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SUPERCONFORMAL QUANTUM MECHANICS

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A B S T R A C T

$N = 1$ and $N = 2$ superconformal quantum mechanics are formulated and solved. Spontaneous symmetry breaking occurs. A de Sitter subalgebra is used to define new supersymmetric evolution operators. The modified system has a discrete spectrum exhibiting new supersymmetries. The $N = 2$ extended de Sitter superalgebra is spontaneously broken to $N = 1$. The relation to the behaviour of finite field theories and Poincaré invariance is discussed.

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1. - INTRODUCTION

The discovery of finite supersymmetric field theories¹⁾ is of great interest. In such field theories, one can dispose of the infinite renormalizations with which only a cold detente has existed during the years. If these theories have also non-perturbatively a zero beta function, one can calculate exactly all Green functions. The finite theories have an additional property, they are conformally symmetric. In the case of renormalizable theories, which were conformally invariant at the classical level, the conformal symmetry was sacrificed in favour of Lorentz invariance in the process of renormalization. For finite theories, it may seem that one can have the best of both worlds. Could this indeed be the case ? While this theory is well defined in the ultra-violet, it may suffer from infra-red problems which would manifest themselves in the properties of the ground state.

Fayet²⁾ has studied the consequences of a spontaneous breaking of the $O(6)$ symmetry of the $N = 4$ supersymmetric model. In that case the colour singlet sector massless sector of the system would consist, between others, of six spin zero particles [a dilaton and the five Goldstone bosons of $O(6)/O(5)$], four Majorana fermions and one gauge boson. $O(6)$ and $SU(N_c)$ are broken, supersymmetry and Poincaré invariance are maintained. Using a large N expansion, Bardeen, Bander and Moshe³⁾ (BBM) have studied ϕ^6 in three dimensions. Perturbatively, the theory has a zero beta function. They have shown that while the vacuum seems to be ill defined perturbatively for large enough couplings, one could stabilize the vacuum by having the beta function non-zero above a certain coupling. The conformal symmetry is broken beyond that coupling. In the supersymmetric ϕ^6_3 , they have obtained results along the lines envisaged by Fayet.

A rather different approach was proposed by one of us⁴⁾. Since the Poincaré group is just one of several subgroups of the conformal $O(D,2)$ group, where D is the number of space-time dimensions, it was suggested that there may be circumstances under which it would become advantageous to pursue the description of the physics of the system in terms of other subgroups of $O(D,2)$, such as $O(D-1,2)$ or $O(D,1)$. This cautious description results from some difficulty in using the current terminology to describe the breaking of Lorentz invariance.

Let us explain: the ground state wave functionals of the various theories are generally unknown. It may be that a normalizable vacuum does not exist in such theories. This has been shown by De Alfaro-Fubini-Furlan⁵⁾ (DFF) to be the case in conformal quantum mechanics and has been conjectured to occur in general conformal invariant models as well as in the two-dimensional Liouville model⁶⁾.

In the absence of a vacuum, one may wish to investigate the consequences of adding an additional prescription to theory, such that the new theory has a well-defined vacuum. That additional rule sacrifices this time the Poincaré invariance in favour of maintaining a well-defined vacuum and a de Sitter subgroup. The details of the construction appear in Refs. 4) and 5). We shall recall some of these constructions later.

It is clear that the procedure breaks scale and Poincaré invariance, but how should we characterize this breaking. If we concentrate on the properties of the breaking, it was shown that these theories have an algebraic structure which results in low-energy theorems, not unlike the low-energy theorems of current algebra. In this sense, the resulting symmetry breaking in the theory is spontaneous, however, it is not spontaneous in the sense of having a theory in which it is necessary to choose ab initio between one of many vacua.

In any case, the procedure will also have an impact on the supersymmetry of the system, this has been studied by De Alfaro-Furlan⁷⁾ (DF) in the days when supersymmetry was originally proposed. We wish now to return to this issue in view of the emergence of quantum conformally invariant field theories. In order to learn the possible emerging structure, we study a well-defined soluble model, which possesses both conformal and supersymmetric invariances. Such a model is conformal supersymmetric quantum mechanics which is the supersymmetric extension of the conformal quantum mechanics which was studied by DFF⁵⁾.

The results in quantum mechanics are interesting in their own right, we will also try to suggest what interpretation should be given in a field theoretic context. The structure of the paper is the following: in Section 2 we describe the conformal supersymmetric structure of the quantum mechanical model. In Section 3 we discuss the $N = 1$ in the presence of an internal quantum number and in particular the case $D = 2$ in which a larger group of supersymmetry is present. Under both circumstances, the original

supersymmetry of the model is spontaneously broken. We will describe how the resulting local structure may also have a new supersymmetric interpretation.

2. - QUANTUM MECHANICAL CONFORMAL SUPERSYMMETRY

In this section, we wish to discuss the fusion of conformal invariance and supersymmetry as it occurs in quantum mechanics. Conformal quantum mechanics has been studied in great detail from the point of interest of this work by DFF⁵⁾. We next review the essential features of the model. The Lagrangian describing the system is:

$$L = \frac{1}{2} \left(\dot{x}^2 - \frac{g}{x^4} \right) \quad (2.1)$$

where g is the dimensionless coupling constant. The action of this system is invariant under the conformal group $O(2,1)$ which is spanned by three generators, H the Hamiltonian, D the dilatation generator and K the conformal generator. These generators form together the algebra

$$[H, D] = i H \quad ; \quad [K, D] = -i K \quad ; \quad [H, K] = 2i D \quad (2.2)$$

The Hamiltonian describes the motion of a particle in a repulsive potential and is given by

$$H = \frac{1}{2} \left(p^2 + \frac{g}{x^4} \right) \quad (2.3)$$

The Hamiltonian can be diagonalized exactly. The eigenspectrum includes all $E > 0$ values for each of which there exists a plane wave normalizable state. The spectrum does not have an endpoint or ground state as the state $E = 0$ is not even plane wave normalizable. We do not know of a principle which requires that a system always needs to have a ground state. However, our understanding of field theory was greatly enhanced by studying the symmetry properties of the vacuum and excitations around it. It is possible to add a new rule⁴⁾ that would enable the choice of a ground state, that rule, however, results in the breakdown of translational invariance in the time t direction. The rule chosen by DFF⁵⁾ was to notice that the quantum evolution of the system can be controlled by a compact operator. Such an

operator would have normalizable eigenstates in general, and a ground state in particular. Using the three symmetry operators, the compact operator can be chosen from any linear combination, G , of the generators

$$G = uH + vD + wK \quad (2.4)$$

as long as

$$\Delta = v^2 - 4uw < 0 \quad (2.5)$$

Any such choice would break translational and scale invariance. DFF have found it convenient to define

$$G = R = \frac{1}{2} \left(\frac{1}{a} K + a H \right) \quad (2.6)$$

where a is a constant (with the dimensions of a length) whose introduction breaks scale invariance. For simplicity, we shall for now on keep $a = 1$. Making use of the algebra

$$\begin{aligned} L_{\pm} &= -\frac{1}{2} (H - K \mp 2D) \\ [R, L_{\pm}] &= \pm L_{\pm} \\ [L_+, L_-] &= -2R \end{aligned} \quad (2.7)$$

it has been shown that the eigenvalues of R are given by a discrete series

$$r_h = r_0 + h \quad h = 0, 1, 2, \dots \quad (2.8)$$

where r_0 is given by

$$r_0 = \frac{1}{2} \left(1 + \sqrt{g + \frac{1}{4}} \right) \quad (2.9)$$

The introduction of the linear combination (2.6) between H and K guarantees that all eigenfunctions are indeed normalizable.

