ONE-LOOP EFFECTS ON THE ROUND SEVEN-SPHERE

G.W. Gibbons

DAMTP, University of Cambridge, U.K.

and

H. Nicolai

CERN - Geneva

A B S T R A C T

We examine the quantum fluctuations of the massive tower of states arising in the spontaneous compactification of 11-dimensional supergravity on the round seven-sphere and show that the Casimir energy and the one-loop β function vanish "floor-by-floor". The Euler number divergence, in contrast, is found to vanish only if all higher modes are taken into account.

It has recently been shown that the 11-dimensional supergravity theory [1] may give rise to effective 4-dimensional theories via the mechanism of spontaneous compactification (for recent reviews, see [2,3]). This means that there exist solutions of the classical equations of motion in which the 11-dimensional spacetime decomposes into the product of four-dimensional Anti-De Sitter space and a seven-dimensional, compact, internal space. However from the point of view of the quantum theory this background should be regarded as a sort of "mean field" approximation. It is therefore important to examine the quantum fluctuations around these classical backgrounds. In this letter we present a calculation of the one loop corrections of the Freund-Rubin parameter for the compactification of the 11-dimensional supergravity on the round seven-sphere [4,5]. This parameter is related to the size of the internal manifold and thus to the gauge-coupling constant of the four-dimensional theory at the classical level. Therefore our calculation determines the contribution to its β -function from the infinite tower of massive states arising from the higher modes on the seven-sphere. Another quantity of interest is the Casimir energy of these massive states. The formal sum of the Casimir energies of all the states on the round seven-sphere equals the ground-state energy which must vanish if the ground-state is supersymmetric. The central result of this paper is that, for the round seven-sphere, both the β -function and the Casimir energy vanish. In fact, there occur more cancellations than are actually required since the various contributions cancel "floor by floor". From these results it follows that the $N = 8$ supersymmetry of the classical ground-state [5] is maintained to lowest non-trivial order in the quantized theory, and therefore supersymmetry is not broken by quantum corrections at this order. The relation between the $SO(8)$ gauge coupling constant and the cosmological constant is preserved at one loop. We also consider the contributions to the Euler number divergence. In contrast to the β -function

and Casimir energies each floor contributes a non-vanishing amount but their total contribution vanishes.

So far the β -function and Casimir energy have been calculated only for the massless modes. At the linearised level the truncation of the 11-dimensional theory on the round 7-sphere to the massless supermultiplet coincides with the gauged N=8 supergravity in 4 dimensions[6,5]. The renormalization of the cosmological constant, which is proportional to the gauge coupling constant, was calculated for all gauged supergravity theories in 4 dimensions by an explicit computation of the coefficient of the relevant counter-term [7]. For $N > 4$ the coefficient was found to vanish. Later this result was rederived by Curtright [8] who observed that the contribution of a particle of helicity s is given by

$$\beta(g) = (-1)^{2s} \frac{1}{96\pi^2} g^3 C(1-12s^2) \quad (1).$$

where C is the Casimir of the corresponding representation, i.e. the eigenvalue of the Casimir operator times the dimension of the representation. The β -function is then seen to vanish by spin-sum rules. It is interesting to note that formula (1), when applied to the massive tower of states arising in the compactification, gives negative contributions for charged gravitons and spin one fields, whereas in ordinary 4-dimensional gauge theories, negative contributions can only come from self interacting spin one fields.

The Casimir energies were calculated by Allen and Davis [9] who used ζ -function regularization of the divergent sum of the energy eigenvalues. Again spin-sum rules imply that the Casimir energy vanishes for $N > 4$ and that the ζ -function has no poles indicating the absence of one loop divergences. It should be pointed out that the relation of the

energy ζ -function regularization to the ζ -function regularization of functional determinants has only been fully worked out in the case when the metric component g_{00} is independent of space and time and $g_{0i} = 0$ (the "ultrastatic" case [10]). Its use in Anti-De Sitter space is thus somewhat of an extrapolation from known results. For some comments on this problem see [21]

