Θ -DEPENDENT STATISTICS FOR GAUGE THEORY MONOPOLES

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A B S T R A C T

Using the Finkelstein-Rubinstein definition, we show that monopoles may admit fractional spin and exotic statistics. Statistics interpolates between the usual Bose and Fermi cases which are attained for $\Theta = 0$, and $\Theta = \pm\pi$, respectively.

A Dirac dyon, a system consisting of one particle with electric charge Q_E , and one with magnetic charge Q_B , each considered as bosons, has half-integral angular momentum if $Q_E Q_B$ due to electric and magnetic fields is half-integral, a condition which is necessary for the non-observation of the Dirac string¹⁾. The 't Hooft-Polyakov monopole²⁾ is an extended object that looks from afar like a Dirac monopole. Thus it is natural that by coupling a charged (isodoublet) boson field to the extended monopole, one obtains an extended dyon in which the angular momentum is again half-integral³⁾. It is worth emphasizing that the sole role of the extra charged scalar was that a dyon may appear as a stable bound state (stated in a different way, to excite charge degrees of freedom about the topologically non-trivial magnetic background).

However, the present understanding of the quantum monopole has changed dramatically with the (unexpected) emergence of the Θ angle in the monopole sectors⁴⁾. Quantum monopole is a dyon, and the charge depends on the angle Θ . It is thus proper to forward the spin-statistics questions directly to the monopole without resorting to couplings with extra charged fields^{f1)}.

In this work we will use the Finkelstein-Rubinstein definition of spin statistics⁶⁾ which was tailored for soliton field theories, and successfully applied⁷⁾ to the Skyrme model⁸⁾. We show that a gauge theory monopole may possess fractional angular momentum and obey peculiar quantum statistics⁹⁾, which interpolates between the usual Bose and Fermi cases.

For definiteness, we will work with the $SO(3)$ Georgi-Glashow model which involves an isotriplet Higgs field²⁾:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + \frac{1}{2} (D_\mu \Phi)^a (D^\mu \Phi)^a - \frac{1}{4} (\Phi^2 - a^2)^2 \quad (1)$$

here a is the Higgs vacuum expectation value.

Monopoles being finite energy static configurations impose that the fields approach the Higgs vacuum as $\vec{x} \rightarrow \infty$, which is defined as

$$F_{\mu\nu} = 0, \quad D_\mu \hat{\Phi} = 0, \quad \hat{\Phi}^2 - a^2 = 0 \quad (2)$$

The $SO(3)$ symmetry is broken down to a $U(1)$ which, in the hedgehog gauge, is the group of rotations around the $\hat{\Phi}$ direction. The topological classification of the fields is based on their behaviour in the asymptotic region, which is topologically equivalent to S_∞^2 . Equations (2) imply $A_\mu = g^{-1} \partial_\mu g$, $\hat{\Phi}^a = \hat{a}x^a$ on S_∞^2 , where g is an $SO(3)$ valued function. Thus we have

$$\begin{aligned} \hat{\Phi} : S_\infty^2 &\rightarrow G/H \approx S_\Phi^2 \\ A_i : S_\infty^2 &\rightarrow G \end{aligned} \quad (3)$$

$\Pi_2(G) = 0$ for any compact simple Lie group. Thus the gauge field is topologically trivial. The topological nature of the magnetic charge is solely determined by the Higgs fields alone

$$Q_m = \frac{1}{8\pi e} \oint_{S_\infty^2} d\sigma_i \varepsilon_{ijk} \varepsilon^{abc} \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \quad (4)$$

and represents the degree of the map $S_\infty^2 \rightarrow S_\Phi^2$. The interesting fact is that Q_m is not the only topological invariant associated with the monopole. There is another invariant which becomes manifest only after the quantization. Using the standard method of quantization around a classical solution, we first observe that the monopole configuration as described by the 't Hooft-Polyakov (tH-P) ansatz is not unique but is parametrized by all possible $U(1)$ gauge rotations (in the $A_0 = 0$ gauge)¹⁰⁾

$$\begin{aligned} A_i &= h^{-1} a_i h + h^{-1} \partial_i h \\ h &= \exp[i\omega(r,t) \hat{x} \cdot \vec{T}] \end{aligned} \quad (5)$$

