NON-INTEGRAL CHARGES IN TWO AND FOUR DIMENSIONS

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A B S T R A C T

We show directly the connection between two and four dimensional soliton theories coupled to fermions, thus explaining the equality of their respective fractional charges. The dependence of these charges on the boundary conditions of the "half fermions" is also computed.

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It has been known for some time that in certain theories, where a fermion is coupled to a topological excitation (soliton), the fermionic vacuum develops a non-integer fermion number^{1),2)}. In Ref. 1), massless fermions were considered and a half-integer charge had been shown to arise due to a zero mode. In the second, a mass was included for the fermion resulting in equal transcendental charge for two-dimensional soliton-fermion and four-dimensional monopole-fermion systems. Some of the mysteries of this equality were unravelled³⁾ by showing the connection between the fractional charge and the chiral phase at infinity, via the use of anomalous commutators in both theories.

Here we shall explain directly the equality, by establishing the equivalence of the $j = 0$ sector of the monopole system in the limit $M_W \rightarrow \infty$ to the soliton theory, showing in fact that these two theories are identical. We use here the fact that the fermion charge in four dimensions is independent of M_W (see later).

For the monopole system⁴⁾, we take the case of an SU(2) gauge theory with the gauge fields A_i^a and the Higgs fields ϕ^a ($i = 1, 2, 3$ space index; $a = 1, 2, 3$ SU(2) adjoint index)

$$\begin{aligned} e A_i^a(x) &= \epsilon_{aij} \dot{X}_j A(r) \\ A_0^a(x) &= 0 \\ \phi^a(x) &= \dot{X}^a \phi(r) \end{aligned} \quad (1)$$

The asymptotic behaviour of the fields at large r is $A(r) = 1/r + O(e^{-M_W r})$ and $\phi(\infty) = v$. $A(r)$ and $\phi(r)$ vanish at $r = 0$. The fermionic Lagrangian is

$$\begin{aligned} \mathcal{L}_F &= \bar{\psi} i \not{D} \psi - \bar{\psi} (m + i g \gamma_5 \tau^a \phi^a) \psi \\ D_\mu &= \partial_\mu + \frac{i}{2} e A_\mu^a \tau^a \end{aligned} \quad (2)$$

This system is known to have²⁾ the fermionic charge

$$q = \frac{1}{\pi} \arctg \frac{g v}{m} \quad (3)$$

The same as for the two-dimensional theory

$$\mathcal{L}_F = \bar{\psi} i \not{\partial} \psi - \bar{\psi} (m + i g \gamma_5 \phi) \psi \quad (4)$$

where, here, $\phi(\infty) = -\phi(-\infty) = v$.

In order to show the equivalence of (2) and (4), we shall first use a method devised by Callan^{5),6)} [see also Ref. 1)] to reduce a $j = 0$ monopole-fermion system to a two-dimensional half-line theory.

The reason why the fractional charge is supposed to arise from the $j = L+S+I = 0$ sector (in the limit $M_W \rightarrow \infty$) is the following. It is only for the $j = 0$ partial wave that there is no centrifugal barrier around the monopole. Since the fractional charge arises from the monopole-fermion interaction the whole contribution will come from $j = 0$.

To complete our proof, we shall demonstrate that q is actually independent of M_W . We decompose the $j = 0$ Fermi field as

$$\psi_j^{(\pm)} = \begin{pmatrix} X_{\pm} \\ \pm X_{\pm} \end{pmatrix} \quad (5)$$

where the $\pm$ refer to helicity. X is a 2×2 matrix where one index is spin and the other is isospin. Then, for $j = 0$,

$$X_{\pm} = \frac{1}{\sqrt{8\pi} r} (g_{\pm} + p_{\pm} \hat{x} \cdot \hat{z}) \tau_z \quad (6)$$

We group $g_{\pm}(r)$ and $p_{\pm}(r)$ into two-dimensional fermions in a way which will be useful in the following:

$$\chi_{\pm} = \begin{pmatrix} g_{\pm} \\ \pm i p_{\pm} \end{pmatrix} \quad (7)$$

