SINGLETON REPRESENTATIONS OF $Osp(N,4)$

H. Nicolai

CERN -- Geneva

and

E. Sezgin

International Centre for Theoretical Physics, Trieste

A B S T R A C T

We demonstrate the existence of singleton supermultiplets in $Osp(N,4)$ for all N . These represent extreme cases of multiplet shortening because they contain only particles of spin $s = 0$ and $s = \frac{1}{2}$. We also discuss some of their field theoretic aspects.

It is well known that the natural background geometry of gauged extended supergravity theories¹⁾ is anti-de Sitter space (=AdS) with an $SO(3,2)$ group of isometries. The bosonic symmetry group of these theories also contains an $SO(N)$ group and therefore the relevant superalgebra is the graded extension of $SO(3,2) \times SO(N)$ which is $Osp(N,4)$. It is clearly important to study the representations of $Osp(N,4)$ which describe particles in AdS and which are relevant for supersymmetric field theories in an AdS background. These must be unitary and hence are infinite-dimensional since $SO(3,2)$ is non-compact. Positive energy representations of $SO(3,2)$ and their relation to wave equations in AdS have been extensively studied by Fronsdal²⁾. These representations are characterized by their lowest energy eigenvalue E_0 and the total angular momentum number s of their lowest energy state and are denoted by $D(E_0, s)$. Unitarity requires that $E_0 \geq s+1$ for $s = 1, 3/2, \dots$, and $E_0 \geq s+\frac{1}{2}$ for $s = 0, \frac{1}{2}$. All of these representations become ordinary massless representations of the Poincaré group in the limit where the radius of AdS becomes infinite except those with $E_0 = s+\frac{1}{2}$ for $s = 0, \frac{1}{2}$ which are exceptional: the singleton representations $D(\frac{1}{2}, 0)$ and $D(1, \frac{1}{2})$ ³⁾ possess no Poincaré limit. In this note, we demonstrate the existence of singleton supermultiplets in $Osp(N,4)$ for arbitrary $N \geq 1$ and investigate some of their field theoretic properties. In the case $N = 1$, the existence of a singleton multiplet was already shown by Heidenreich⁴⁾ who constructed all particle representations of $Osp(1,4)$. Representations of $Osp(N,4)$ for arbitrary N were investigated in Refs. 5) and 6). In Ref. 6), it was shown that there is a new type of multiplet shortening in these algebras for $N > 1$. The N -extended singleton multiplets constitute the extreme case of multiplet shortening as their $SO(3,2)$ vacuum states contain only representations with $s = 0$ and $s = \frac{1}{2}$ for arbitrary N ; this is to be contrasted with ordinary representations of both Poincaré supersymmetry and $Osp(N,4)$ which require particles of helicity $s \geq 1$ for $N \geq 3$.

For the convenience of the reader we now briefly summarize the essential properties of the $Osp(N,4)$ algebra and refer to Ref. 6), whose conventions and notations we follow in the first part of this paper, for further details and explanations. The even elements of $Osp(N,4)$ are the 10 Hermitean $SO(3,2)$ generators $M_{AB} = -M_{BA}$ where $A, B = 0, 1, \dots, 4$ and $N(N-1)/2$ Hermitean $SO(N)$ generators $T_{ij} = -T_{ji}$ where $i, j = 1, \dots, N$. The odd elements are given by $4N$ Majorana spinor charges Q_α^i , where $\alpha = 1, \dots, 4$ is a Dirac index. The part of

the algebra containing the spinor charges is characterized by the relations

$$\{Q_\alpha^i, \bar{Q}_\beta^j\} = 2\delta^{ij} \ell_{\alpha\beta}^{AB} M_{AB} + i\delta_{\alpha\beta} T^{ij} \quad (1)$$

$$[M_{AB}, Q_\alpha^i] = -i(\ell_{AB})_{\alpha\beta} Q_\beta^i \quad (2)$$

$$[T^{ij}, Q_\alpha^k] = i\delta^{ik} Q_\alpha^j - i\delta^{jk} Q_\alpha^i \quad (3)$$

