a different energy which we call E'_μ ; the corresponding Hermitian adjoint spinor is denoted by $\phi_\lambda'^+$:

$$\psi_e'^+ \left[k'^2 + V^2 - 2V(E'_\mu - k'^2 \mu / 2m E'_\mu) - \vec{\pi}^{*2} - ec \vec{\sigma} \vec{B}^- \right] = 0 \quad (\text{B.12})$$

where $\vec{\pi}^*$ operates to the left. We multiply this by ψ_r from the right, integrate over all space and transform $\vec{\pi}^2$ into $\vec{\pi}^{*2}$ by partial integrations. Then we multiply (B.9) by $\psi_\lambda'^+$ from the left, integrate that expression and subtract from it the former integral, obtaining:

$$\int d^3r \psi_e'^+ \left[k^2 - k'^2 - 2V(E_\mu - E'_\mu)(1 - \frac{\mu}{m}) \right] \psi_r = 0 \quad (\text{B.13})$$

Extracting a factor $E_\mu - E'_\mu$ and normalizing ψ in agreement with the non-relativistic expression, we find the orthonormality relations for discrete states

$$\int d^3r \psi_e'^+ \left[\frac{1}{2}(E_\mu + E'_\mu) - V(1 - \frac{\mu}{m}) \right] \psi_r = \mu \delta_{nn'} \quad (\text{B.14})$$

where n and n' are the relevant quantum numbers. Now comes the perturbation calculation of $\delta\epsilon$: we multiply (B.10) by ψ_λ^+ from the left and integrate over all space, transform $\vec{\pi}^2$ into $\vec{\pi}^{*2}$ by partial integration and observe that the left-hand side of (B.10) vanishes because of (B.12), taken at $\epsilon' = \epsilon$. The right-hand side is simplified by the normalization integral (B.14) for $n = n'$, leaving us with

$$\delta E = \delta E_\mu = \int d^3r \psi_e^+ \left[\delta V(E_\mu - V) - \frac{1}{2} \delta D \right] \psi_r \quad (\text{B.15})$$

Before continuing, let us discuss two special cases of this formalism, namely the Klein-Gordon and Dirac equations for a single particle in a static four-vector potential $A^\mu(\vec{r})$. In both cases, we set $\mu/m = 0$ but keep the notation μ for the particle mass. The Klein-Gordon equation has $c = 0$ and therefore $\psi_\lambda = \psi_r$. With these simplifications, (B.14) and (B.15) do in fact reproduce the known Klein-Gordon results. But what about the Dirac equation? How can $\delta\epsilon$, as given by (B.15), be identical to (B.4)? The question is most easily answered in the high-energy representation (ψ_r, ψ_λ) , but let us use here the more familiar low-energy representation (ψ_g, ψ_k) :

$$\psi_r = \psi_g + \psi_k, \quad \psi_e = \psi_g - \psi_k \quad (\text{B.16})$$

Then the Dirac orthogonality integral is rewritten as

$$\int d^3r (\psi_g'^+ \psi_g + \psi_k'^+ \psi_k) = \frac{1}{2} \int d^3r (\psi_r'^+ \psi_r + \psi_e'^+ \psi_e) \quad (B.17)$$

Using now the Dirac equation (2.2), one sees immediately that (B.17) is identical with (B.14), for $\mu/m = 0$.

The factor $\frac{1}{2}$ in (B.17) deserves a special discussion. In going from one representation of Dirac spinors to the other, one normally takes a unitary transformation:

$$\psi_r = 2^{-\frac{1}{2}} (\psi_g + \psi_k), \quad \psi_e = 2^{-\frac{1}{2}} (\psi_g - \psi_k). \quad (B.18)$$

With that transformation, the factor $\frac{1}{2}$ would be absent in (B.17), but then the non-relativistic limit of the normalization integral (B.14) would only be half the usual value, which is

$$\int d^3r \psi^+ \psi = 1, \quad (B.19)$$

where ψ has one component for each spin orientation. Thus the transformation (B.16) is dictated by the requirement of non-relativistic normalisation limits both for ψ_g and ψ_r . (As long as the "high-energy representation" was used only at high energies, that requirement was of course unnecessary.)