The extra interaction we have, as compared with Callan, is $i\phi \bar{\psi} \gamma_5 \vec{\tau} \cdot \vec{x} \psi$, and we obtain the Lagrangian (not the Lagrangian density)

$$\begin{aligned}
 L(t) = \int_0^\infty dr \left\{ \bar{\chi}_+ i \bar{\gamma}^\alpha \partial_\alpha \chi_+ + \bar{\chi}_- i \bar{\gamma}^\alpha \partial_\alpha \chi_- \right. \\
 + m (\bar{\chi}_+ \chi_- + \bar{\chi}_- \chi_+) + i(A - \frac{1}{r}) (\bar{\chi}_+ \bar{\gamma}_5 \chi_+ + \bar{\chi}_- \bar{\gamma}_5 \chi_-) \\
 \left. + i g \phi (\bar{\chi}_+ \bar{\gamma}_5 \chi_- + \bar{\chi}_- \bar{\gamma}_5 \chi_+) \right\} \quad (8)
 \end{aligned}$$

where the $\bar{\gamma}_\mu$ matrices are the two-dimensional ones:

$$\bar{\gamma}_0 = \tau_3 \quad \bar{\gamma}_1 = -i \tau_1 \quad \bar{\gamma}_5 = \bar{\gamma}_0 \bar{\gamma}_1 = \tau_2$$

$(A-1/r)$ vanishes exponentially rapidly outside the monopole core, and inside it goes like $-1/r$. For large M_W this term can be eliminated provided we impose the boundary condition $\tau_+ \chi_\pm(0) = 0$ ⁽⁵⁾⁻⁽⁷⁾.

It would be convenient then to perform the transformation:

$$\lambda_{1,2} = \frac{1}{\sqrt{2}} (\chi_+ \pm \chi_-) \quad (9)$$

in which case we obtain a flavour-diagonal interaction

$$L(t) = \int_0^\infty dr \sum_{n=1}^2 \left[\bar{\lambda}_n i \not{\partial} \lambda_n - (-)^n \bar{\lambda}_n (m + i g \not{\gamma}_5 \phi) \lambda_n \right] \quad (10)$$

and the boundary conditions $\tau_+ \lambda_n(0) = 0$, $n = 1, 2$. Now, in order to show that the half-line Lagrangian (10) is equivalent to the full line one (4), one further transformation must be made:

$$\begin{aligned}
 \rho(x) &= \frac{1}{\sqrt{2}} [\lambda_2(x) + \bar{\gamma}_5 \lambda_1(x)] ; \quad x > 0 \\
 \rho(x) &= \frac{1}{\sqrt{2}} \bar{\gamma}_0 [\lambda_2(-x) - \bar{\gamma}_5 \lambda_1(-x)] ; \quad x < 0 \quad (11)
 \end{aligned}$$

The boundary conditions $\tau_{+n}^{\lambda}(0) = 0$ ensure the continuity of ρ at $x = 0$. The Lagrangian (10) now takes the form

$$\mathcal{L}(t) = \int_{-\infty}^{\infty} dx \left[\bar{\psi} i \not{\partial} \psi - \bar{\psi} (m + i g \tilde{\phi}) \psi \right] \quad (12)$$

where $\tilde{\phi}(x)$ is defined as $\tilde{\phi}(x) = \begin{cases} \phi(x); & x > 0 \\ -\phi(-x); & x < 0 \end{cases}$. The Lagrangian (12) is identical to the two-dimensional theory (4).

We now have to see what happens to the fermion number current. The four-dimensional charge density can be translated as:

$$4\pi n^2 \psi^\dagger \psi = \chi_+^\dagger \chi_+ + \chi_-^\dagger \chi_- = j_0(\chi_+) + j_0(\chi_-) \quad (13)$$

We can see also that after the transformation (11), j_0 will obey:

$$j_0(\psi, x=r) + j_0(\psi, x=-r) = j_0(\chi_+, x=r) + j_0(\chi_-, x=r) \quad (14)$$

The vacuum is also translated into vacuum all the way. So, finally, we have for the fractional charge

$$q(2) = q(4) = q$$

where $q(2)$, $q(4)$ are the vacuum charge of the Lagrangians (2) and (4), respectively.

We have thus shown that the two theories are equivalent, and in such a way that their respective vacuum charges are identical. In fact, this holds for their charge densities as well as can be seen from (13) and (14). The latter is, however, particular to the $M_W \rightarrow \infty$ limit, as can be seen, for example, from the explicit expressions in Ref. 2).