with corresponding relations for the adjoint operators $\bar{Q}_\alpha^i$. The $SO(3,2)$ and $SO(N)$ structure relations are, respectively,

$$[M_{AB}, M_{CD}] = i(\eta_{AD} M_{BC} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{BC} M_{AD}) \quad (4)$$

$$[T^{ij}, T^{kl}] = -i(\delta^{jk} T^{il} - \delta^{ik} T^{jl} - \delta^{jl} T^{ik} + \delta^{il} T^{jk}) \quad (5)$$

where $\eta_{AB} = \text{diag}(+, -, -, -, +)$. As in Ref. 6), the Majorana spinor Q_α^i can be parametrized as

$$Q_\alpha^i = \begin{pmatrix} a_\alpha^i \\ \epsilon_{\alpha\beta} \bar{a}_\beta^i \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (6)$$

where the operators a_α^i and $\bar{a}_\alpha^i = (a_\alpha^i)^\dagger$ are fermionic lowering and raising operators whose properties have been discussed in Ref. 6). We also recall the method employed in Ref. 6) to construct positive energy representations of the algebra (1)-(5). One starts from a vacuum state $|\text{vac}\rangle$ which is annihilated by the energy lowering operators a_α^i

$$a_\alpha^i |\text{vac}\rangle = 0 \quad (7)$$

As a consequence of the $Osp(N,4)$ algebra, the vacuum state is also annihilated by the $SO(3,2)$ energy de-boost operators $M_a^- \equiv iM_{0a} + M_{a4}$ ($a = 1,2,3$). The representation space is spanned by all vectors of the form

$$(M_1^+)^{n_1} (M_2^+)^{n_2} (M_3^+)^{n_3} \bar{B} |vac\rangle \quad (8)$$

Here, $M_a^+ \equiv iM_{0a} - M_{a4}$ are the $SO(3,2)$ boost operators, n_i are non-negative integers and the set $\bar{B}$ consists of all combinations of products of the operators a_{α}^{-i} which are antisymmetric under interchange of the index pairs $(\frac{1}{\alpha})$; the latter set contains 2^{2N} operators. A unitary representation of $Osp(N,4)$ is defined on the span of (8) by imposing a Hilbert space structure. The vacuum state is assumed to be orthonormal. One then calculates the norms of all higher states (8) by using the structure relations (1)-(5). The requirement that the norms be non-negative, which is equivalent to the requirement of unitarity, imposes restrictions on the quantum numbers of the vacuum. In the limiting case where some norms vanish, the corresponding states are absent from this representation. This is the phenomenon of multiplet shortening.

We are here interested in singleton representations and will therefore assume from the outset that the vacuum state is characterized by the relations $E_0 = \frac{1}{2}$ and $s = 0$. Furthermore, the vacuum must belong to the $2^{[N/2]}$ dimensional spinor representation of $SO(N)$, and we adopt the Gel'fand-Zeythin labelling^{1*}. Thus, (7) becomes

$$\begin{aligned} a_{\alpha}^i |\phi\rangle &= \\ &= a_{\alpha}^i |(E_0 = \frac{1}{2}, s = 0) (\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{[N/2] \text{ times}})\rangle = 0 \end{aligned} \quad (9)$$

$(\frac{1}{2}, \dots, \frac{1}{2})$ denotes the highest weight. It is not difficult to label the $2^{[N/2]}$ components of the vacuum states, but for our purposes it suffices to consider only one label, M , which is the eigenvalue of T^{12} , i.e.,

^{1*})See, e.g., the fourth paper in Ref. 9).

$$\begin{aligned} T^{12} |(\frac{1}{2}, 0) (\frac{1}{2}, \dots, \frac{1}{2}) M\rangle &= \\ &= M |(\frac{1}{2}, 0) (\frac{1}{2}, \dots, \frac{1}{2}) M\rangle, \quad M = \pm \frac{1}{2} \end{aligned} \quad (10)$$