We are interested in studying the supersymmetric version of conformal quantum mechanics. To achieve this we shall make use of the general

construction by Witten^{8),9)}, which enables one to convert any quantum mechanical model into its supersymmetric counterpart. The supersymmetric algebra is given by:

$$\begin{aligned}\frac{1}{2} \{Q, Q^+\} &= H \\ \{Q, Q\} &= \{Q^+, Q^+\} = 0\end{aligned}\tag{2.10}$$

A minimal realization of this algebra is given by a point particle in one dimension which has two degrees of freedom

$$\begin{aligned}Q &= \psi^+ \left(-Lp + \frac{dW}{dx} \right) \\ Q^+ &= \psi \left(Lp + \frac{dW}{dx} \right)\end{aligned}\tag{2.11}$$

where $p = -i(\partial/\partial x)$; the superpotential $W(x)$ is any function of x and the spinor operators ψ and ψ^+ obey the anticommutation relations

$$\{\psi^+, \psi\} = 1\tag{2.12}$$

We define

$$B = \frac{1}{2} [\psi^+, \psi]\tag{2.13}$$

B is the generator of the $U(1)$ transformation $\psi \rightarrow e^{i\alpha}\psi$, $\psi^+ \rightarrow e^{-i\alpha}\psi^+$ and has eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$. We see that the algebra of ψ^+ , ψ and B is isomorphic to that of the Pauli spin matrices

$$\begin{aligned}\psi &\rightarrow \frac{\sigma_1 - i\sigma_2}{2} \\ \psi^+ &\rightarrow \frac{\sigma_1 + i\sigma_2}{2} \\ B &\rightarrow \frac{1}{2} \sigma_3\end{aligned}\tag{2.14}$$

The Hamiltonian governing the time behaviour of the system is

$$H = \frac{1}{2} \{Q, Q^*\} = \frac{1}{2} \left\{ p^2 + \left(\frac{dW}{dx} \right)^2 - 2B \frac{d^2 W}{dx^2} \right\} \quad (2.15)$$

The conformal supersymmetric quantum mechanics is given by the superpotential

$$W(x) = \frac{f}{2} g x^2 \quad (2.16)$$

where f is the dimensionless coupling constant. In this case, the system has a richer algebraic structure, it is the superconformal algebra of Haag-Lopuszanski and Sohnius¹⁰⁾. This leads to the introduction of an extra pair of spinor operators, S and S^+ which are, in some way, the "square roots" of the operator K ¹¹⁾. We define

$$\begin{aligned} S &= \psi^* x \\ S^+ &= \psi x \end{aligned} \quad (2.17)$$

The enlarged algebra is now given by

$$\begin{aligned} \frac{1}{2} \{Q, Q^*\} &= H & \frac{1}{2} \{S, S^+\} &= K \\ \frac{1}{2} \{Q, S^+\} &= \frac{f}{2} - \frac{1}{2} B + \frac{1}{2} D \\ \frac{1}{2} \{Q^*, S\} &= \frac{f}{2} - \frac{1}{2} B - \frac{1}{2} D \end{aligned} \quad (2.18)$$

All other anticommutators like $\{Q, Q\}$, $\{Q, S\}$, etc., vanish

$$H = \frac{1}{2} \left(P^2 + \frac{f^2 + 2fB}{x^2} \right) \quad (2.19)$$

$$K = +\frac{1}{2} x^2 \quad D = -\frac{1}{4} \{x, P\}$$

$$B = \frac{1}{2} [\psi^+, \psi]$$

Let us now examine in which way does the system realize the supersymmetric algebra. Because of the relation $H = \frac{1}{2} \{Q, Q^+\}$, supersymmetry is unbroken, i.e., Q and Q^+ annihilate the vacuum if and only if the Hamiltonian annihilates the vacuum as well. In other words, supersymmetry is broken if the system does not have a normalizable zero energy state. We already know that in conformal quantum mechanics, the zero energy state is not even plane wave normalizable. Endowing the system with supersymmetry, one does not introduce a taming scale. We would thus suspect that the zero energy state will remain non-renormalizable. This is indeed checked by an exact calculation. Witten⁸⁾ has shown that a zero energy state always exists and is given by solving the first order differential equations

$$Q|\psi\rangle = Q^+|\psi\rangle = 0 \quad (2.20)$$

which can be expressed as

$$\left(\frac{dW}{dx} + 2fB \right) |\psi\rangle = 0 \quad (2.20')$$

The general solution of Eq. (2.20') is

$$|\psi\rangle = \exp(-2B W(x)) | \rangle \quad (2.21)$$

where $| \rangle$ is any x independent two component state vector. In the conformal case

$$|\psi(x)\rangle = x^{-2fB} | \rangle$$

We thus have the x dependence:

$$\psi \sim x^{-f} \quad \text{for} \quad B = \frac{1}{2}$$

$$\psi \sim x^f \quad \text{for} \quad B = -\frac{1}{2}$$

Neither choice of B can result in a normalizable state: the wave function will blow up for either large or small values of x . Supersymmetry is broken for superconformal quantum mechanics.

In fact, superconformal quantum mechanics can be solved exactly in the same way as conformal quantum mechanics. For a positive f , one looks at the $B = \frac{1}{2}$ sector, in which the eigenvalue equation reduces to that of the conformal quantum mechanical case with $g = f^2 + f$. The $B = \frac{1}{2}$ sector has a positive definite spectrum, every $E > 0$ is an eigenvalue with a corresponding plane wave normalizable Bessel wave function $x^{\frac{1}{2}} J_{f+\frac{1}{2}}(x\sqrt{2E})$. As H commutes with Q , each $B = \frac{1}{2}$ state has a $B = -\frac{1}{2}$ state with the same energy as supersymmetric partner. At this stage, the diagonalization of the Hamiltonian has been completed. Once again one is faced with an exactly solvable system whose energy spectrum is $E > 0$, the ground state at $E = 0$ being non-normalizable.

We follow here the point of view of DDF that it is possible to study the time evolution of the system by means of a compact operator that has a discrete set of normalizable eigenfunction. It is first instructive to diagonalize the operator R defined by Eq. (2.6)

$$R = \frac{1}{2} \left(a H + \frac{1}{a} K \right)$$

where a is a scale factor required by dimensional analysis which is set to be unity.

As already mentioned, Eq. (2.8), the R spectrum in conformal quantum mechanics is equally spaced. This equal spacing is the quantum mechanical analogue of the low-energy theorems related to the breakdown of a symmetry, in this case the dilatation invariance⁴⁾. In the case of conformal supersymmetric quantum mechanics, the appearance of a zero energy fermionic excitation (the quantum mechanical "Goldstone fermion") was already apparent in the spectrum of H . Once R is diagonalized, an explicit difference between the energy necessary to excite bosons and fermions is expected.