It may be instructive to compute the fractional charges that arise from each of the "half" fermions λ_1, λ_2 . This will show the dependence on the boundary conditions. Suppose λ_2^ε is a classical solution with energy ε . Then from (10) we get

$$-i\bar{\gamma}_5 \partial_1 \lambda_2^\epsilon - (\bar{\gamma}_0 m + ig\phi\bar{\gamma}_1) \lambda_2^\epsilon = \epsilon \lambda_2^\epsilon \quad (15)$$

and the boundary condition $\tau_+ \lambda_2^\epsilon(0) = 0$. Define

$$\rho_2^\epsilon(x) = \begin{cases} \lambda_2^\epsilon(x) & ; \quad x > 0 \\ \bar{\gamma}_0 \lambda_2^\epsilon(-x) & ; \quad x < 0 \end{cases} \quad (16)$$

Then $\rho_2^\epsilon(x)$ is a solution of the equation derived from (12) with energy ϵ . It will obey as well $\tau_+ \rho_2^\epsilon(0) = 0$. The solutions of (12) can be seen to be of two types (due to the antisymmetry of $\tilde{\phi}$):

$$\text{Type 1: } \rho(x) = \begin{pmatrix} \text{antisymmetric} \\ \text{symmetric} \end{pmatrix}; \quad \text{Type 2: } \rho(x) = \begin{pmatrix} \text{symmetric} \\ \text{antisymmetric} \end{pmatrix}.$$

The transformation (16) has as its range all the solutions of Type 2. So the fractional charge coming from λ_2 is seen to be:

$$q_2 = \lim_{\lambda \rightarrow \infty} \int_{-\lambda}^{\lambda} dx \frac{1}{2} \sum_{n=\text{type 2}} [\rho^{(n)\dagger} \rho^{(n)} - \tilde{\rho}^{(n)\dagger} \tilde{\rho}^{(n)}] \quad (17)$$

Here $\tilde{\rho}^{(n)}$ are negative energy solutions. Care was taken in the definition here to achieve a proper regularization⁸⁾. Similarly, we can make the transformation $\lambda_1^\epsilon \rightarrow \bar{\gamma}_5 \lambda_1^\epsilon$, then use $-\bar{\gamma}_0$ for $x < 0$ in (16), and the sum now extends over Type 1 solutions. We obtain again

$$q = q_1 + q_2$$

Expression (17), and its analogue for ρ_1 , enable us to compute the charge for each of the "half" fermions separately. This can be done by solving (15) in the special case $\tilde{\phi}(x) = \epsilon(x)v$ and substituting the solution in (17). (For justification, see below.) It turns out that in this case there is a continuum of energies

$$[\epsilon(k)]^2 = k^2 + (gv)^2$$

and also one bound state. The continuum contributions for q_1 and q_2 turn out to be equal:

$$(q_1)_{\text{cont.}} = (q_2)_{\text{cont.}} = \frac{1}{2\pi} \arctg \frac{g\nu}{m} - \frac{1}{4} \epsilon(\nu)$$

For $\nu > 0$, there is a bound state with energy $\epsilon = m$ in λ_1 , while for $\nu < 0$ the bound state is with energy $\epsilon = -m$ in λ_2 . Thus

$$\begin{aligned} q_1 &= \frac{1}{2\pi} \arctg \frac{g\nu}{m} + \frac{1}{4} \\ q_2 &= \frac{1}{2\pi} \arctg \frac{g\nu}{m} - \frac{1}{4} \end{aligned} \tag{18}$$

Note that the different boundary conditions caused these two charges to be different.

The values of the charges q_1, q_2 depend only on the asymptotic behaviour of $\tilde{\phi}$ as is demonstrated in Ref. 9). This is why we can choose $\tilde{\phi}(x) = \epsilon(x)\nu$.

From a similar reasoning, the charge will be independent of M_W since it does not affect the asymptotic and $r = 0$ behaviour of the fields. Thus it is enough to treat the $M_W \rightarrow \infty$ limit, as we do here.

ACKNOWLEDGEMENTS

One of us (D.G.) would like to acknowledge discussions with G. Date, and another (Y.F.) with J. Ellis.

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