Acting on the vacuum state with $\bar{a}_1^{-1}$, we obtain a new set of states with $E_0 = 1$ and $s = \frac{1}{2}$. Hence,

$$\bar{a}_\alpha^i |\phi\rangle = c_1 \Gamma^i |\chi\rangle + c_2 |\lambda^i\rangle \quad (11)$$

where c_1 and c_2 are constants and the Γ^i 's form a representation of the $SO(N)$ Clifford algebra (the Γ^i 's play the role of Clebsch-Gordan coefficients here). Furthermore,

$$|\chi\rangle = |(1, \frac{1}{2}) (\frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})\rangle \quad (12)$$

with + sign for odd N and - sign for even N , and

$$|\lambda^i\rangle = |(1, \frac{1}{2}) (\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})\rangle \quad (13)$$

with $\Gamma^i |\lambda_i\rangle = 0$. Again, it will not be necessary to exhibit a labelling system. We shall assume, however, that the vacuum state and $|\lambda_i\rangle$ are normalized to one. Specializing to $i = 1, 2$, we get from (11)

$$\begin{aligned} (\bar{a}_1^1 + i\bar{a}_1^2) |\phi\rangle &= c_1 (\Gamma^1 + i\Gamma^2) |\chi\rangle + \\ &+ c_2 (|\lambda^1\rangle + i|\lambda^2\rangle) \end{aligned} \quad (14)$$

Using the $Osp(N, 4)$ algebra, we can easily calculate the norm of both sides in this equation. The result is

$$\begin{aligned}
 \langle \phi | (M_{04} + J^3 - T^{12}) | \phi \rangle &= \\
 &= |c_1|^2 \langle \chi | (1 + i \Gamma^{12}) | \chi \rangle + \\
 &\quad + |c_2|^2 \| (|\lambda^1\rangle + i|\lambda^2\rangle) \|^2
 \end{aligned} \tag{15}$$

where $\Gamma^{12} = \frac{1}{2}[\Gamma^1, \Gamma^2]$. Remembering that $J^3|\phi\rangle = 0$, $M_{04}|\phi\rangle = \frac{1}{2}|\phi\rangle$ from (9), and noting that $T^{12} = -(i/2)\Gamma^{12}$, we find

$$\frac{1}{2} - M = (1 - 2M)|c_1|^2 + \frac{1}{2}|c_2|^2 \| (|\lambda^1\rangle + i|\lambda^2\rangle) \|^2 \tag{16}$$

where we have made use of the assumed orthonormality as well as Eq. (10). $M = \frac{1}{2}$ now implies $c_2 = 0$ while $M = -\frac{1}{2}$ yields $|c_1|^2 = \frac{1}{2}$. Choosing $c_1 = 1/\sqrt{2}$, we thus have

$$\bar{a}_i' | \phi \rangle = \frac{1}{\sqrt{2}} \Gamma^i | \chi \rangle \tag{17}$$

Continuing in this manner, we now consider states obtained by applying an antisymmetrized product of two $\bar{a}_\alpha^{-1}$'s to the vacuum state. There are only three operators of this type

$$\begin{aligned}
 A &\equiv \epsilon_{\alpha\beta} \bar{a}_\alpha^i \bar{a}_\beta^i \\
 B &\equiv \epsilon_{\alpha\beta} \left(\bar{a}_\alpha^i \bar{a}_\beta^j + \bar{a}_\alpha^j \bar{a}_\beta^i - \frac{2}{N} \delta^{ij} \bar{a}_\alpha^k \bar{a}_\beta^k \right) \\
 C &\equiv \bar{a}_{(\alpha}^i \bar{a}_{\beta)}^j - \bar{a}_{(\alpha}^j \bar{a}_{\beta)}^i
 \end{aligned} \tag{18}$$