To procede we recall that the spectrum on the round 7-sphere has recently been determined completely ([3,11,12,13], for the fermions see also [14]). The results are summarized in table 1. We use Dynkin labels to designate irreducible representations of $SO(8)$. In the first column of table 1 we give the spin and in the last column we show the associated energy labels. For each n table 1 represents an irreducible representation of $Osp(8|4)$. Thus n labels the "floors" of the infinite tower of massive states. The restrictions on n are such that the Dynkin labels only contain non-negative integers. In particular there are two states for which n may take the value -1 ("the basement"). These two states constitute the $N = 8$ singleton multiplet [13,15,24] but the corresponding modes on the round 7-sphere vanish identically. Therefore they certainly do not appear at the linearised level. In what follows we will also need the formulas for the dimension and the Casimir of an irreducible $SO(8)$ representation (a,b,c,d) . They are given by [11]

$$\begin{aligned}
 D(a,b,c,d) = & (1+a)(1+b)(1+c)(1+d) \left(1+\frac{a+b}{2}\right) \left(1+\frac{b+c}{2}\right) \left(1+\frac{b+d}{2}\right) \\
 & \cdot \left(1+\frac{a+b+c}{3}\right) \left(1+\frac{a+b+d}{3}\right) \left(1+\frac{b+c+d}{3}\right) \\
 & \cdot \left(1+\frac{a+b+c+d}{4}\right) \left(1+\frac{a+2b+c+d}{5}\right)
 \end{aligned} \tag{2}$$

and

$$C(a,b,c,d) = \frac{1}{28} D(a,b,c,d) \left[(a(a+b+\frac{c+d}{2}+6) + b(a+2b+c+d+10) + c(\frac{a+d}{2}+b+c+6) + d(\frac{a+c}{2}+b+d+6)) \right] \quad (3)$$

It is now straightforward to determine the contribution of the massive states to the one-loop β -function using the above results. For arbitrary n , one simply inserts (3) into (1) and sums all the contributions in a given $Osp(8|4)$ multiplet, i.e. for fixed n . We emphasize that for massive states the associated Higgs-states of lower helicity must be taken into account. As a consequence of (2), (3), this expression is an eighth order polynomial in the "floor"-label n , and we have therefore resorted to the algebraic programme REDUCE for its actual evaluation. The result is that the contributions cancel within each $Osp(8|4)$ multiplet, i.e. "floor by floor".

An alternative method of performing this calculation is by ζ -function regularization. This method is more elaborate in that it requires the knowledge of the energy spectrum of a given $SO(3,2)$ representation [16] but it has the advantage that it allows the determination of the Casimir energies as well. In figures 1 and 2 we have reproduced the weight diagrams of the unitary irreducible massless representations of $SO(3,2)$. These are labelled by the energy E_0 of the state of lowest energy and by its total spin s . Each dot in the weight diagram represents $2j+1$ states where $j=l+s$ denotes the total angular momentum of the corresponding states. For a massless representation, the representation labels, E_0 and s obey the relation $E_0 = s+1$. Note that the weight diagram for $s=0$ differs from the weight diagrams for higher spin in that it contains only "half as many" states. The weight diagrams for massive spin $s > 1/2$, for

which $E_0 > s+1$, are obtained by adjoining the lower helicity Higgs states such that the lowest energy is increased by one unit as the spin is decreased by one unit. For example, the weight diagram for a massive spin-1 particle is the superposition of helicity-1 and helicity-0 states shown in figure 3. The zero-point energy corresponding to these weight diagrams is given by a formally divergent sum. In order to regularize it, we define generalized ζ -functions as follows. For spin $s > 0$, we define

$$\begin{aligned}\zeta(z; E_0, s) &= \sum_{k, l=0}^{\infty} (2(l+s)+1)(E_0+k+l)^{-z} \\ &= \zeta(z-2, E_0) + 2(s+1-E_0)\zeta(z-1, E_0) + (E_0-1)(E_0-2s-1)\zeta(z, E_0)\end{aligned}\quad (4)$$

where

$$\zeta(z, \alpha) = \sum_{m=0}^{\infty} (m+\alpha)^{-z} \quad \text{Re } z > 1 \quad (5)$$

is the usual generalized ζ -function (see e. g. [17]). Formula (4) was first obtained by Hull [18]. From (4) and (5), one immediately verifies that $\zeta(z; E_0, s)$ is holomorphic in the region $\text{Re } z > 3$ of the complex plane. To the left of this region, the ζ -function is defined by analytic continuation. For the case of spin-0, the correct definition reads

$$\begin{aligned}\tilde{\zeta}(z; E_0,) &= \sum_{k, l=0}^{\infty} (2l+1)(E_0+2k+l)^{-z} \\ &= \frac{1}{2} \left[\zeta(z-2, E_0) + (3-2E_0)\zeta(z-1, E_0) + (E_0^2-3E_0+2)\zeta(z, E_0) \right]\end{aligned}\quad (6)$$

