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ON THE EQUIVALENCE BETWEEN THE WESS-ZUMINO ACTION
AND THE FREE FERMI THEORY IN TWO DIMENSIONS

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ABSTRACT

Using the functional technique we prove the bosonization rules of Witten for the currents in a non-Abelian two-dimensional theory with a particular regularization of the Fermi theory.

Recently, Witten¹⁾ has derived bosonization rules for non-Abelian theories in two dimensions by using arguments based on the uniqueness of the representation of certain Kac-Moody algebras and showing the equivalence between a free Fermi theory and a bosonic chiral theory described by the Wess-Zumino Lagrangian in two dimensions²⁾.

The bosonization rules for the currents have been also derived³⁾ by using directly the functional integral technique, but some limitations in the application of these rules have been noticed.

In this paper, a study of the properties of the bosonic chiral model is presented and by using its invariance under the Kac-Moody algebra a certain operator identity is shown. This implies that this theory is equivalent to a free Fermi theory provided that it is regularized in a very specific way in which the vector current relative to the $SU(N)$ part of $U(N)$ is not conserved. The bosonization rules for the currents proposed in Ref. 1) follow then trivially.

Let us consider the two-dimensional Wess-Zumino action in a two-dimensional Minkowski space-time^{1),2)}:

$$S(U) = \frac{1}{8\pi} \int d^2x \operatorname{Tr}(\partial_\mu U \partial_\mu U^{-1}) + \frac{1}{12\pi} \int_Q d^3\xi \epsilon^{ijk} \operatorname{Tr} \left[U^{-1} \partial_i U \partial_j U \partial_k U^{-1} \right] \quad (1)$$

where Q is a three-dimensional hemisphere with compactified two-dimensional space as boundary. The field U_{ij} is taken to be an element of the $U(N)$ group, such that $U^+ = U^{-1}$. Later, we shall consider the extension of our result to $O(N)$ groups.

We define the following currents:

$$J_\mu = \frac{i}{8\pi} \left\{ [U^{-1} \partial_\mu U + U \partial_\mu U^{-1}] - \epsilon_{\mu\nu} [U^{-1} \partial_\nu U - U \partial_\nu U^{-1}] \right\}$$

$$J_+ = \frac{i}{4\pi} U^{-1} \partial_+ U \quad J_- = \frac{i}{4\pi} U \partial_- U^{-1} \quad (2)$$

with the notations

$$J_{\pm} = J_0 \pm J_1 \quad \partial_{\pm} = \partial_0 \pm \partial_1 \quad (3)$$

The action (1) possesses a local $U(N) \times U(N)$ symmetry corresponding to its invariance under the transformation

$$U \rightarrow a(x^-) U b^{-1}(x^+) \quad (4)$$

where $a(x^-)$ and $b(x^+)$ are arbitrary unitary matrices depending only on one of the two light cone co-ordinates. A special case of this symmetry is the global $U(N) \times U(N)$ invariance whose generators $J_{\pm}$ are, however, not covariant under the transformations (4):

$$\begin{aligned} J_+ &\rightarrow b(x^+) J_- b^{-1}(x^+) + \frac{i}{4\pi} b(x^+) \partial_+ b^{-1}(x^+) \\ J_- &\rightarrow a(x^-) J_- a^{-1}(x^-) + \frac{i}{4\pi} a(x^-) \partial_- a^{-1}(x^-) \end{aligned} \quad (5)$$

The local invariance [Eq. (4)] generates the manifold of the classical solutions (local minima of the classical action)

$$U_{cl}(x) = a(x^-) b^{-1}(x^+) \quad (6)$$

such that $S(U_{cl}) = 0$ and the classical conservation equations

$$\partial_+ J_-^{\alpha} = \partial_- J_+^{\alpha} = 0 \quad (7)$$

are satisfied.

We are interested in the generating function for the Green functions of the currents $J_{\pm}$. For this purpose, we couple them to external sources $A_{\pm}$, that we define to be Hermitean matrices belonging to the algebra of $U(N)$ and we parametrize in terms of unitary matrices A, B , by means of

$$iA_+ = A^{-1} \partial_+ A \quad iA_- = B^{-1} \partial_- B \quad (8)$$

Equations (8) have a trivial reparametrization invariance

$$A \rightarrow \alpha(x^-) A \quad B \rightarrow \beta(x^+) B \quad (9)$$

We can now define

$$\begin{aligned} e^{iW(A_+, A_-)} &\equiv \langle e^{i \int d^2x \text{Tr} [A_+ J_- + A_- J_+]} \rangle \equiv \\ &\equiv \frac{\int e^{iS(U) + i \int d^2x \text{Tr} [A_+ J_- + A_- J_+]} \mathcal{D}U}{\int e^{iS(U)} \mathcal{D}U} \end{aligned} \quad (10)$$