Using the $Osp(N,4)$ algebra again, we may straightforwardly calculate the norms of the states $A|\phi\rangle$ and $B|\phi\rangle$ with the results

$$\begin{aligned}
 \| A | \phi \rangle \|^2 &= \\
 &= \langle \phi | \left(2 N M_{04} (2 M_{04} + N - 2) - \right. \\
 &\quad \left. - 4 T^{ij} T^{ij} - 4 N J^a J^a \right) | \phi \rangle
 \end{aligned} \tag{19}$$

and

$$\begin{aligned}
 \| B | \phi \rangle \|^2 &= \| A | \phi \rangle \|^2 + \\
 &+ \langle \phi | \left(4 N (N+1) M_{04}^2 - 4 N^2 M_{04} - \right. \\
 &\quad \left. - 4 N (N+1) J^a J^a + 4 T^{ij} T^{ij} \right) | \phi \rangle
 \end{aligned} \tag{20}$$

Inserting the values $E_0 = \frac{1}{2}$, $s = 0$ and

$$\langle \phi | T^{ij} T^{ij} | \phi \rangle = \frac{1}{4} N (N-1) \tag{21}$$

one verifies that both states have zero norm for all N .

The calculation is slightly different for the state $C | \phi \rangle$ because of boost admixtures. We have

$$\bar{a}_{(\alpha}^{[i} \bar{a}_{\beta)}^{j]} | \phi \rangle = c_3 \bar{a}_{(\alpha}^k \bar{a}_{\beta)}^k T^{ij} | \phi \rangle + | \overset{ij}{\underset{\alpha\beta}{} } \rangle \tag{22}$$

where c_3 is a constant. The first term on the right-hand side of (22) is not genuinely new because $\bar{a}_{(\alpha}^k \bar{a}_{\beta)}^k$, being a symmetric combination, is proportional to the $SO(3,2)$ boost operators M_a^+ and therefore belongs to the Regge trajectory of the vacuum state to which the state $| \overset{ij}{\underset{\alpha\beta}{} } \rangle$ is orthogonal by

construction. To determine c_3 , we observe that $|_{\alpha\beta}^{ij}>$ is annihilated by the $SO(3,2)$ de-boosts as it is a new state in the supermultiplet. Applying the de-boost to both sides of (22), we obtain, after a little calculation,

$$\begin{aligned} i(\sigma_2 \sigma_a)_{\alpha\beta} T^{ij} |\phi> &= \\ &= c_3 N (\sigma_2 \sigma_a)_{\alpha\beta} (M_{04} + \sigma^a J^a) T^{ij} |\phi> \end{aligned} \quad (23)$$

from which we read off that $c_3 = i/NE_0$. Substituting this value and $E_0 = \frac{1}{2}$ into (22) and calculating the norms on both sides, we find that,

$$\langle \begin{smallmatrix} ij \\ \alpha\beta \end{smallmatrix} | \begin{smallmatrix} ij \\ \alpha\beta \end{smallmatrix} \rangle = 0 \quad \text{for all } N \quad (24)$$

Thus, we have established that no new states are obtained by applying products of two fermion creation operators to the vacuum since $A|\phi>$ and $B|\phi>$ have vanishing norm, and $C|\phi>$ is an element of the space belonging to the Regge trajectory of the vacuum. Applying products of three and more fermionic operators will then lead to no new states. Altogether, we have thus shown that the states $|\phi>$ and $|\chi>$ already form a supermultiplet by themselves for arbitrary N . For $N = 1$, one recovers the singleton multiplet discussed in Ref. 4); for $N = 2$, the $s = 0$ state has charge $+\frac{1}{2}$ whereas the $s = \frac{1}{2}$ state has charge $q = -\frac{1}{2}$ while for $N = 3$, both states belong to the $J = \frac{1}{2}$ representation of $SO(3)$, etc. For $N = 8$, the supercharge, and the two singleton states belong to the three inequivalent eight-dimensional representations of $SO(8)$. The weight diagram of the $N = 8$ singleton representation has already been constructed by Günaydin⁵⁾. It is obvious from our construction that singleton representations also exist for all $N > 8$.