where a_i is the tH-P ansatz, and $A_i = -iA_i^a T^a$. ω is chosen to be spherically symmetric because the monopole is invariant under simultaneous spin and isospin rotations. Motions of the system in this new configuration space produce radial electric fields, which, together with the magnetic field, yield

$$q = \frac{e^2}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = n \quad (6)$$

where n is the winding number of h , which measures the number of full rotations around the $\hat{\Phi}$ direction. This quantity is known (from instanton physics) to label the topology of the gauge fields, and has a simple topological explanation in the present context. Due to the degeneracy depicted in (5), the gauge field topology is now determined by

$$A_i : S^1 \otimes S_\infty^2 \rightarrow G \quad (7)$$

The index therefore is $\Pi_3(G) = \mathbb{Z}$ ¹¹⁾. The new topology associated with the gauge fields forces upon us that the monopole Lagrangian is to be supplemented by the so-called Θ term:

$$\mathcal{L} \rightarrow \mathcal{L}_\Theta = \mathcal{L} + \Theta \frac{e^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} \quad (8)$$

Clearly the Pontryagin index, q , is gauge invariant (small and large). Under large gauge transformations, such as 2π rotations around the $\hat{\Phi}$ direction, $W = \int d^3x K_0^\mu$, where $\partial_\mu K^\mu = e^2/32\pi^2 F_{\mu\nu}^a \tilde{F}^{\mu\nu,a}$, changes by an amount equal to the winding number of the transformations, but $q = W(T) - W(0)$ is clearly invariant. Thus Θ is not quantized.

It was previously observed that these two different indices which label the homotopy classes of the Higgs and gauge fields (Q_m and q) are equivalent^{11), f2)}. Here what we will show is that the Pontryagin index measures the connectivity of the one-monopole sector of the space of maps $\Phi: S_\infty^2 \rightarrow S_\Phi^2$. This connectivity is known to be the crucial quantity for the admittance of spin in the sense of Finkelstein and Rubinstein. We now turn to this problem.

We denote the set of all mappings described by $\Phi: S_\infty^2 \rightarrow S_\Phi^2$ by Ω . Ω is divided into a collection of homotopy classes, Ω_i , labelled by $\Pi_2(S^2) = \mathbb{Z}$, Ω_1 denoting the one-monopole sector. We shall investigate the conditions under which wave functionals may be defined on Ω_1 which are multivalued under 2π spatial rotations. Since gauge fields are classically topologically trivial, we have to consider only the Higgs fields. Let us

consider $\Phi_1 \in \Omega_1$. If it is possible to define a multivalued functional on Ω_1 , then it must be possible to follow a path in Ω_1 , starting and ending at Φ_1 , such that the value of the functional along the path changes continuously and does not return to its original value when the end of the path is reached. (It is crucial that the path in Ω_1 must not be contractible.) We will consider the paths which correspond to a spatial 2π rotation of the monopole as it was shown in Ref. 5) that kink exchange is homotopic to kink rotation. The quantity which contains information about the path structure of Ω_1 is $\Pi_1(\Omega_1)$ [set of different classes of closed path in Ω_1]. A necessary and sufficient condition for the existence of multivalued functionals in Ω_1 is that this group has at least an element of order 2 (the least value corresponds to ordinary spinors). A 2π rotation loop in Ω_1 can be defined as $\Phi(R_\omega \vec{x})$ where R_ω is a 3×3 rotation matrix. This sequence of maps from Ω_1 into S^2 describes a mapping $S^1 \times S_\omega^2 \approx S_\Phi^2$ which is classified by $\Pi_3(S^2)$ due to the extra rotation angle which labels the path in Ω_1 . There is an important homeomorphism in Ω . Loops which corresponds to 2π rotations of a single monopole, and loops which correspond to monopole-antimonopole creation, separating and rotating the monopole through an angle 2π without touching the antimonopole and bringing them together to annihilate. This second class of loops proved to be very useful in proving the spin of Skyrmions, and will be even more useful because of the very nature of the maps $S_\omega^3 \rightarrow S^2$ [which is known as the Hopf map¹³⁾] which we shall briefly review.