After this discussion, it is clear that the energy shifts $\delta\epsilon$ due to δV in the two equations (B.4) and (B.15) are identical. For the part due to $\delta\vec{A}$, we insert (B.2) into (B.4):

$$\begin{aligned} \int d^3r \psi_D^+ \vec{\sigma} \delta\vec{A} \psi_D &= \int d^3r (\psi_g^+ \vec{\sigma} \delta\vec{A} \psi_k + \psi_k^+ \vec{\sigma} \delta\vec{A} \psi_g) \\ &= \frac{1}{2} \int d^3r \delta\vec{A} (\psi_r^+ \vec{\sigma} \psi_r - \psi_e^+ \vec{\sigma} \psi_e) \\ &= \frac{1}{2\mu} \int d^3r \psi_e^+ (\vec{\sigma} \vec{\pi}^* \vec{\sigma} \delta\vec{A} + \vec{\sigma} \delta\vec{A} \vec{\sigma} \vec{\pi}) \psi_r \\ &= \int d^3r \psi_e^+ H' \psi_r, \\ H' &\equiv \{ \vec{\sigma} \vec{\pi}, \vec{\sigma} \delta\vec{A} \} / 2\mu \end{aligned} \quad (B.20)$$

which agrees with the findings from (B.6) and (B.15). Moreover, H' is pre-

cisely the non-relativistic Hamiltonian perturbation due to $\delta\vec{A}$. Its appearance in a fully relativistic expression shows that in this case (which includes the hyperfine structure), the "high-energy representation" is in fact more closely related to the Schrödinger equation than the "low-energy representation".

For the application of second-order perturbation theory, one would like to have relativistic Coulomb Green's functions. These have been constructed for the Dirac equation precisely on the basis of the quadratic Dirac equation¹⁸⁾. Similar Green's functions may be constructed for the other equations which have exact solutions, in particular for the Klein-Dirac equation.

APPENDIX C - CONNECTION WITH DIRAC WAVE FUNCTIONS FOR

$$S_a = \frac{1}{2}, S_b = 0$$

The solutions of Eq. (2.5) for $V = \hat{\alpha}/r$ are

$$\Psi_{r\pm} = [\chi_e^{jm_j} \pm i y_c \chi_{\tilde{e}}^{jm_j}] R_{\pm}(r), \quad y_c = (\gamma_c - j - \frac{1}{2})/\hat{\alpha}c, \quad (C.1)$$

for $\tilde{\lambda} = 2j - \lambda$, and with $R_{\pm}$ given by (A.1). Comparison with (B.2) and

$$\Psi_g = g(r) \chi_e^{jm_j}, \quad \Psi_k = i f(r) \chi_{\tilde{e}}^{jm_j} \quad (C.2)$$

shows $f = \pm y_c g$. It is easily checked that these are solutions of the coupled quadratic equations (2.5). But in the limit $c \rightarrow 1$, they are not the normal solutions of the Dirac equation (B.1). This complication is due to the parity degeneracy of the Dirac energies and has a peculiar history. The quadratic Dirac equation with the Coulomb potential $\hat{\alpha}/r$ was first solved by Martin and Glauber¹⁸⁾, who also found that their solutions were parity mixtures. Kolsrud¹⁹⁾ expressed the standard Dirac wave functions g_D and f_D in terms of the solutions $R_{\pm}$:

$$g_D = [(\varepsilon + \mu)(\gamma - \kappa) - \hat{\alpha} \kappa_n] R_- + \frac{\gamma + i\eta}{2\gamma(2\gamma + 1)} [(\varepsilon + \mu)(\gamma + \kappa) - \hat{\alpha} \kappa_n] R_+,$$

$$\eta = \hat{\alpha} \varepsilon / k, \quad k = i \kappa_n, \quad \kappa = \pm j \pm 1/2, \quad (C.3)$$

using recurrence relations between the different confluent hypergeometric functions contained in g_D and f_D . (The form of (C.3) differs from Kolsrud's relation because it uses a different normalization factor.) Independently of this, the parity invariance of the general two-component Kramers equation was discussed by Feynman and Gell-Mann²⁰⁾. For this reason, the equation is also called Feynman-Gell-Mann equation. Its solutions for the Coulomb potential were rederived by Auvil and Brown²¹⁾. See also Biedenharn et al.²²⁾.