Having established the existence of the singleton representations of $Osp(N,4)$, we now consider the problem of building a Lagrangian field theory for them. For $N = 1$, Fronsdal has already shown that³⁾ no physical information is lost if one writes the singleton action as a surface integral over spatial infinity of AdS , which is $S^2 \times S^1$. This is not surprising, since in $d = 3$, a Majorana fermion has one physical degree of freedom, which matches the single bosonic degree of freedom in three dimensions. The AdS group $SO(3,2)$ acts as the conformal group on $S^2 \times S^1$. Accordingly, one can introduce only interaction terms that are consistent with this conformal invariance³⁾.

One way to derive an action for singletons is the following. Consider the free action of a massive $Osp(1,4)$ multiplet in AdS, which has been given by Breitenlohner and Freedman and which reads⁷⁾

$$S = \frac{1}{2} \int d^4x \sqrt{-\bar{g}} \left\{ \bar{g}^{\mu\nu} \partial_\mu A \partial_\nu A + \bar{g}^{\mu\nu} \partial_\mu B \partial_\nu B + \right. \\ \left. + i \bar{\chi} \gamma^\mu \bar{D}_\mu \chi + (2a^2 + am - m^2) A^2 + \right. \\ \left. + (2a^2 - am - m^2) B^2 - m \bar{\chi} \chi \right\} \quad (25)$$

where a is the inverse AdS radius and m an arbitrary mass parameter. $\bar{g}^{\mu\nu}$ is the AdS background metric and $\bar{D}_\mu$ denotes the AdS-covariant derivative [in what follows, we will rely on the results, conventions and notation of Ref. 7)]. The action (25) is invariant under

$$\delta A = \frac{1}{\sqrt{2}} \bar{\epsilon} \chi \quad \delta B = \frac{i}{\sqrt{2}} \bar{\epsilon} \gamma^5 \chi \\ \delta \chi = - \frac{1}{\sqrt{2}} \left[i \gamma^\mu \partial_\mu (A + i \gamma^5 B) + \right. \\ \left. + a (A - i \gamma^5 B) + m (A + i \gamma^5 B) \right] \epsilon \quad (26)$$

Here, $\epsilon(x)$ is a Killing spinor in AdS which obeys $\tilde{D}_\mu \epsilon = 0$ and can be written as

$$\epsilon(x) = S(x) \xi \quad (27)$$

with constant ξ and

$$S(x) = (\cos \vartheta)^{-1/2} \left[\cos \frac{\vartheta}{2} + i \gamma^i \hat{x}^i \sin \frac{\vartheta}{2} \right] \left[\cos \frac{t}{2} - i \gamma^0 \sin \frac{t}{2} \right] \quad (28)$$

where we have introduced polar co-ordinates $x^1 = \rho \sin \theta \cos \phi$, $x^2 = \rho \sin \theta \sin \phi$, $x^3 = \rho \cos \theta$ and the radial unit vector $\hat{x}^i \equiv x^i / \rho$. In these co-ordinates, the

AdS metric is given by

$$ds^2 = (a \cos \varphi)^{-2} \left[dt^2 - d\varphi^2 - \sin^2 \varphi (d\theta^2 + \sin^2 \theta d\phi^2) \right]^{(29)}$$

The angular variables θ and ϕ satisfy the usual constraints, whereas $0 \leq \rho \leq \pi/2$ and spatial infinity corresponds to $\rho = \pi/2$. The time co-ordinate t is also periodic with period 2π , but from the dependence of $S(x)$ on t in (28), it is obvious that one should formulate the theory on the double covering of AdS, i.e., identify t with $t+4\pi$. The results of Ref. 7) guarantee that the commutator algebra following from (26) coincides with the algebra of $Osp(1,4)$, i.e., Eqs. (1)-(5) for $N = 1$. We next observe that, in the limit $\rho \rightarrow \pi/2$, the matrix S tends to

$$S(x) \xrightarrow{\varphi \rightarrow \frac{\pi}{2}} \sqrt{2} (\cos \varphi)^{-1/2} P_+ M(t) \quad (30)$$