Indeed looking at the form of the Hamiltonian given in Eq. (2.19), we see that it reduces to the standard form (2.3) with $g = f(f+1)$ for $B = \frac{1}{2}$ and $g = f(f-1)$. One can thus apply Eq. (2.8) to both cases. The lowest level, given by Eq. (2.9), is given by $r_0 = \frac{1}{2}(3/2+f)$ for $B = \frac{1}{2}$ and $r_0 = \frac{1}{2}(\frac{1}{2}+f)$ for

$B = -\frac{1}{2}$ respectively. We see that two such series corresponding to the two different fermionic numbers are shifted by $\frac{1}{2}$. This semi-integer shift can be understood algebraically in the following manner. Define the fermion lowering and raising operators by

$$\begin{aligned} M &= Q - S & M^+ &= Q^+ - S^+ \\ N &= Q^+ + S^+ & N^+ &= Q + S \end{aligned} \quad (2.22)$$

which obey the anticommutator algebra:

$$\frac{1}{4} \{M, M^+\} = R + \frac{1}{2}B - \frac{1}{2}f = T_1 \quad (2.23a)$$

$$\frac{1}{4} \{N, N^+\} = R - \frac{1}{2}B + \frac{1}{2}f = T_2 \quad (2.23b)$$

$$-\frac{1}{4} \{M, N\} = L_- \quad (2.23c)$$

$$-\frac{1}{4} \{M^+, N^+\} = L_+ \quad (2.23d)$$

The explicit form of our operators is

$$\begin{aligned} M &= \psi^* \left(-\epsilon P + \frac{dm}{dx} \right) \\ M^+ &= \psi \left(\epsilon P + \frac{dm}{dx} \right) \\ N &= \psi \left(\epsilon P + \frac{dn}{dx} \right) \\ N^+ &= \psi^* \left(-\epsilon P + \frac{dn}{dx} \right) \end{aligned} \quad (2.24)$$

to be compared to

$$Q = \psi^+ \left(-L P + \frac{dW}{dx} \right)$$

$$Q^\dagger = \psi \left(L P + \frac{dW}{dx} \right)$$

where

$$m(x) = \frac{1}{2} (f g x^2 - x^2)$$

$$n(x) = \frac{1}{2} (f g x^2 + x^2) \quad (2.25)$$

$$W(x) = \frac{1}{2} f g x^2$$

We are now ready to study the level structure of our system, in particular the doublet structure present in standard supersymmetry. Let us first notice a striking algebraic analogy between the operators defined in Eq. (2.24) and the original operators of standard supersymmetry. This is embodied in the formulae

$$\frac{1}{4} \{Q, Q^\dagger\} = \frac{1}{2} H$$

$$\frac{1}{4} \{M, M^\dagger\} = T_1$$

$$\frac{1}{4} \{N, N^\dagger\} = T_2$$

These relations follow from a de Sitter subalgebra present in the superconformal algebra¹¹⁾ [Eq. (2.18)].

The doublet structure of the eigenstates of H follows from the fact that H commutes with Q and $Q^\dagger$. In a similar way, a doublet structure for T_1 follows from:

$$[T_1, M] = [T_1, M^\dagger] = 0$$

and a doublet structure for T_2 follows from:

$$[T_2, N] = [T_2, N^\dagger] = 0$$

It is, however, important to notice that when we look at the explicit expressions (2.24) and (2.25) of the three sets of operators, we find them to have a different type of spectrum. This is due to the three different forms of the superpotentials $W(x)$, $m(x)$ and $n(x)$. Indeed when we search for

the lowest eigenstates of the operators H_1 , T_1 and T_2 , we have to consider wave functions whose space dependence should be $e^{\pm W}$, $e^{\pm m}$, $e^{\pm n}$ respectively. We see that out of the six possible choices only

$$e^{m(x)} = x^{\frac{1}{2}} e^{-\frac{1}{2}x^2}$$

gives rise to a normalizable wave function.

We thus conclude that out of the three operators, only T_1 has a normalizable lowest eigenstate $|\psi\rangle$ obeying

$$M|\psi\rangle = M^+|\psi\rangle = 0 \quad (2.26)$$

whose solution is

$$|\psi\rangle = e^{-2m(x)B} | \rangle \quad (2.27)$$

In order to get a normalizable state, we chose $| \rangle = |\downarrow\rangle$ where $|\downarrow\rangle$ is the no fermion state corresponding to $B = -\frac{1}{2}$. We thus have

$$|\psi\rangle = e^{m(x)} |\downarrow\rangle = x^{\frac{1}{2}} e^{-\frac{1}{2}x^2} |\downarrow\rangle \quad (2.28)$$

The level spectrum of the operator T_1 can easily be obtained by considering the two operators N and N^+ which play the rôle of lowering and raising operators respectively, since

$$\begin{aligned} [T_1, N] &= -N \\ [T_1, N^+] &= N^+ \end{aligned} \quad (2.29)$$

It is easy to verify that

$$N|\psi\rangle = 0 \quad (2.30)$$

The spectrum is thus given by

$$\begin{aligned} T_1 &= 0, 1, 2, 3, \dots & B &= -\frac{1}{2} \\ &= 1, 2, 3, \dots & B &= \frac{1}{2} \end{aligned}$$

A similar rôle is taken by the operators L_- and L_+ with commutators

$$\begin{aligned} [T_1, L_-] &= -L_- \\ [T_1, L_+] &= L_+ \end{aligned} \quad (2.31)$$

Notice that L_+ raises T_1 by keeping B constant whereas N^+ does it by changing the value of B . The operators N^+, N can be viewed as creation and destruction operators of a Goldstone fermion whereas L_+ and L_- as creation and destruction operators of a Goldstone boson.

It is useful to describe the spectrum in terms of the original operators R used in DDF. We use the relation (2.23a) between T_1 and R and recall that N_1 and M decrease R by $\frac{1}{2}$ whereas N^+, M^+ increase R by $\frac{1}{2}$:

$$\begin{aligned} [R, N] &= -\frac{1}{2} N & [R, M] &= -\frac{1}{2} M \\ [R, N^+] &= \frac{1}{2} N^+ & [R, M^+] &= \frac{1}{2} M^+ \end{aligned} \quad (2.32)$$

The lowest eigenstate $|\psi\rangle$ given by Eq. (2.28) corresponds to:

$$V_0 = \frac{1}{2} \left(l + \frac{1}{2} \right) \quad (2.33)$$

The higher levels are obtained by repeated applications of the M^+, N^+ operators. We thus get

$$\begin{aligned} B = -\frac{1}{2} & \quad R = \frac{1}{2} \left(l + \frac{1}{2} \right) + n \\ B = \frac{1}{2} & \quad R = \frac{1}{2} \left(l + \frac{1}{2} \right) + \frac{1}{2} + n \end{aligned} \quad (2.34)$$

The level structure of the operator R is depicted in Fig. 1. Having solved the model exactly, we summarize the results displayed in Fig. 1, by viewing it from different angles.

In conformal supersymmetrical quantum mechanics, supersymmetry is spontaneously broken, the Hamiltonian has a continuous $E > 0$ spectrum. Under such circumstances, it is useful to observe the time evolution of the system using as a complete set the eigenfunctions of a compact operator. In general, the diagonalization of any compact operator will do, however, under particular circumstances, it may turn out that using different operators may emphasize different symmetry structures of the model. Such circumstances arise in the case at hand as we view the system through eigenfunctions of the operators R , T_1 and T_2 .

The first way is described in Fig. 1 itself; the spectrum of R consists of two B sectors. In each B sector the bosonic excitations are equally spaced by one unit. The "Goldstone boson" operators L_+ , L_- relating the states. Fermionic excitations are one half of a unit and are carried by the "Goldstone fermion" operators N , N^+ resulting from Q supersymmetry breaking.

The second way is to assign to T_1 the rôle of the observation operator. In that case, while dilatation invariance is still broken, the supersymmetry generated by M , M^+ is unbroken. The states will exhibit a supersymmetrical structure. This new supersymmetric operator mixing Q and s is responsible for the twin degeneracy of all non-zero T_1 states. There exists a single state of zero T_1 . Bosonic and fermionic excitations from the ground state are equal and non-zero. The equal level spacing is again unity. The equal bosonic spacing is the analogue of the low-energy theorems related to the breakdown of dilatational invariance. The fermionic equal spacing is the analogue of a fermionic companion of a Goldstone boson, such companions appear when supersymmetry is unbroken.