The Hopf map can be expressed in terms of the homogeneous coordinates of S_ω^3 and S_Φ^2 as

$$\Phi^a = \frac{1}{R^2} \bar{z} \sigma^a z \quad (9)$$

where $z = (z_1 z_2)$ is a complex two-component spinor with the constraint $|z_1|^2 + |z_2|^2 = R^2$, and R is the radius of S_R^3 (R is taken to ∞ eventually). z and $e^{i\alpha} z$ yield the same Φ^a , thus the inverse image of any point on S_Φ^2 is a circle on S^3 . The feature is extremely important for the spin property. The particular map, (9), has a Hopf invariant one.

Let us now briefly recall the construction of the Hopf invariant in such a way that the equivalence to the Pontryagin index is manifest. This

in turn will make the relation between the Θ angle and spin statistics of the monopole transparent. Let us note that a different manifestation of the same phenomenon already occurs in the Skyrme model¹⁴⁾. There, the one-Skyrmion sector of the space of maps was doubly connected, which meant spinor structure for the Skyrmion. Furthermore, the model has a mod-2 Θ vacuum structure. That is, the coefficient of the Wess-Zumino term is quantized to be an integer.

In constructing the Hopf invariant^{15),f3)}, the first step is to define a two-form area element on S^2_Φ (minimum use of differential form language makes the presentation easier), such that $\int_{S^2_\Phi} \omega_2 = 1$. The next step is to use the pull-back of the Hopf map to define a new two-form, $\sigma_2 = \Phi^* \omega_2$, on S^3 . Pull-backs of the closed forms are closed forms. So σ_2 is closed, $d\sigma_2 = 0$. Because all closed two-forms on S^3 are exact [the mathematical expression for this is that the second cohomology group of S^3 is trivial, $H^2(S^3)=0$], one can find a (non-unique) one-form σ_1 on S^3 which satisfies $\sigma_2 = d\sigma_1$. The Whitehead integral representation for the Hopf invariant is, then

$$H = \int_{S^3} \sigma_1 \wedge \sigma_2 \quad (10)$$

Now, the pull-back of the area two-form on S^2_Φ is

$$\sigma_2 = \frac{1}{8\pi} \varepsilon^{abc} \phi^a d\phi^b \wedge d\phi^c \quad (11)$$

Using (1), this can be rewritten as

$$\sigma_2 = \frac{1}{4\pi} \phi^a dA^a \quad (12)$$

Thus σ_1 is

$$\sigma_1 = \frac{1}{4\pi} \phi^a A^a \quad (13)$$

Plugging (11) and (13) in (10), we finally get the expression for the Hopf invariant

$$H = \frac{1}{16\pi^2} \oint_{S^3} A^1 \wedge A^2 \wedge A^3 \quad (14)$$

which is nothing but the Pontryagin index. Thus the Θ term, which is forced to be included by the phase ambiguity of the monopole states under large gauge transformations, is nothing but the index which measures the connectivity of the one-monopole sector of the space of maps.

Let us recall once more that the pair creation and annihilation after rotating one around the other by an angle 2π , was the measure of the connectivity of the one-monopole sector of the space of maps. This is actually what is described by the Hopf map $S^3 \rightarrow S^2$. Its invariant describes the linking number between two great circles on S^3 , which can be thought of as two curves in R^3 by stereographic projection. The map depicted in (9) has linking number 1, it thus corresponds to a 2π rotation of the pair.

We now turn to the physical question of the spin of the monopole. To determine the spin, we spatially rotate the monopole adiabatically through 2π . At the end of the rotation, the wave functional acquires a phase factor e^{iS} , where S is the action corresponding to the adiabatic rotation. So the angular momentum J is given by $e^{iS} = e^{2\pi i J}$. The only term in $\mathcal{L}$ which is linear in the momenta is the q term. Thus this is the term which has a non-zero contribution during an adiabatic rotation.