We shall compute exactly $W(A_+, A_-)$. For this purpose, we must recall the following relation³⁾ that is a fundamental property of the Wess-Zumino action $S(U)$:

$$\begin{aligned} S(AB^{-1}) &= S(U) + S(AB^{-1}) + \\ &+ \int d^2x \text{Tr} [A_+ J_- + A_- J_+] + \frac{1}{4\pi} \int d^2x \text{Tr} [A_+ A_- - A_+ U A_- U^{-1}] \end{aligned} \quad (11)$$

This equation allows us to rewrite (10) in the form:

$$e^{iW(A_+, A_-)} = \frac{\int \mathcal{D}U e^{iS(AB^{-1}) + \frac{i}{4\pi} \int d^2x \text{Tr} (A_+ U A_- U^{-1}) - i[S(A) + S(B)]}}{\int \mathcal{D}U e^{iS(U)}} \quad (12)$$

We can now make a change of variable in the functional integral by exploiting the local $U(N) \times U(N)$ invariance of the Haar measure and we get

$$e^{iW(A_+, A_-)} = \left\langle e^{\frac{i}{4\pi} \int d^2x \text{Tr} [(AA_+ A^{-1}) U (BA_- B^{-1}) U^\dagger] - iS(A) - iS(B^{-1})} \right\rangle \quad (13)$$

In order to compute the expectation value indicated in (13), we can make use of the invariance under the transformation (4) and write

$$\begin{aligned} \left\langle e^{\frac{i}{4\pi} \int d^2x \text{Tr} [A'_+ U A'_- U^{-1}]} \right\rangle &= \left\langle e^{\frac{i}{4\pi} \int d^2x \text{Tr} [(a^{-1} A'_+ a) U (b^{-1} A'_- b) U^\dagger]} \right\rangle = \\ &= \left\langle \int da(x^-) \int db(x^+) e^{\frac{i}{4\pi} \int d^2x \text{Tr} [(a^{-1} A'_+ a) U (b^{-1} A'_- b) U^\dagger]} \right\rangle \quad (14) \end{aligned}$$

where we have defined for simplicity of notations

$$A'_+ = A A_+ A^{-1} \quad A'_- = B A_- B^{-1} \quad (15)$$

and the integration over a, b is intended to be normalized

$$\int da(x^-) = \int db(x^+) = 1 \quad (16)$$

In the derivation of Eq. (14), we have assumed that the physical vacuum is invariant under the symmetry described by Eq. (4). This symmetry generates any classical solution starting from the trivial $U = 1$. Therefore, our assumption implies that the invariant vacuum to vacuum amplitudes can be computed taking any classical solution as boundary condition at $t = \pm\infty$ or equivalently averaging over all solutions. The last step of Eq. (14) can be consistently interpreted in this way.

Amazingly enough, we are able to perform explicitly this integration by means of a lattice regularization where $\int d^2x \rightarrow a^2 \sum_{x^+ x^-}$, where a is the lattice spacing. The main ingredient of our calculation is a result due to

Itzykson and Zuber⁴⁾, who performed a $U(N)$ group integration dV for any given couple of Hermitean matrices M_1, M_2 :

$$\int dV e^{\beta \text{Tr}(M_1 V M_2 V^\dagger)} = \sum_{\{n\}} \frac{\beta^{|n|} \prod_{k=0}^{N-1} k!}{\prod_{k=0}^{N-1} (k+n_k)!} \chi_{\{n\}}(M_1) \chi_{\{n\}}(M_2) \quad (17)$$

where $\{n\}$ is a set of N non negative integers characterizing a polynomial representation of $U(N)$, $|n| = \sum_{k=0}^{N-1} n_k$ and $\chi_{\{n\}}$ is the corresponding character of $U(N)$ analytically continued to an arbitrary $N \times N$ matrix.