The third way to view the system is through T_2 . In that case, the system is equally spaced by one unit, however, there is no normalizable zero T_2 state. The supersymmetry apparent in that form, N , is spontaneously broken. The ground state is to be chosen between the lowest T_2 states having $B = -\frac{1}{2}$ and $\frac{1}{2}$, once the choice is made, a zero " T_2 energy" fermionic excitation connecting the ground state and its degenerate partners, exists.

This corresponds to a "massless" Goldstone fermion of N supersymmetry breaking. The equal bosonic spacing has already been discussed. The different points of view are shown in Fig. 2.

The basic features uncovered seem to be rather general, in order to be certain we shall study conformal supersymmetrical examples in which the particle may move in more than one dimension, we shall also study an $N = 2$ system in which one expects to encounter difficulties when attempting to break supersymmetry. A more technical reason for extending the study is related to the fact that the variable x can achieve all positive as well as all negative values. We wish to be certain that some regularization prescription at the origin will not invalidate the qualitative results. With a larger number of dimensions, this problem would not occur as the radial variable is never negative.

Let us end this section with an amusing technical remark related to the Witten index⁸⁾. The Witten index is defined to be the number of bosonic zero energy solutions minus the number of fermionic zero energy solutions. When supersymmetry is broken, there exists no zero energy state and thus the Witten index is zero. If the Witten index is non-zero supersymmetry is broken. The Witten index may be zero also in a supersymmetrical case. Assume that f were negative then it would appear that the state $|0\rangle \approx x^{-|f|} \exp x^{-2}$ is still normalizable as long as $|f| < \frac{1}{2}$, thus the model would have a supersymmetric transition at $|f| = \frac{1}{2}$, the Witten index⁸⁾ is zero in the broken supersymmetric phase and is non-zero in this case in the supersymmetric phase. It would thus seem that by changing f (and not at $f = 0$) one can change discontinuously the Witten index. However, one shall note that when the state $|0\rangle$ is included in the Hilbert space, for $f < -\frac{1}{2}$, the Hamiltonian ceases to be Hermitean (and physical) as long as there is no source at the origin and thus the index theorem need not hold for it.

3. SUPERCONFORMAL QUANTUM MECHANICS IN MANY DIMENSIONS

a) The general case

The supersymmetric quantum mechanics of one particle in many dimensions (or alternatively one particle with many degrees of freedom in one dimension) is posed in a similar manner to the supersymmetric quantum mechanics of one particle in one dimension¹³⁾. The algebra as stated in Eq. (2.10) is unchanged. The operators realizing the algebra are generalized

Eqs. (2.11)-(2.13) to be:

$$\begin{aligned}
 Q &= \sum_{\alpha=1}^D \psi_{\alpha}^+ \left(-\epsilon p_{\alpha} + \frac{\partial W}{\partial x_{\alpha}} \right) \\
 Q^+ &= \sum_{\alpha=1}^D \psi_{\alpha} \left(\epsilon p_{\alpha} + \frac{\partial W}{\partial x_{\alpha}} \right) \\
 H &= \frac{1}{2} \{ Q, Q^+ \} = \frac{1}{2} \sum_{\alpha} p_{\alpha}^2 + \left(\frac{\partial W}{\partial x_{\alpha}} \right)^2 - \sum_{\alpha\beta} B_{\alpha\beta} \frac{\partial^2 W}{\partial x_{\alpha} \partial x_{\beta}}
 \end{aligned} \tag{3.1}$$

where ψ 's obey the anticommutation relations

$$\{ \psi_{\alpha}^+ \psi_{\beta} \} = \delta_{\alpha\beta} \quad \{ \psi_{\alpha} \psi_{\beta} \} = \{ \psi_{\alpha}^+ \psi_{\beta}^+ \} = 0 \tag{3.2}$$

and $B_{\alpha\beta}$ is defined by

$$B_{\alpha\beta} = \frac{1}{2} [\psi_{\alpha}^+ \psi_{\beta}] \tag{3.3}$$

We chose the superpotential $W(x_{\alpha})$ to depend only on $x^2 = x_{\alpha} x_{\alpha}$. In this case, the theory will be invariant with respect to simultaneous rotations of x_{α} , p_{α} , ψ_{α} , ψ_{α}^+ . The theory will also be invariant with respect to inversions, i.e., under transformations like

$$x_i \rightarrow -x_i, \quad p_i \rightarrow -p_i, \quad \psi_i \rightarrow -\psi_i, \quad \psi_i^+ \rightarrow -\psi_i^+$$

We now add the fundamental requirement of conformal invariance. The superpotential is then uniquely defined as

$$W(x_{\alpha}) = \frac{1}{2} f g x^2 \tag{3.4}$$

In this case we can introduce the extended algebra of Eqs. (2.17)-(2.19) where the relevant operator will now have the form

$$\begin{aligned}
 S &= \int \psi_a^* x_a & S^* &= \int \psi_a x_a \\
 K &= \frac{1}{2} \int x_a^2 & D &= -\frac{1}{4} \int \{ p_a x_a \} \\
 H &= \frac{1}{2} \left\{ p^2 + \frac{f^2}{x^2} + 2f B_{\alpha\beta} \frac{2x_\alpha x_\beta - x^2 \delta_{\alpha\beta}}{x^4} \right\}
 \end{aligned} \tag{3.5}$$

where

$$B = \sum B_{\alpha\alpha} \tag{3.5'}$$

We wish to show that, as in the previous case of Section 2, the Hamiltonian has only non-normalizable zero energy states. Let us discuss, following Affleck¹²⁾, the general method of obtaining solutions to the ground state equations:

$$Q |\psi\rangle = Q^* |\psi\rangle = 0 \tag{3.6}$$

We concentrate on $O(D)$ invariant states, we also distinguish between even (+) and odd (-) states which decouple because of the parity conservation. We start by defining the empty and the totally full states as:

$$\psi_a |0\rangle = 0 \quad \psi_a^+ |D\rangle = 0 \tag{3.7}$$

We also introduce the two even scalar states

$$\begin{aligned}
 |\downarrow +\rangle &= r^{-\frac{D-1}{2}} |0\rangle & \hat{x}_a &= \frac{x_a}{|x|} \\
 |\uparrow +\rangle &= r^{-\frac{D-1}{2}} (\hat{x}_a \psi_a) |0\rangle
 \end{aligned} \tag{3.8}$$

and the odd doublet

$$\begin{aligned} |\downarrow -\rangle &= r^{-\frac{D-1}{2}} (\hat{x}_\alpha \varphi_\alpha) |D\rangle \\ |\uparrow -\rangle &= r^{-\frac{D-1}{2}} |D\rangle \end{aligned} \quad (3.9)$$

We can now introduce the most general even and odd states $|\psi_+(x)\rangle$ and $|\psi_-(x)\rangle$ defined as linear combinations of $|\uparrow+\rangle$ and $|\downarrow+\rangle$:

$$\begin{aligned} |\psi_+(x)\rangle &= \varphi_{+\downarrow}(r) |\downarrow+\rangle + \varphi_{+\uparrow}(r) |\uparrow+\rangle \\ |\psi_-(x)\rangle &= \varphi_{-\downarrow}(r) |\downarrow-\rangle + \varphi_{-\uparrow}(r) |\uparrow-\rangle \end{aligned} \quad (3.10)$$

We transform Eq. (3.6) into explicit equations for the ϕ functions

$$\left(2b \frac{d}{dr} + \frac{dW_\pm}{dr} \right) |\psi_\pm\rangle = 0 \quad (3.11)$$

where b is an operator defined by

$$b |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle \quad b |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle \quad (3.12)$$

and the effective potentials $w^\pm$ are given by:

$$W_\pm = \pm \frac{1}{2}(D-1) g r + W(r) \quad (3.13)$$

We thus get the general solutions

$$\begin{aligned} |\psi_+ \rangle &= \exp -2b w_+ |+\rangle \\ |\psi_- \rangle &= \exp -2b w_- |-\rangle \end{aligned} \quad (3.14)$$

where $|+\rangle$ and $|-\rangle$ are r independent combinations of $|\uparrow_-\rangle$, $|\downarrow_+\rangle$ and $|\uparrow_-\rangle$, $|\downarrow_+\rangle$ respectively.