where $M(t) \equiv \cos(t/2) - i\gamma^0 \sin(t/2)$. Furthermore,

$$\begin{aligned} P_{\pm} &\equiv \frac{1}{2} (1 \pm i\gamma^i \hat{x}^i) \\ P_{\pm}^2 &= P_{\pm}, \quad P_+ P_- = P_- P_+ = 0 \\ P_+ + P_- &= 1 \end{aligned} \quad (31)$$

are two projection operators. Their existence enables us to truncate the action (25) by imposing a "chirality condition" at spatial infinity with respect to the operator P_+ (or, equivalently, P_-); it is important that such a condition does not interfere with the Majorana properties of the spinors unlike the usual chirality condition because here γ^5 is replaced by $i\gamma^i \hat{x}^i$. Since the "chirality" of $\varepsilon(x)$ at $\rho = \pi/2$ is already fixed by (27), we see that a possible truncation is

$$B = 0, \quad \chi_+ \equiv P_+ \chi = 0, \quad \varepsilon_- \equiv P_- \varepsilon = 0 \quad (32)$$

Consistency then requires that $\delta B = \delta \chi_+ = 0$. Evidently, (26) and (32) imply $\delta B = 0$, but for $\delta \chi$, the argument is more subtle. Noting that $\gamma^\mu = \gamma^a V_a^\mu$, where V_a^μ is given in Eq. (3.13) of Ref. 7), and applying the projector P_+ to $\delta \chi$ in (26), we obtain

$$\begin{aligned} \delta \chi_+ &= \\ &= -\frac{1}{\sqrt{2}} \left[a \cos \varrho \frac{\partial}{\partial \varrho} A + (m + a) A \right] \varepsilon_+ \end{aligned} \quad (33)$$

It has been shown in Ref. 3) that the singleton action can be written as a surface integral over $S^2 \times S^1$. In order to yield a non-trivial contribution at spatial infinity, the singleton Lagrangian must scale as $(\cos \varrho)^3$. To determine the scaling behaviour of A , it is sufficient to look at the kinetic term. This gives

$$A(\varrho, \theta, \varphi, t) \rightarrow (a \cos \varrho)^{1/2} \phi(\theta, \varphi, t) \quad (34)$$

as $\varrho \rightarrow \pi/2$. Inserting this into (33), we see that (33) will only vanish for the special mass value

$$m = -\frac{a}{2} \quad (35)$$

Remarkably, this is precisely the value m must have in a singleton representation, because the field equation for A reads

$$\{ E_0(E_0 - 3)a^2 + 2a^2 + am - m^2 \} A = 0 \quad (36)$$

since the eigenvalue of the $SO(3,2)$ Casimir operator for a spinless particle is just $E_0(E_0 - 3)a^2$. Substitution of $E_0 = \frac{1}{2}$ into (36) immediately yields (35) (the alternative value $m = 3a/2$ is obtained if, instead of (32), we impose $A = \chi_- = 0$).

One can further clarify the structure by rewriting the action and transformation rules in a manifestly $S^2 \times S^1$ covariant form. To this end, we express $\chi = \chi_-$ in the form

$$\chi_- = a \cos \varphi \begin{pmatrix} \lambda \\ i \sigma^i \hat{x}^i \lambda \end{pmatrix} \quad (37)$$

where λ is a two component Majorana-spinor on $S^2 \times S^1$. The scaling behaviour in (37) is dictated by the same requirement as in (34). The supersymmetry transformation parameter $\varepsilon = \varepsilon_+$ may be re-expressed in an analogous fashion

$$\varepsilon_+ = \sqrt{2} (a \cos \varphi)^{-1/2} \begin{pmatrix} \eta \\ -i \sigma^i \hat{x}^i \eta \end{pmatrix} \quad (38)$$

From (26), (32), (34), (37) and (38), it now follows that

$$\delta \phi = 2 \bar{\eta} \lambda \quad (39)$$

and

$$\begin{aligned} \delta \lambda = & -i \partial_0 \phi \eta - \\ & - i \varepsilon^{ijk} \hat{x}^j \sigma^k \left(\varphi \frac{\partial \phi}{\partial x^i} - x^i \hat{x}^j \frac{\partial \phi}{\partial x^j} \right) \eta \end{aligned} \quad (40)$$