Spatial rotation for a single monopole is equivalent to an isospin rotation. Thus we need to evaluate q for the gauge transformed monopole configuration given in (5), which describes a map $S^1 \times S_\infty^2 \rightarrow S_\Phi^2$. Clearly a gauge transformation rotates all components of the gauge field; it generates a time component also

$$A_0 = h^{-1} \partial_t h = i \dot{\omega}(r,t) \hat{r} \cdot \vec{\tau}$$

Using the Bianchi identity, q can be written as

$$q = -\frac{1}{4\pi^2} \int d^4x \text{Tr} [B_i \partial_0 A_i] - \frac{1}{4\pi^2} \int dt \oint d^2\sigma_i \text{Tr} (B_i A_0) \quad (15)$$

For the configuration

$$\begin{aligned}\omega(0, r) &= 0 \\ \omega(\infty, r) &= \begin{cases} \pi, & r \rightarrow \infty \\ 0, & r = 0 \end{cases}\end{aligned}\quad (16)$$

we get (the surface contribution goes to zero like R^{-2})

$$q = \omega(\infty, \infty) - \omega(0, \infty) = 1 \quad (17)$$

Thus we see that the monopole has angular momentum $\Theta/2\pi$.

Finally, we will present a Hamiltonian method derivation of our results which will make them more transparent. Because space rotation is equivalent to a gauge rotation, we can write a monopole-antimonopole pair creation operator where the monopole is rotated by an amount ω as

$$\begin{aligned}G_\omega &= \Phi(R_\omega \vec{x}) \Phi^\dagger(\vec{x}) \\ &= U_\omega^\dagger \Phi(x) U_\omega \Phi^\dagger(x)\end{aligned}\quad (18)$$

Here $\Phi^\dagger$ is the monopole creation operator. G_ω describes a loop that cannot be contracted. The pair is created at $\omega = 0$, in which case $G_\omega = I$. They are brought back $\omega = 2\pi$. $G_{2\pi}|\text{vac}\rangle$ is the state obtained as a result of 2π rotation of the monopole, because of the homeomorphism between the path structures in Ω_0 and Ω_1 . U_ω is the unitary operator which implements gauge transformations around the direction $\hat{\Phi}$ by ω .

$$U_\omega = \exp \left[i \frac{\omega}{2\pi e} \oint d^2\sigma_i (\Pi_i^a \hat{\Phi}^a) - \frac{i}{2} \int d^3x (\omega^a G^a) \right] \quad (19)$$

The Gauss law constraint

$$G^a = (D_i \Pi_i)^a + e \varepsilon^{abc} \phi^b \pi^c \approx 0 \quad (20)$$

is not sufficient to ensure the gauge invariance of the states, and $U_{2\pi}$ picks a phase, $\exp(-i\Theta)$, when acting on the monopole states

$$U_{2\pi} |Mon\rangle = e^{-i\theta} |Mon\rangle \quad (21)$$

This phase is the θ angle. Φ annihilates the monopole state; thus the last factor does not create any further phase, and we get

$$G_{2\pi} |vac\rangle = e^{-i\theta} |vac\rangle \quad (22)$$

Thus the monopole wave functional goes back to its original value if $\theta = 0$ at the end of the 2π rotation loop. It changes sign for $\theta = \pm\pi$; it is a spinor. For other values of θ , the statistics are not well defined, and interpolates between these two cases.

This Hamiltonian proof clearly shows that what we observe here is by no means surprising. It is almost trivial that this result was bound to come out given the fact that the lowest quantum monopole state has the charge $\theta/2\pi$.

The implications of this result are far-reaching, given the fact that charge $\theta/2\pi$ has led to the very interesting Callan-Rubakov phenomenon^{4),10)}. Could one perhaps claim that the quantum monopole has spin-0, in the light of recent electric dipole moment estimates for the neutron¹⁷⁾. Although $\theta = \pm\pi$ is also good as far as CP invariance goes, that estimate seems to yield a value for θ very close to $\theta = 0$. If the electric dipole moment for the neutron is zero, then this clearly would mean that the monopole has spin 0.

A similar phenomenon was recently discovered for the 1+2 dimensional $O(3)$ non-linear σ model which describes phenomenologically the Heisenberg ferromagnet in a two-dimensional system¹⁸⁾.

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FOOTNOTES

- f1) The question of monopole spin from a topological point of view was previously considered in Ref. 5).
- f2) Similar arguments for the instanton sector were raised in Ref. 12).
- f3) The relevance of the Hopf map for the magnetic charge quantization was previously noted in Ref. 16).