The range of summation in (4) and (6) is obtained by inspection of the

corresponding weight diagrams in figures 1 and 2. For the calculation of the Casimir energies, we have to evaluate these ζ -functions at $z=-1$. The relevant values of the ordinary ζ -function are given by [17]

$$\zeta(-n, \alpha) = - \frac{B_{n+1}(\alpha)}{n+1} \quad (7)$$

with

$$B_2(\alpha) = \alpha^2 - \alpha + \frac{1}{6} \quad (8)$$

$$B_3(\alpha) = \alpha^3 - \frac{3}{2} \alpha^2 + \frac{\alpha}{2}$$

$$B_4(\alpha) = \alpha^4 - 2 \alpha^3 + \alpha^2 - \frac{1}{30}$$

Substituting these formulas into equs. (4) and (6), we get for spin $s>0$

$$\begin{aligned} \zeta(-1; E_0, s) = & -\frac{1}{12} E_0^4 + \frac{s+1}{3} E_0^3 - \frac{2s+1}{2} E_0^2 \\ & + \frac{5s+2}{6} E_0 - \frac{2s+19}{120} \end{aligned} \quad (9)$$

which reduces to the correct result for $E_0 = s+1$ [10], namely

$$\zeta(-1; s+1, s) = \frac{1}{4} s^4 - \frac{1}{6} s^2 + \frac{1}{120} \quad (10)$$

For spin-0 particles on the other hand, we obtain

$$\tilde{\zeta}(-1; E_0) = -\frac{1}{24} E_0^4 + \frac{1}{4} E_0^3 - \frac{1}{2} E_0^2 + \frac{3}{8} E_0 - \frac{19}{240} \quad (11)$$

These formulas are sufficient for the massless case. For the massive

states, the contributions of the lower helicity states must be added. For massive spin 2 this procedure yields the result

$$\zeta(z; E_0, 2) + \zeta(z; E_0+1, 1) + \tilde{\zeta}(z; E_0+2) \quad (12)$$

whereas, for massive spin $\frac{3}{2}$, it leads to

$$\zeta(z; E_0, \frac{3}{2}) + \zeta(z; E_0+1, \frac{1}{2}) \quad (13)$$

Collecting all these terms for a fixed $Osp(8|4)$ multiplet characterized by the integer n , we arrive at the following expression

$$\begin{aligned} \zeta_{MULTIPLY}(z) = & \\ & D(n, 0, 0, 0) \left[\zeta(z; 3+\frac{n}{2}, 2) + \zeta(z; 4+\frac{n}{2}, 1) + \tilde{\zeta}(z; 5+\frac{n}{2}) \right] \\ & - D(n, 0, 0, 1) \left[\zeta(z; \frac{5}{2}+\frac{n}{2}, \frac{3}{2}) + \zeta(z; \frac{7}{2}+\frac{n}{2}, \frac{1}{2}) \right] \\ & - D(n-1, 0, 1, 0) \left[\zeta(z; \frac{7}{2}+\frac{n}{2}, \frac{3}{2}) + \zeta(z; \frac{9}{2}+\frac{n}{2}, \frac{1}{2}) \right] \\ & + D(n, 1, 0, 0) \left[\zeta(z; 2+\frac{n}{2}, 1) + \tilde{\zeta}(z; 3+\frac{n}{2}) \right] \\ & + D(n-1, 0, 1, 1) \left[\zeta(z; 3+\frac{n}{2}, 1) + \tilde{\zeta}(z; 4+\frac{n}{2}) \right] \\ & + D(n-2, 1, 0, 1, 1) \left[\zeta(z; 4+\frac{n}{2}, 1) + \tilde{\zeta}(z; 5+\frac{n}{2}) \right] \\ & - D(n+1, 0, 1, 0) \zeta(z; \frac{3}{2}+\frac{n}{2}, \frac{1}{2}) \\ & - D(n-1, 1, 1, 0) \zeta(z; \frac{5}{2}+\frac{n}{2}, \frac{1}{2}) \end{aligned}$$