The second term in (40) can be further simplified by noting that it is actually ρ -independent and therefore contains only derivatives with respect to the angular variables θ and ϕ . Thus, (40) can be rewritten in the form

$$\delta \lambda = -i \partial_0 \phi \cdot \eta - i \tilde{\gamma}^\alpha \partial_\alpha \phi \cdot \eta \quad (41)$$

where $\alpha = 1, 2$ labels the co-ordinates on S^2 and the two-by-two matrices $\tilde{\gamma}^\alpha$ generate the Clifford algebra in two dimensions, i.e., $\{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\tilde{g}^{\alpha\beta}$ with the S^2 -metric $\tilde{g}^{\alpha\beta}$. Applying the same procedure to (25), one can obtain the singleton action as an integral over $S^2 \times S^1$. There is, however, a further subtlety. In proving the supersymmetry of the action (25), partial integrations were carried out at liberty and surface terms were discarded. If, however, the fields scale as (34) and (37), surface terms cannot be neglected. Substituting the value (35) into (25) and using (34), one finds that the sum of the mass term and the part of the kinetic term containing derivatives with respect to the radial variable ρ have the wrong scaling

behaviour. It turns out that this term is precisely cancelled by the corresponding term with the wrong scaling behaviour in

$$\frac{1}{2} \int d^4x \sqrt{-\bar{g}} \partial_\mu (\bar{g}^{\mu\nu} A \partial_\nu A) \quad (42)$$

Therefore, the kinetic term in (25) should read

$$- \frac{1}{2} \int d^4x \sqrt{-\bar{g}} \bar{g}^{\mu\nu} A \bar{D}_\mu \partial_\nu A \quad (43)$$

instead. The final Lagrangian on $S^2 \times S^1$ now reads

$$\begin{aligned} \mathcal{L}(\text{singleton}) = & (\partial_0 \phi)^2 + \tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \\ & + 2i \bar{\lambda} \partial^0 \lambda + 2i \bar{\lambda} \tilde{g}^\alpha \partial_\alpha \lambda \end{aligned} \quad (44)$$

Note that there is no need for covariantizing the ∂_α -derivative, which acts on λ , because a possible connection term drops out as λ is a Majorana spinor. In accordance with the remark after (29), the singleton Lagrangian (44) has to be integrated over the double covering of $S^2 \times S^1$, i.e.,

$$S[\text{singleton}] = \frac{1}{2} \int_0^{4\pi} dt \int_{S^2} d\Omega \mathcal{L}(\theta, \varphi, t) \quad (45)$$

We now briefly comment on the question of self-interactions. Fronsdal has shown³⁾ how to extend (39), (41) and (44) by including supersymmetric interactions. The only terms with the correct scaling behaviour are ϕ^6 and $\phi^2 \bar{\chi} \chi$. To see whether such interactions could also be present for higher $N > 1$, we consider the case $N = 8$ and utilize some results of Marcus and Schwarz⁸⁾ who have classified three-dimensional supergravity theories.

Adopting their notation, we assign the supersymmetry transformation parameter $\eta^{\dot{A}}$ and the fields $\phi^{\dot{A}}$ and $\lambda^{\dot{A}}$ to the three inequivalent representations of $SO(8)$. Equations (39) and (41) are consequently replaced by

$$\begin{aligned}\delta\phi^A &= 2\Gamma_{A\dot{A}}^i \bar{\eta}^i \lambda^{\dot{A}} \\ \delta\lambda^{\dot{A}} &= (-i\partial_0\phi^A - i\tilde{\gamma}^\alpha\partial_\alpha\phi^A)\Gamma_{A\dot{A}}^i \eta^i\end{aligned}\tag{46}$$