If we apply the general formulation to our conformal choice (3.4), we get for the effective potential:

$$\begin{aligned} w_+ &= \left(f + \frac{D-1}{2} \right) \lg r \\ w_- &= \left(f - \frac{D-1}{2} \right) \lg r \end{aligned} \quad (3.15)$$

This shows that the lowest state wave functions are in general non-normalizable and exhibit the familiar power behaviour

$$\begin{aligned} |\psi_+ \rangle &= r^{-2b \left(f + \frac{D-1}{2} \right)} |+\rangle \\ |\psi_- \rangle &= r^{-2b \left(f - \frac{D-1}{2} \right)} |-\rangle \end{aligned} \quad (3.16)$$

Supersymmetry is spontaneously broken.

In conclusion, as in Section 2, we are led to study the evolution of the system in terms of a compact operator.

The fundamental operators M and N will now have the form

$$\begin{aligned}
 M &= \sum_{\alpha} \psi_{\alpha}^+ \left(-L p_{\alpha} + \frac{\partial m}{\partial x_{\alpha}} \right) \\
 M^+ &= \sum_{\alpha} \psi_{\alpha} \left(+i p_{\alpha} + \frac{\partial m}{\partial x_{\alpha}} \right) \\
 N &= \sum_{\alpha} \psi_{\alpha} \left(+L p_{\alpha} + \frac{\partial n}{\partial x_{\alpha}} \right) \\
 N^+ &= \sum_{\alpha} \psi_{\alpha}^+ \left(-L p_{\alpha} + \frac{\partial n}{\partial x_{\alpha}} \right)
 \end{aligned}
 \tag{3.17}$$

where

$$\begin{aligned}
 m(x) &= \frac{1}{2} \left(f l_g x^2 - x^2 \right) \\
 n(x) &= \frac{1}{2} \left(f l_g x^2 + x^2 \right)
 \end{aligned}
 \tag{3.18}$$

and we have again

$$\frac{1}{4} \{M M^+\} = R + \frac{B}{2} - \frac{f}{2} = T_1
 \tag{3.19}$$

We apply Affleck's procedure to obtain the zero T_1 state solutions of $T_1 |\psi\rangle = 0$, i.e.,

$$M |\psi\rangle = M^+ |\psi\rangle = 0
 \tag{3.20}$$

We finally get

$$\begin{aligned}
 |\psi_+ \rangle &= \exp -2b m_+ |+\rangle \\
 |\psi_- \rangle &= \exp -2b m_- |-\rangle
 \end{aligned}
 \tag{3.21}$$

where

$$m_{\pm} = \left(f + \frac{D-1}{2}\right) \ln r - \frac{r^2}{2} \tag{3.22}$$

Again, in both cases the normalization requirement forces $b = -\frac{1}{2}$ so that we finally get

$$\begin{aligned}
 |\psi_+ \rangle &= r^{\left(f + \frac{D-1}{2}\right)} e^{-\frac{r^2}{2}} |\downarrow + \rangle \\
 |\psi_- \rangle &= r^{\left(f - \frac{D-1}{2}\right)} e^{-\frac{r^2}{2}} |\downarrow - \rangle
 \end{aligned}
 \tag{3.23}$$

i.e.,

$$\begin{aligned}
 |\psi_+ \rangle &= r^f e^{-\frac{r^2}{2}} |0 \rangle \\
 |\psi_- \rangle &= r^{f-(D-1)} (\psi_a \hat{x}_a) e^{-\frac{r^2}{2}} |D \rangle
 \end{aligned}
 \tag{3.24}$$

It is important to notice that there is a substantial difference between the even and the odd zero eigenstates of T_1 . Indeed, from Eq. (3.24), we see that whereas $|\psi_+ \rangle$ is normalizable for all positive values of f , $|\psi_- \rangle$ starts being normalizable only for $f > D-1$ (recall that radial normalization integral is $\int \psi^\dagger \psi r^{D-1} dr$). M supersymmetry is unbroken.

The two lowest levels $|\psi_+\rangle$ and $|\psi_-\rangle$ are, of course, also eigenstates of R . Using Eq. (3.13) and the fact that

$$\begin{aligned} B |\psi_+\rangle &= -\frac{D}{2} |\psi_+\rangle \\ B |\psi_-\rangle &= \frac{D-2}{2} |\psi_-\rangle \end{aligned} \quad (3.25)$$

we obtain

$$\begin{aligned} R |\psi_+\rangle &= \frac{1}{2} \left(f + \frac{D}{2} \right) |\psi_+\rangle \\ R |\psi_-\rangle &= \frac{1}{2} \left(f - \frac{D-2}{2} \right) |\psi_-\rangle \end{aligned} \quad (3.26)$$

Starting from Eq. (3.26), using the same procedure of last section, we can construct the spectrum of scalar eigenstates of R .

$$B = -\frac{D}{2} \quad \text{EVEN} \quad r_n = \frac{1}{2} \left(f + \frac{D}{2} \right) + n \quad (3.27)$$

$$B = -\frac{D-2}{2} \quad \text{EVEN} \quad r_n = \frac{1}{2} \left(f + \frac{D}{2} \right) - 1 + n$$

$$B = \frac{D-2}{2} \quad \text{ODD} \quad r_n = \frac{1}{2} \left(f - \frac{D}{2} - 1 \right) + n$$

$$B = \frac{D}{2} \quad \text{ODD} \quad r_n = \frac{1}{2} \left(f - \frac{D}{2} + 2 \right) + n$$

b) The D = 2 model

In the case of D = 2, we have

$$\begin{aligned}
 B = -1 & \quad \text{EVEN} & r_n &= \frac{1}{2}(f+1) + n \\
 B = 0 & \quad \text{EVEN} & r_n &= \frac{1}{2}(f+1) + n \\
 B = 0 & \quad \text{ODD} & r_n &= \frac{1}{2}f + n \\
 B = 1 & \quad \text{ODD} & r_n &= \frac{1}{2}(f+1) + n
 \end{aligned} \tag{3.28}$$

We see that for D = 2, the eigenvalues of R corresponding to B = -1 and B = 1 and the two B = 0 states are the same, it would be easy to check that the eigenstates have also a similar form. This suggests that in the case of D = 2, we are in presence of a new kind of symmetry relating even to odd states. We want to show that this new property amounts to an extended N = 2 supersymmetry.