$$\begin{aligned}
& -D(n-2,1,0,1)\zeta(z;\frac{7}{2},\frac{n}{2},\frac{1}{2}) \\
& -D(n-2,0,0,1)\zeta(z;\frac{9}{2},\frac{n}{2},\frac{1}{2}) \\
& +D(n+2,0,0,0)\tilde{\zeta}(z;1+\frac{n}{2}) \\
& +D(n,0,2,0)\tilde{\zeta}(z;2+\frac{n}{2}) \\
& +D(n-2,2,0,0)\tilde{\zeta}(z;3+\frac{n}{2}) \\
& +D(n-2,0,0,2)\tilde{\zeta}(z;4+\frac{n}{2}) \\
& +D(n-2,0,0,0)\tilde{\zeta}(z;5+\frac{n}{2})
\end{aligned} \tag{14}$$

Again, using REDUCE we have shown that $\zeta_{\text{MULTIPL}}(z)$ vanishes at $z = -1$ for arbitrary n . In a similar way one may analyse the pole structure of $\zeta_{\text{MULTIPL}}(z)$. The generalized Riemann ζ -function has a single first order pole at $z = 1$, with residue one. Therefore the function $\zeta_{\text{MULTIPL}}(z)$ could, a priori, have poles at $z = 3, 2, 1$. By use of (4) and (6) together with (14) the residues of $\zeta_{\text{MULTIPL}}(z)$ at these values of z may be easily evaluated. Again they are found to vanish separately for each n . Hence $\zeta_{\text{MULTIPL}}(z)$ is an entire function for arbitrary n . This is significant because the residues depend upon the asymptotic density of states at high energy which enter the divergent integrals of the quantized theory.

As we mentioned above the states belonging to the $N = 8$ singleton multiplet do not appear at the linearized level. Nonetheless, one may enquire whether they contribute to the zero-point energy. In fact, their

possible contribution to the Casimir energy may be calculated using the ζ -function method, whereas Curtright's method is inapplicable since these multiplets have no Poincare limit. From the weight diagram for the singleton multiplet depicted in figure 4 one may derive their contribution to the ζ -function as

$$\zeta_{SINGLETON}(z) = 16\zeta(z-1, \frac{1}{2}) - 16\zeta(z-1, 1) \quad (15)$$

where a factor of 8 is included because the states belong to 8-dimensional representations of $SO(8)$. Evidently the residues at $z=2$ cancel and one easily checks from formulas (7) and (8) that $\zeta_{SINGLETON}(-1) = 0$.

The vanishing of the β -function is certainly related to the absence of one-loop counter-terms in the 11-dimensional theory[19]. It should be pointed out, however, that the Freund-Rubin parameter is not a parameter of the original Lagrangian of the 11-dimensional theory. The obvious analogy is that of quantum electrodynamics with an external electromagnetic field ("electrovac"). In order to render this theory finite at one loop, the background field must be renormalized in the same way as the fluctuations, and therefore the renormalization properties of the theory in an electromagnetic background coincide with those in a vanishing background [20]. Although this argument indicates that the total contribution of the massless and the massive modes to the β -function vanishes, it is surprising to discover that the cancellations actually take place within each $Osp(8|4)$ multiplet, i.e. "floor by floor". The argument also implies that the β -function will vanish at one loop for any other background. In some cases, this may be explicitly checked: for instance, using Curtright's formula, we have verified the vanishing at one loop for some of the

stationary points of the $N = 8$ potential.

While the one-loop contribution to the β -function does not depend upon the background around which one expands, the Casimir energies do depend upon it. Obviously it would be of interest to investigate this problem in a more general context but this requires knowledge of the full spectrum of fluctuations around compactified solutions other than the round 7-sphere which is not yet available. Of special importance is the question of whether or not the classical relation between gauge coupling constant and cosmological constant is preserved in broken phases of the quantized theory. The vanishing of the quantum corrections to the round seven-sphere indicates that, at least if $\Lambda \neq 0$, the mechanism suggested by Candelas and Raine [22] for "flattening" the curvature of spacetime will not work.