To include interactions, a further term must be added to $\delta\lambda^{\dot{A}}$, namely

$$\delta'\lambda^{\dot{A}} = \delta\lambda^{\dot{A}} + f(\phi)\Gamma_{A\dot{A}}^i \phi^A \eta^i\tag{47}$$

with $f(\phi)$ an arbitrary scalar function of ϕ . In the commutator ϕ^A , this leads to an additional term

$$\begin{aligned}[\delta_1, \delta_2]\phi^A &= \dots + \\ &+ 2f(\phi)\bar{\eta}_1^i \eta_2^j (\Gamma^i \Gamma^{jT} - \Gamma^j \Gamma^{iT})_{AB} \phi^B\end{aligned}\tag{48}$$

For $f(\phi) = 1$, the additional term is just the $SO(8)$ rotation in the commutator (1). However, proper scaling behaviour requires $f(\phi) \propto \phi^2$ and there appears to be no way to reconcile the required scaling with the closure of the algebra. This argument is easily seen to apply to all $N > 1$. We therefore conclude that, for $N > 1$, there is no supersymmetric self-interaction for singletons.

One may, however, enquire whether $N > 1$ singletons could be made to interact with other fields such as gravity. This would require a formulation of singleton Lagrangians inside AdS and not just at its boundary. A natural setting where such couplings might occur is Kaluza-Klein supergravity. Indeed, inspection of the recently determined mass spectrum of 11-dimensional supergravity on $S^{7,9)}$ reveals the possible existence of two states with the quantum numbers of the $N = 8$ singleton multiplet in the "basement" of the infinite tower of states on S^7 . Unfortunately, the associated modes vanish identically and therefore do not appear at the linearized level in the compactification. Whether they could appear at the non-linear level and

interact with the other states that arise in the compactification or whether they appear in other compactifications such as the squashed $S^{7|0}$, remains to be investigated.

ACKNOWLEDGEMENTS

E.S. is grateful to CERN for the kind hospitality extended to him during his stay there. We would like to thank Professors Abdus Salam, J. Strathdee, F. Englert and P. Spindel for enlightening discussions, and C.N. Pope for his helpful comments.

REFERENCES

- 1) D.Z. Freedman and A. Das, Nucl. Phys. B120 (1977) 221;
E. Fradkin and M.A. Vasiliev, Lebedev preprint (1976);
D.Z. Freedman and J. Schwarz, Nucl. Phys. B137 (1978) 333;
E. Cremmer, S. Ferrara and J. Scherk, unpublished;
A. Das, M. Fischler and M. Rocek, Phys. Rev. D16 (1977) 3427;
B. de Wit and H. Nicolai, Nucl. Phys. B188 (1981) 98; Nucl. Phys. B208 (1982) 323.
- 2) C. Fronsdal, Phys. Rev. D12 (1975) 3819.
- 3) C. Fronsdal, "The Dirac Supermultiplet", UCLA preprint 81/TEP/6 (1981);
M. Flato and C. Fronsdal, Phys. Lett. 97B (1980) 236.
- 4) W. Heidenreich, Phys. Lett. 110B (1982) 461.
- 5) M. Günaydin, in Group Theoretical Methods in Physics, Istanbul, 1982, (Lecture Notes in Physics 180, Springer Verlag).
- 6) D.Z. Freedman and H. Nicolai, Nucl. Phys. B237 (1984) 342.
- 7) P. Breitenlohner and D.Z. Freedman, Ann. Phys. 144 (1982) 249.
- 8) N. Marcus and J. Schwarz, CALTECH preprint-68-1018 (1983).
- 9) B. Biran, A. Casher, F. Englert, M. Rومان and P. Spindel, Phys. Lett. 134B (1984) 179;
F. Englert and H. Nicolai, CERN preprint TH.3711 (1983);
E. Sezgin, Trieste preprint IC/83/220 (1983);
A. Casher, F. Englert, H. Nicolai and M. Rومان, CERN preprint TH.3794 (1984);
The spectrum of the fermions has also been given in:
R. D'Auria and P. Fré, Torino preprint IFTT 430 (1983).
- 10) M.A. Awada, M.J. Duff and C.N. Pope, Phys. Rev. Lett. 50 (1983) 294;
B.E.W. Nilsson and C.N. Pope, Phys. Lett. 133B (1984) 67.