For $c \neq 1$, one has no degeneracy, which raises the following problem: is it possible to construct from R_- alone or from R_+ alone a function which approaches g_D for $c = 1$? The answer is yes, by one differentiation. One may use the following differential relations

$$F'(a, b) = \frac{a(b-a)}{b^2(b+1)} \times F(a+1, b+2) + \frac{a}{b} F(a, b)$$

$$\times F'(a+1, b+2) = \left(\frac{b-a}{b} - \frac{b+1}{x} \right) \times F(a+1, b+2) + (b+1) F(a, b) \quad (C.4)$$

to derive

$$\partial_r R_- = \left(\frac{\gamma-1}{r} + 2 \frac{\varepsilon}{\gamma} \right) R_- - \frac{\gamma+i\eta}{2\gamma(2\gamma+1)} \left(2 \frac{\varepsilon}{\gamma} + \kappa_n \right) R_+$$

$$\partial_r R_+ = - \left(\frac{\gamma+1}{r} + 2 \frac{\varepsilon}{\gamma} \right) R_+ + 2\kappa_n(2\gamma+1) R_-$$

(C.5)

The following approximation is sufficient for our precision:

$$g_0(\ell = j \pm \frac{1}{2}) = N_{\pm} \left[1 \mp \frac{\hat{\alpha}}{4\mu\gamma} \left(\partial_r - \frac{\vec{L} \cdot \vec{\sigma}}{r} \right) \right] R_{\pm} \chi_{\ell}^{jm_j}$$

(C.6)

It expresses the relativistic wave function entirely in terms of one confluent hypergeometric function.

REFERENCES

- 1) I.T. Todorov - Phys.Rev. D3 (1971) 2351.
- 2) I.T. Todorov - in "Properties of Fundamental Interactions", A. Zichichi Ed., Vol. 9C, Bologna: Editrice Compositori (1973).
- 3) H. Pilkuhn - "Relativistic Particle Physics", Springer (1979).
- 4) G. Breit - Phys.Rev. 34 (1929) 553.
- 5) G. Breit - Phys.Rev. 36 (1930) 383.
- 6) C. Schwartz - Ann.Phys.(N.Y.) 6 (1959) 156;
D. Beder - Nuclear Phys. A305 (1978) 411, and Phys.Rev. B21 (1980) 3861.
- 7) V.G. Baryshevsky and S.A. Kuten - Phys.Letters A67 (1978) 355.
- 8) H. Grotch and D.R. Yennie - Z.Phys. 202 (1967) 425, and Revs.Modern Phys. 41 (1969) 350.
- 9) G.P. Lepage - Phys.Rev. A16 (1977) 863.
- 10) J. Fröhlich and H. Pilkuhn - J.Phys.B. 17 (1983) 147.
- 11) H.A. Kramers - "Quantentheorie des Electrons un der Strahlung", Akad.Verlagsges, Leipzig (1933).
- 12) L.M. Brown - in "Lecture Notes in Theoretical Physics", Vol. IV (Interscience, New York, 1962).
- 13) R.P. Feynman - Phys.Rev. 84 (1951) 108.
- 14) H. Pilkuhn - Z.Physik A305 (1982) 241.
- 15) S. Barmo, H. Pilkuhn and H.G. Schlaile - Z.Physik A301 (1981) 283.
- 16) K. Bechert and J. Meixner - Ann.Phys. (Leipzig) 22 (1935) 525.
- 17) H. Feshbach and F. Villars - Revs.Modern Phys. 30 (1958) 24.
- 18) P.C. Martin and R.J. Glauber - Phys.Rev. 109 (1958) 1307.
- 19) M. Kolsrud - Physica Norvegica 2 (1966) 43.
- 20) R.P. Feynman and M. Gell-Mann - Phys.Rev. 109 (1958) 193.
- 21) P.R. Auvil and L.M. Brown - Am.J.Phys. 46 (1978) 679.
- 22) L.C. Biedenharn, M.Y. Han and H. van Dam - Phys.Rev. D6 (1972) 500.

FIGURE CAPTION

- Fig. 1 (a) The two-body scattering in the centre-of-mass.
- (b) The corresponding scattering of the effective particle on an external potential V . The effective particle energy ϵ , mass μ_E and scattering amplitude F are given in Eqs. (1.14) , (1.15) and (1.12) below.

(a)

(b)