Finally we turn to the Euler number divergence. For the massless modes this known not to vanish [7]. From this it is obvious that there can be no floor by floor cancellation. In fact one finds, using the results tabulated in [7], that each floor contributes an amount equal to $D(n,0,0,0)$ times the contribution of the massless sector. The total contribution is therefore formally

$$-5 \cdot \sum_{n=0}^{\infty} D(n,0,0,0) \quad (16)$$

Just as before this can be regularized by introducing yet another ζ -function by

$$\zeta_{\Delta}(z) = 1 + \sum_{n=1}^{\infty} D(n,0,0,0) \left[n(n+6) \right]^{-z} \quad (17)$$

This ζ -function is that associated with the scalar laplacian Δ on the round 7-sphere whose eigenvalues are $n(n+6)$ with multiplicity $D(n,0,0,0)$. We are interested in $\zeta_{\Delta}(0)$ which represents the "renormalized" total number of modes and which is obtained by analytically continuing $\zeta_{\Delta}(z)$ from the region $\text{Re}z > \frac{7}{2}$ where the series in (17) converges. By use of the following integral representation

$$\zeta_{\Delta}(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} \text{Tr} \exp(-t\Delta) dt \quad (18)$$

It follows that $\zeta_{\Delta}(0)$ is proportional to the constant term in the asymptotic expansion of $\exp(-t\Delta)$ at small t . It is well known that this expansion is of the form $(4\pi t)^{\frac{-d}{2}} \times (\text{series in positive integral powers of } t)$ where d is dimension of the space over which Δ is defined [23]. Since in our case $d=7$ there is no constant term on the 7-sphere and thus^{*}

$$\zeta_{\Delta}(0) = 0 \quad (19)$$

Therefore the sum of the contributions of all modes to the Euler counter-term vanishes. That is, the massive modes exactly cancel the contribution of the massless sector. Similar arguments apply to any other 7-dimensional internal manifold.

Acknowledgements H.N. is grateful to D.A.M.T.P. for the hospitality extended to him during his stay. G.W.G. would like to thank C.M. Hull for some useful discussions.

*A similar cancellation has been shown to take place in the simpler 5-dimensional Kaluza-Klein theory [19]

Spin	SO(8) content		E_0
2^+	$(n \ 0 \ 0 \ 0)$	$n \geq 0$	$3 + n/2$
$3/2^{(1)}$	$(n \ 0 \ 0 \ 1)$	$n \geq 0$	$5/2 + n/2$
$3/2^{(2)}$	$(n-1 \ 0 \ 1 \ 0)$	$n \geq 1$	$7/2 + n/2$
$1^{-(1)}$	$(n \ 1 \ 0 \ 0)$	$n \geq 0$	$2 + n/2$
1^+	$(n-1 \ 0 \ 1 \ 1)$	$n \geq 1$	$3 + n/2$
$1^{(-2)}$	$(n-2 \ 1 \ 0 \ 0)$	$n \geq 2$	$4 + n/2$
$1/2^{(1)}$	$(n+1 \ 0 \ 1 \ 0)^*$	$n \geq 0$	$3/2 + n/2$
$1/2^{(2)}$	$(n-1 \ 1 \ 1 \ 0)$	$n \geq 1$	$5/2 + n/2$
$1/2^{(3)}$	$(n-2 \ 1 \ 0 \ 1)$	$n \geq 2$	$7/2 + n/2$
$1/2^{(4)}$	$(n-2 \ 0 \ 0 \ 1)$	$n \geq 2$	$9/2 + n/2$
$0^{+(1)}$	$(n+2 \ 0 \ 0 \ 0)^*$	$n \geq 0$	$1 + n/2$
$0^{-(1)}$	$(n \ 0 \ 2 \ 0)$	$n \geq 0$	$2 + n/2$
$0^{+(2)}$	$(n-2 \ 2 \ 0 \ 0)$	$n \geq 2$	$3 + n/2$
$0^{-(2)}$	$(n-2 \ 0 \ 0 \ 2)$	$n \geq 2$	$4 + n/2$
$0^{+(3)}$	$(n-2 \ 0 \ 0 \ 0)$	$n \geq 2$	$5 + n/2$

TABLE 1 - The spectrum of supergravity on the 7-sphere. The states marked by an asterisk constitute the $N = 8$ singlet multiplet when $n = -1$.