Let us first identify an internal SU(2) group, not unlike quasi-spin¹³⁾ in nuclear physics, whose generators are:

$$\begin{aligned}
 Y &= \psi_1 \psi_2 & Y^+ &= -\psi_1^+ \psi_2^+ \\
 B &= \frac{1}{2} \left\{ [\psi_1^+ \psi_1] + [\psi_2^+ \psi_2] \right\}
 \end{aligned} \tag{3.29}$$

which obey the SU(2) algebra

$$\begin{aligned}
 [B, Y^+] &= 2 Y^+ & [B, Y] &= -2 Y \\
 [Y^+, Y] &= B
 \end{aligned} \tag{3.30}$$

We also need the commutators

$$\begin{aligned} [\gamma^+ \psi_\alpha] &= \varepsilon_{\alpha\beta} \psi_\beta^+ = \bar{\psi}_\alpha \\ [\gamma \psi_\alpha^+] &= -\varepsilon_{\alpha\beta} \psi_\beta = -\bar{\psi}_\beta \end{aligned} \quad (3.31)$$

where $\varepsilon_{\alpha\beta}$ is the 2×2 antisymmetric tensor

$$\begin{aligned} [\gamma B_{\alpha\beta}] &= \gamma \delta_{\alpha\beta} \\ [\gamma^+ B_{\alpha\beta}] &= -\gamma^+ \delta_{\alpha\beta} \end{aligned} \quad (3.32)$$

It is now important to introduce the unitary operator U which performs the transformation

$$\begin{aligned} U^{-1} \psi_\alpha U &= \varepsilon_{\alpha\beta} \psi_\beta^+ = \bar{\psi}_\alpha \\ U^{-1} \psi_\alpha^+ U &= \varepsilon_{\alpha\beta} \psi_\beta = -\bar{\psi}_\alpha^+ \end{aligned} \quad (3.33)$$

we also have

$$\begin{aligned} U^{-1} \gamma U &= -\gamma^+ \\ U^{-1} \gamma^+ U &= -\gamma \\ U^{-1} B U &= -B \end{aligned} \quad (3.34)$$

If we introduce the isomorphism with angular momentum operators

$$B \Rightarrow 2\sigma_3 \quad \gamma^+ \Rightarrow \sigma_+ \quad \gamma \Rightarrow \sigma_-$$

Equations (3.34) read

$$U^{-1} \sigma_1 U = \sigma_1, \quad U^{-1} \sigma_2 U = \sigma_2$$

$$U^{-1} \sigma_3 U = -\sigma_3$$

The unitary transformation U represents a 180° rotation around the (2) axis.

We finally have the fundamental transformation property:

$$U^{-1} B_{\alpha\beta} U = -\sigma_{\alpha\beta} B + B_{\alpha\beta} \quad (3.35)$$

Let us now look at the effect of our transformation on the Hamiltonian. One obtains

$$[B, H] = 0$$

$$[Y, H] = -2Y \square W \quad (3.36)$$

$$[Y^+, H] = 2Y^+ \square W$$

In the generalized conformal case, one has

$$\square W = f \frac{D-2}{x^2} \quad (3.37)$$

In particular, in the case of $D = 2$ one has $\square W = 0$. The same result would be correct for any analytic function W . Thus, for analytic functions the quasi-spin is an extra invariance group of the system. This can also be seen directly by

$$U^{-1} H U = H + 2B \cdot \square W \quad (3.38)$$

again for $\square W = 0$, one obtains

$$U^{-1} H U = H \quad (3.39)$$

Our extended supersymmetry is now obtained by combining the usual operator Q , Q^+ , S and S^+ with the generators of our internal $SU(2)$ group. The $N = 2$ extended supersymmetry group will be obtained by associating to each even supersymmetry operator A and odd partner $\bar{A}$ defined as

$$\bar{A} = U^{-1} A U \quad (3.40)$$

In this way standard $N = 2$ supersymmetry will contain besides Q and Q^+ :

$$\bar{Q} = \sum \bar{\Psi}_\alpha^+ (-i p_\alpha + \frac{\partial W}{\partial x_\alpha}) = \sum \epsilon_{\alpha\beta} \Psi_\beta (-i p_\alpha + \frac{\partial W}{\partial x_\alpha}) \quad (3.41)$$

$$Q = \sum \bar{\Psi}_\alpha (i p_\alpha + \frac{\partial W}{\partial x_\alpha}) = \sum \epsilon_{\alpha\beta} \Psi_\beta^+ (i p_\alpha + \frac{\partial W}{\partial x_\alpha})$$

One can also cast the operators Q in a form more readily related to an $N = 2$ model in two dimensions, from which $N = 2$ quantum mechanics can be obtained by dimensional reduction¹⁴⁾⁻¹⁷⁾. That is

$$\begin{aligned} Q_1 &= Q^+ + i \bar{Q} = i(p_1 + i p_2)(\psi_1 - i \psi_2) + \left(\frac{\partial W}{\partial x_1} - i \frac{\partial W}{\partial x_2}\right)(\psi_1 + i \psi_2) \\ Q_2 &= Q^+ - i \bar{Q} = i(p_1 - i p_2)(\psi_1 + i \psi_2) + \left(\frac{\partial W}{\partial x_1} + i \frac{\partial W}{\partial x_2}\right)(\psi_1 - i \psi_2) \end{aligned} \quad (3.42)$$

where W is an analytic function.

Let us now discuss the extended supersymmetry algebra containing the operators, Q , Q^+ , $\bar{Q}$, $\bar{Q}^+$. The only non-vanishing anticommutators are:

$$\begin{aligned} \frac{1}{2} \{Q, Q^+\} &= H \\ \frac{1}{2} \{\bar{Q}, \bar{Q}^+\} &= H \end{aligned} \quad (3.43)$$

From Eqs. (3.42), we learn that Q , Q^+ , $\bar{Q}$, $\bar{Q}^+$, all commute with H . We thus expect for a general $N = 2$ model¹⁴⁾⁻¹⁷⁾ a quartet structure, each quartet containing two even and two odd levels. The zero energy states:

$$H|\psi\rangle = 0 \quad (3.44)$$

have to obey the equations

$$Q|\psi\rangle = Q^+|\psi\rangle = \bar{Q}|\psi\rangle = \bar{Q}^+|\psi\rangle = 0 \quad (3.45)$$

and are, in general, $|\downarrow\rangle$ and $|\uparrow\rangle$ type states.

Let us now make the usual choice

$$W = \frac{1}{2} \epsilon_{ij} x^i x^j$$

In this case, the theory will be invariant with respect to $O(2)$ rotations of $x_i, p_i, \psi_i, \psi_i^+$. The generator of this group will be the angular momentum operator

$$J = p_1 x_2 - p_2 x_1 + i(\psi_1^+ \psi_2 - \psi_2^+ \psi_1) \quad (3.46)$$

We shall then obtain $N = 2$ supersymmetry which contains the extra operators $S, \bar{S}^+$ and $\bar{S}, S^+$ given by

$$\begin{aligned} \bar{S} &= \sum \bar{\psi}_\alpha x_\alpha \\ \bar{S}^+ &= \sum \psi_\alpha x_\alpha \end{aligned} \quad (3.47)$$

We already know from our general analysis of the $N = 1$ model that in the case of superconformal quantum mechanics, supersymmetry is broken and there is no normalizable ground state. This result is unaltered by the discovery that for $D = 2$, the model has extra symmetries. Note that, in general, it is very difficult to break $N = 2$ supersymmetry¹⁵⁾. We are faced again with a model which has a spectrum which is bounded from below but has no ground state. Supersymmetry is broken spontaneously. We are thus led again to the algebra of the M, N operators together with their odd partners $\bar{M}, \bar{N}$. We shall discuss the general anticommutator algebra in terms of those operators. It is convenient to classify them into two families