After little thought, it is easy to convince oneself that:

$$\begin{aligned} & \int \prod_{x^+ x^-} da(x^-) db(x^+) e^{\frac{i a^2}{4\pi} \sum_{x^+ x^-} \text{Tr} [a^-(x^-) A'_+ a(x^-) U b^-(x^+) A'_- b(x^+) U^\dagger]} \\ &= e^{\sum_{x^+ x^-} \log \left[\sum_{\{n\}} \left(\frac{i a^2}{4\pi} \right)^{|n|} \frac{\prod_{k=0}^{N-1} k!}{\prod_{k=0}^{N-1} (k+n_k)!} \chi_{\{n\}}(A'_+(x^+, x^-)) \chi_{\{n\}}(A'_-(x^+, x^-)) \right]} \end{aligned} \quad (18)$$

In the continuum limit, only the term proportional to a^2 in the exponent is non-vanishing and therefore

$$\begin{aligned}
 & \int da(x^-) db(x^+) e^{\frac{i}{4\pi} \int d^2x \text{Tr}[(a^- A'_+ a) U (b^- A'_- b) U^{-1}]} = \\
 & = e^{\frac{i}{4\pi} \int d^2x \frac{1}{N} \text{Tr}(A'_+(x)) \text{Tr}(A'_-(x))} = \\
 & = e^{\frac{i}{4\pi N} \int d^2x \text{Tr}(A_+) \text{Tr}(A_-)}
 \end{aligned} \tag{19}$$

Notice that the dependence on U has already disappeared at the level of Eq. (18). Therefore, taking the expectation value in (14) is a trivial operation and we reach the following final result:

$$W(A_+, A_-) = -S(A) - S(B^{-1}) + \frac{1}{4\pi N} \int d^2x \text{Tr}(A_+) \text{Tr}(A_-) \tag{20}$$

We can also reinterpret our result as indicating that the proper normal ordering prescription needed to define the operator $U_{ij}(x) U_{hk}^{-1}(x)$, appearing in (13), would lead to the operator identity:

$$: U_{ij}(x) U_{hk}^{-1}(x) : = \frac{1}{N} \delta_{ik} \delta_{jh} = \int dU U_{ij} U_{hk}^\dagger \tag{21}$$

It is interesting to rewrite our main result, Eq. (20), by considering the natural decomposition of the $U(N)$ group element in terms of an Abelian factor belonging to $U(1)$ and an element of the $SU(N)$ group $\tilde{U}$:

$$U_{ij} = e^{\frac{i\varphi}{N}} \tilde{U}_{ij}, \quad \det \tilde{U} = 1 \tag{22}$$

Under this decomposition, the action $S(U)$ splits into separate factors:

$$S(U_1 \tilde{U}) = S(U_1) + S(\tilde{U}) ; \quad S(U_1) = \frac{1}{8\pi} \int d^2x \partial_\mu \varphi \partial_\mu \varphi \tag{23}$$

When we apply the same procedure to the currents and sources, we find that they decompose into irreducible tensors

$$J_{\mu}^{ij} = \tilde{J}_{\mu}^{ij} + \frac{\epsilon_{\mu\nu} \partial^{\nu} \varphi}{8\pi\sqrt{N}} \delta^{ij}$$

$$J_{\pm}^{ij} = \tilde{J}_{\pm}^{ij} \mp \frac{1}{4\pi\sqrt{N}} \partial_{\pm} \varphi \delta^{ij}, \quad \text{Tr}(\tilde{J}_{\pm}) = 0 \quad (24)$$

and

$$A_{+} = \tilde{A}_{+} + \frac{1}{\sqrt{N}} \partial_{+} \alpha; \quad A_{-} = \tilde{A}_{-} + \frac{1}{\sqrt{N}} \partial_{-} \beta; \quad \text{Tr}(\tilde{A}_{\pm}) = 0 \quad (25)$$

where we have defined $A_{\pm} = e^{i\alpha/\sqrt{N}}$ and $B_{\pm} = e^{i\beta/\sqrt{N}}$. Inserting these expressions in (20), we find that it also decomposes according to

$$W(A_{+}, A_{-}) = -S(\tilde{A}) - S(\tilde{B}^{-1}) - S(A_{\pm} B_{\pm}^{-1}) \quad (26)$$

This formulation makes it evident that there is an invariance of the generating functional, corresponding to the transformation:

$$A \rightarrow A a_1(x), \quad B \rightarrow B a_1(x); \quad a_1^{ij} = e^{\frac{i\alpha(x)}{\sqrt{N}}} \delta^{ij} \quad (27)$$

This local U(1) invariance is a reflection of the fact that the Abelian current $(1/8\pi\sqrt{N})\epsilon_{\mu\nu} \partial^{\nu} \phi \delta^{ij}$, introduced in Eq. (24), is automatically conserved and this feature must survive in the quantized version of the model.