REFERENCES

- 1) E. Cremmer, B. Julia and J. Scherk - Phys.Lett. 76B (1978) 409.
- 2) L. Castellani, R. D'Auria and P. Fré - in "Supersymmetry and Supergravity 83", Proc. 19th Winter School on Theoretical Physics, Karpacz, ed. B. Milewski (World Scientific);
M.J. Duff, B.E.W. Nilsson and C.N. Pope - Preprint ICTP/82-83/29 (1983);
B. de Wit - Preprint NIKHEF-H/83-18 (1983);
P. Van Nieuwenhuizen - Les Houches Lecture Notes, to appear (1983).
- 3) F. Englert and H. Nicolai - CERN Preprint TH.3711 (1983).
- 4) P.G.O. Freund and M.A. Rubin - Phys.Lett. 97B (1980) 233.
- 5) M.J. Duff and C.N. Pope - in "Supergravity '82", eds. S. Ferrara, J.G. Taylor and P. Van Nieuwenhuizen (World Scientific, 1983).
- 6) B. Biran, B. de Wit, F. Englert and H. Nicolai - Phys.Lett. 124B (1983) 45.
- 7) S.M. Christensen, M.J. Duff, G.W. Gibbons and M. Rocek - Phys.Rev.Lett. 45 (1980) 161.
- 8) T. Curtright - Phys.Lett. 102B (1981) 17.
- 9) B. Allen and S. Davis - Phys.Lett. 124B (1983) 353.
- 10) G.W. Gibbons - in "Differential Geometric Methods in Mathematical Physics II", eds. K. Bleuler, H.R. Petry and A. Reetz, Lecture Notes in Mathematics 676 (Springer Verlag).
- 11) D.Z. Freedman and H. Nicolai - Nucl.Phys. B237 (1984) 342.
- 12) B. Biran, A. Casher, F. Englert, M. Rومان and P. Spindel - Phys.Lett. 134B (1984) 179;
A. Casher, F. Englert, H. Nicolai and M. Rومان - CERN Preprint TH. 3794 (1984).
- 13) E. Sezgin - Trieste Preprint IC/83/220 (1983).
- 14) R. D'Auria and P. Fré - Torino Preprint IFTT 430 (1983).
- 15) M. Günaydin - in "Group Theoretical Methods in Physics", Lecture Notes in Physics 180 (Springer Verlag) (1983).
- 16) C. Fronsdal - Rev.Mod.Phys. 37 (1965) 221; Phys.Rev. D10 (1974) 589 and Phys.Rev. D12 (1975) 3819;
C. Fronsdal and R.B. Haugen - Phys.Rev. D12 (1975) 3819.
- 17) W. Magnus, F. Oberhettinger and R.P. Soni - "Formulae and Theorems for the Special Functions of Mathematical Physics" (Springer Verlag).
- 18) C.M. Hull - DAMTP Preprint (1983), to appear in Nucl.Phys. B.
- 19) M.J. Duff and D. Toms - in "Unification of the Fundamental Particle Interaction II", eds. J. Ellis and S. Ferrara (Plenum Press) (1982).

- 20) J. Schwinger - Phys.Rev. 82 (1951) 664.
- 21) J.S. Dowker - Phys.Rev. 28D (1983) 3031.
- 22) P. Candelas and D.J. Raine - Texas Preprint (1983).
- 23) P. Gilkey - "The Index Theorem and the Heat Kernel" (Publish or Perish) (1974).
- 24) H. Nicolai and E. Sezgin - CERN Preprint TH. 3845 (1984).

FIGURE CAPTIONS

Figure 1 The massless spin-0 representations. The diagram is the same for massive spin-0 representations for which $E_0 \neq 1, 2$.

Figure 2 The massless representations for spin greater than 0.

Figure 3 Massive spin-1 representations. The states marked by a cross x decouple for $E_0 = 2$.

Figure 4 Singleton spectrum: bosonic states are marked by a circle 0 and fermionic ones by a triangle Δ .

- Figure 1 -

- Figure 2 -

- Figure 3 -

- Figure 4 -