I - $\underline{M M^+ \bar{N} \bar{N}^+}$

$$\begin{aligned}\frac{1}{4} \{M M^+\} &= R + \frac{B}{2} - \frac{f}{2} = T_1 \\ \frac{1}{4} \{\bar{N} \bar{N}^+\} &= R + \frac{B}{2} + \frac{f}{2} = \bar{T}_2 = T_1 + f \\ \frac{1}{2} \{M N^+\} &= -\frac{1}{2} \{M^+ \bar{N}\} = i J\end{aligned}\quad (3.48)$$

II - $\underline{\bar{M} \bar{M}^+ N N^+}$

$$\begin{aligned}\frac{1}{4} \{\bar{M} \bar{M}^+\} &= R + \frac{B}{2} - \frac{f}{2} = \bar{T}_1 \\ \frac{1}{4} \{N N^+\} &= R + \frac{B}{2} + \frac{f}{2} = T_2 = \bar{T}_1 + f \\ \frac{1}{2} \{N \bar{M}^+\} &= -\frac{1}{2} \{\bar{M} N^+\} = -i J\end{aligned}\quad (3.49)$$

The non-vanishing anticommutators between members of the two different families are

$$\begin{aligned}\frac{1}{2} \{M \bar{M}^+\} &= -\frac{1}{2} \{\bar{N} N^+\} = Y \\ \frac{1}{2} \{M^+ \bar{M}\} &= -\frac{1}{2} \{N \bar{N}^+\} = Y^+ \\ -\frac{1}{4} \{M N\} &= -\frac{1}{4} \{\bar{M} \bar{N}\} = L_- \\ -\frac{1}{4} \{M^+ N^+\} &= -\frac{1}{4} \{\bar{M}^+ \bar{N}^+\} = L_+\end{aligned}\quad (3.50)$$

Let us now proceed to discuss the level spectrum of our problem. The analogy with the supersymmetric version of our model [Eq. (3.44) and (3.45)] suggests that we shall have again a quartet structure. Indeed, if we consider the operator

$$T_1 = R + \frac{B}{2} - \frac{f}{2}$$

we do indeed have a quartet structure since it commutes with the operators $M, M^+, \bar{N}, \bar{N}^+$; two of the levels are even, two are odd. However, the analogy with the Hamiltonian is not complete since, whereas

$$\frac{1}{4} \{M, M^+\} = T_1$$

we have the analogous anticommutator

$$\frac{1}{4} \{\bar{N}, \bar{N}^+\} = T_1 + f$$

This means that the lowest level condition:

$$T_1 |\psi\rangle = 0$$

does only imply

$$M |\psi\rangle = M^+ |\psi\rangle = 0$$

and does not involve at all the operators $\bar{N}, \bar{N}^+$. This weaker condition will be satisfied by a pair of states (one even and one odd). We will later offer an interpretation of this result. The search for those states has already been made for the general case at the beginning of this section, and we shall just adapt Eqs. (3.24) to our case.

The odd state which we shall define as $|F_0\rangle$ will satisfy the equation

$$B |F_0\rangle = 0$$

$$R |F_0\rangle = \frac{f}{2} |F_0\rangle \quad (3.51)$$

and the explicit form of $|F_0\rangle$ is

$$|F_0\rangle = r^{(f-1)} (\psi_a \hat{x}_a) e^{-\frac{x^2}{2}} |2\rangle \quad (3.52)$$

where $|2\rangle$ is the fully occupied state (corresponding to $B = 1$). The new group theoretical feature is that the two states belong to a doublet created by the $\bar{N}$ and $\bar{N}^x$ operators so that

$$\bar{N} |F_0\rangle = 0 \quad (3.53)$$

and the even state of the doublet $|C_0\rangle$ can be obtained by

$$N |F_0\rangle = |C_0\rangle \quad (3.54)$$

so that

$$\begin{aligned} B |C_0\rangle &= -|C_0\rangle \\ R |C_0\rangle &= \left(\frac{f}{2} + \frac{1}{2}\right) |C_0\rangle \end{aligned} \quad (3.55)$$

One can make an analogous classification of states using:

$$\bar{T}_1 = R - \frac{B}{2} - \frac{f}{2} \quad (3.56)$$

Again we have quartets of equal $\bar{T}_1$ states generated by $\bar{M}$, $\bar{M}^+$, N , N^+ . The zero $\bar{T}_1$ condition will again amount to

$$\bar{M} |\psi\rangle = \bar{M}^+ |\psi\rangle \quad (3.57)$$

and will be composed by a doublet of odd states interconnected by N and N^+ . The $B = 0$ level of the doublet will be again the celebrated $|F_0\rangle$ state which will have the added property

$$N |F_0\rangle = 0 \quad (3.58)$$

The $B = 1$ level $|\bar{C}_0\rangle$ will be given by

$$N^+ |F_0\rangle = |\bar{C}_0\rangle \quad (3.59)$$

and will correspond to $R = f/2 + \frac{1}{2}$. $|C_0\rangle$ and $|\bar{C}_0\rangle$ will be connected by the transformation

$$\begin{aligned} Y^\dagger |C_0\rangle &= |\bar{C}_0\rangle \\ Y |\bar{C}_0\rangle &= |C_0\rangle \end{aligned} \quad (3.60)$$

In order to construct a complete table of levels, it is also useful to introduce explicitly the lowest even $B = 0$ level $|E_0\rangle$ defined by

$$\begin{aligned} |E_0\rangle &= M^\dagger |C_0\rangle = \bar{M}^\dagger |\bar{C}_0\rangle \\ R |E_0\rangle &= \left(\frac{f}{2} + 1\right) |E_0\rangle \end{aligned} \quad (3.61)$$

The spectrum of R is shown in Fig. 3.

The states $|F_n\rangle$, $|C_n\rangle$, $|\bar{C}_n\rangle$, $|E_n\rangle$ are just higher recurrences obtained by applying $(L_+)^n$ to the corresponding lowest levels $|F_0\rangle$, $|C_0\rangle$, $|\bar{C}_0\rangle$, $|E_0\rangle$.

It is convenient to use as reference state the lowest level $|F_0\rangle$, since it belongs at the same time to the lowest T_1 and the lowest $\bar{T}_1$ doublet, it obeys to the equations

$$M |F_0\rangle = M^\dagger |F_0\rangle = \bar{M} |F_0\rangle = \bar{M}^\dagger |F_0\rangle = 0 \quad (3.62)$$

If we take the state $|F_0\rangle$ as "vacuum", we see that the symmetry induced by the M operators is preserved whereas that induced by the N operators is spontaneously broken. In order to have a perspective of our full results shown in Figs. 3 and 4, we wish to use different observation operators in the same manner and with similar purpose to what we did in Section 2. Using R as the observation operator, we find the spectrum of Fig. 3. As in the simple $N = 1$, $D = 1$ superconformal system, the splitting in R related to

bosonic excitations ($\Delta B = 0$) is twice that associated with the fermionic excitation ($\Delta B = \pm 1$). The fermionic ($\Delta R = \frac{1}{2}$) and bosonic ($\Delta R = \frac{1}{2}$) splittings are equal all over the spectrum: $\Delta R = \frac{1}{2}$ for a fermionic excitation and $\Delta R = 1$ for bosonic excitations (with $\Delta B = 0$).