We conclude the discussion of the bosonic theory described by (1) with an observation concerning the structure of the quantum effective action of the model. In order to define this quantity, we must introduce the "classical" currents

$$j_+ = \frac{\delta W}{\delta A_-} = \frac{i}{4\pi} (B^{-1} \partial_+ B) + \frac{1}{4\pi N} \text{Tr}(A_+) \mathbb{1} \quad (28)$$

$$j_- = \frac{\delta W}{\delta A_+} = \frac{i}{4\pi} (A^{-1} \partial_- A) + \frac{1}{4\pi N} \text{Tr}(A_-) \mathbb{1}$$

The effective action is defined to be

$$\begin{aligned} \Gamma(j_+, j_-) &= W(A_+, A_-) - \int d^2x \text{Tr} (A_+ j_- + A_- j_+) = \\ &= S(\tilde{A}^{-1}) + S(\tilde{B}) + S(B_1 A_1^{-1}) \end{aligned} \quad (29)$$

Notice that the relationship between the "classical" currents $\tilde{j}_\pm$ and their respective contributions to the effective action Γ is the same as that of the "quantum" currents $\tilde{j}_\pm$ and the action $S(U)$ (where U is thought as a function of $\tilde{j}_\pm$), but the two contributions to Γ decouple, in contrast with what happens in $S(U)$. The Abelian factor behaves as a free field theory, as expected.

In the last part of this work, we want to show that the generating functional for the currents, obtained in the bosonic theory described by the action $S(U)$, is equal to the corresponding one in the free Fermi theory provided that we regularize this theory in a very specific way.

The generating functional $W_F(A_\mu)$ for the Green functions involving currents of the free Fermi theory with Dirac fermions can be written as

$$e^{iW_F(A_+, A_-)} = \int \mathcal{D}(\bar{\Psi}, \Psi) e^{i \int d^2x \bar{\Psi} i \not{D} \Psi} \quad (30)$$

where $D_\mu = \partial_\mu + iA_\mu$.

The fermion determinant appearing in (30) has been exactly computed and the result is

$$W_F(A_+, A_-) = -S(AB^{-1}) \quad (31)$$

where $S(AB^{-1})$ is the same expression defined in (1) and the sources A_+ , A_- have been parametrized as in (8).

The result (31) is based on the fact that one regularizes the Fermi theory preserving the vector symmetry.

However, one could also regularize it in such a way that the Green functions involving right- and left-handed currents are factorized in a product of a Green function with only the right currents and a Green function with only the left currents. In this case, instead of (31), one would get the following expression for the determinant:

$$\begin{aligned} W_F(A_+, A_-) &= -S(A) - S(B^{-1}) = \\ &= -S(AB^{-1}) - \frac{1}{4\pi} \int d^2x \operatorname{Tr}(A_+ A_-) \end{aligned} \quad (32)$$

Finally, one could also regularize it in such a way that only the vector current, corresponding to the $U(1)$ factor of $U(N)$, is conserved. In this case, one would get^{*)}

$$\begin{aligned} W_F(A_+, A_-) &= -S(A) - S(B^{-1}) + \\ &+ \frac{1}{4\pi N} \int d^2x \operatorname{Tr}(A_+) \operatorname{Tr}(A_-) = \\ &= -S(\tilde{A}) - S(\tilde{B}^{-1}) - S(A_1 B_1^{-1}) \end{aligned} \quad (33)$$

^{*)} Notice that the three results, that we find for the fermion determinant, differ by a local polynomial quadratic in the external field A_μ . This is precisely the arbitrariness due to the infinities of the theory, as discussed by Gasser and Leutwyler in Appendix A of Ref. 5).

that is exactly equal to the corresponding expression in the case of the bosonic theory described by the action in (1).

This equality implies the following bosonization rules for the currents in a non-Abelian Fermi theory:

$$\begin{aligned} -\bar{\psi} \gamma_+ \psi &\rightarrow \frac{i}{2\pi} U^{-1} \partial_+ U \\ -\bar{\psi} \gamma_- \psi &\rightarrow \frac{i}{2\pi} U \partial_- U^{-1} \end{aligned} \quad (34)$$

that are the same proposed by Witten in Ref. 1). However, it must be stressed that the equivalence is exactly valid only if the fermion theory is regularized in the way implied by (33). If we regularize the Fermi theory as in (31), the bosonization rules (34) are not exactly valid anymore and some caution must be used in bosonizing an interacting theory as, for instance, the non-Abelian Thirring model. Notice also that the regularization implied by (33) destroys vector gauge invariance and therefore it is not allowed in a locally invariant theory as QCD₂. This is reflected in the fact that the generating functional (10) is not vector gauge invariant³⁾.

Finally, the equivalence shown in the case of U(N) is also true for an O(N) group that was considered by Witten in Ref. 1). The only differences are that the action (1) must have an additional factor $\frac{1}{2}$ to be equivalent to a Fermi theory with Majorana fermions and that for $N > 2$ there is no Abelian factor group and therefore the last term in (26) is absent. Notice also that for $N = 2$, the theory is completely equivalent to the U(1) case.

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