This equal spacing is an analogue, in a system evolving with R , for a Goldstone boson and a Goldstone fermion. Note, however, that all excited states are two-fold degenerate, for $R = f/2 + (2n+1)/2$, these are degenerate bosons, for $R = f/2 + 2n/2$ these are degenerate fermions. The doubling follows from the fact that when $N = 2$ supersymmetry is broken two Goldstone like fermions should appear, an even fermion related to M, M^+ and an odd fermion related to $\bar{M}, \bar{M}^+$. When $T_2, \bar{T}_1$ ($T = B/2 \pm f/2$) are chosen as observation operators, the fermionic operators which commute with $R - B/2$ are candidates for supersymmetric operators. The lowest eigenvalue of $\bar{T}_1$ is zero and it corresponds to a normalizable eigenfunction. The equal spacing between bosonic levels reflects the breakdown of time translational invariance. It is tempting to say that the quartet degeneracy of the excited states reflects a redefined $N = 2$ extended supersymmetry. Note, however, that only $\bar{M}, \bar{M}^+$ are supersymmetric operators fulfilling an algebra in which H is replaced by $\bar{T}_1$, this is more like an $N = 1$ algebra, the quartet degeneracy results from the fact that also N and N^+ commute with $\bar{T}_1$, even though the anticommutator $\{N, N^+\}$ differs from $\bar{T}_1$ [Eq. (3.48)]. This is a rather peculiar situation. The vacuum state is two-fold degenerate but contains also a bosonic ($B = 1$) state and thus has a zero Witten index. This is not possible for a model with unbroken $N = 2$ supersymmetry. In this example, a system with an $N = 1$ supersymmetry displays an excited $N = 2$ spectrum.

In the case one chooses T_2 as an evolution operator, one finds that its lowest eigenvalue is $f > 0$. The anticommutator of N, N^+ is

$$\{N, N^+\} = T_2 \quad (3.63)$$

N has a zero eigenvalue non-normalizable state. Let us now take a slightly different view of the situation. Assume that both N, N^+ and $\bar{M}, \bar{M}^+$ are the supersymmetric generators of an $N = 2$ de Sitter supersymmetry which contains central charges (operators which commute with the supersymmetry operators). We would now view the pairs of anticommutators of $\bar{M}, \bar{M}^+$ and N, N^+ as

$$\begin{aligned}
 \{N \ N^+\} &= U + \frac{f}{2} & \{N \ \bar{M}\} &= \{N^+ \ \bar{M}^+\} = 0 \\
 \{\bar{M} \ \bar{M}^+\} &= U - \frac{f}{2} & \{N \ \bar{M}^+\} &= \{N^+ \ \bar{M}\} = -2iJ
 \end{aligned}
 \tag{3.64}$$

where $f/2$ is the central charge (which is non-zero in this case for all states in the spectrum). $U = R - B/2$ is chosen to be the evolution operator (one could use $U = R - B/2 - f/2$ just as well). In this case, we interpret the results as follows, the $N = 2$ supersymmetry was broken down to an $N = 1$ supersymmetry. N is the broken operator while $\bar{M}$ is the surviving operator. The breakdown of the supersymmetry associated with N is responsible for the degenerate ground states $B = 0$ and $B = 1$. The "Goldstone fermion" connects them. The Witten index is thus zero. The excited spectrum, as in any supersymmetric theory, reflects the symmetry of the Hamiltonian, which is the full $N = 2$ supersymmetry, that explains the existence of the quartets. Note that the lowest eigenvalue of $U = R - B/2$ is $f/2 > 0$ (choosing $U = R - B/2 - b/2$) would set it zero but would not change the Witten index. The spectrum is depicted in Fig. 5. The fact that $N = 2$ was not totally broken, is rather surprising since one usually concludes⁸⁾ that since $H = \sum_{\alpha} Q_{\alpha}^i Q_{\alpha}^i$ (fixed i) once there exists an i_0 such that $Q^{i_0}|0\rangle \neq 0$, the same is true for all i . This results in a total breakdown of the supersymmetry. However, in the presence of a central charge, we have shown that there can be a gradual breakdown of an extended supersymmetry. In this case, however, the central charge does not seem to have any direct topological meaning. The same discussion is valid for the pair of operators $R + B/2 \pm b/2$ and their corresponding supersymmetry generators $M, M^+, \bar{N}, \bar{N}^+$.

Our last effort will be to discuss the excited states corresponding to eigenvalues of J^2 different from zero. We notice that to any $J^2 = j^2$, (j integer) correspond the levels $|+j\rangle$ and $|-j\rangle$ which transform into each other under inversion.

In order to get the level spectrum for $j \neq 0$, we go back to our starting equations (3.48) and (3.49). We immediately see that we shall still have a quartet structure since T_1 commutes with $M, M^+, \bar{N}, \bar{N}^+$. However, the previous procedure for determining the lowest eigenstate has to

be modified. Indeed, suppose that T_1 has a $|\psi_0\rangle$ zero eigenstate subject to the equations

$$M|\psi_0\rangle = M^\dagger|\psi_0\rangle = 0 \quad (3.65)$$

using the third of Eqs. (3.48), we shall have

$$\frac{i}{2} \langle \psi_0 | \{M, \bar{N}^\dagger\} | \psi_0 \rangle = \langle \psi_0 | J | \psi_0 \rangle = 0 \quad (3.66)$$

in contrast with our assumption of $j \neq 0$ [a $j \neq 0$ state would break a bosonic $O(2)$ symmetry]. We are thus led to look for another pair of operators say μ and ν such that the mixed anticommutators involving μ and ν vanish. Let us define

$$\begin{aligned} \tan \theta &= \frac{j}{f} \\ \cos \frac{\theta}{2} &= \frac{1}{2} [1 + f(\delta^2 + f^2)^{-\frac{1}{2}}]^{\frac{1}{2}} \\ \sin \frac{\theta}{2} &= \frac{1}{2} [1 - f(\delta^2 + f^2)^{-\frac{1}{2}}]^{\frac{1}{2}} \end{aligned} \quad (3.67)$$

and

$$\begin{aligned} \mu &= \cos \frac{\theta}{2} M + i \sin \frac{\theta}{2} \bar{N} \\ \nu &= -i \sin \frac{\theta}{2} M + \cos \frac{\theta}{2} \bar{N} \end{aligned} \quad (3.68)$$

it is easy to show that

$$\begin{aligned}\frac{1}{4} \{ \mu^+ \mu \} &= R + \frac{B}{2} - \frac{1}{2} \sqrt{f^2 + j^2} = T(j) \\ \frac{1}{4} \{ v^+ v \} &= R + \frac{B}{2} + \frac{1}{2} \sqrt{f^2 + j^2} = T(j) + \sqrt{f^2 + j^2}\end{aligned}\quad (3.69)$$

$$\{ \mu^+ v \} = \{ \mu v^+ \} = 0$$

The form (3.68) of μ and v tells us that only the states $|\psi_0\rangle$ obeying

$$\mu |\psi_0\rangle = \mu^+ |\psi_0\rangle \quad (3.70)$$

are acceptable. They correspond to

$$T(j) |\psi_0\rangle = \left(R + \frac{B}{2} - \sqrt{f^2 + j^2} \right) |\psi_0\rangle = 0 \quad (3.71)$$

In particular, the lowest level $|F_0(j)\rangle$, corresponding to $B = 0$, will correspond to the lowest eigenvalue of R , i.e.,

$$R |F_0(j)\rangle = \frac{1}{2} \sqrt{f^2 + j^2} |F_0(j)\rangle \quad (3.72)$$

It is easy to apply the same procedure to the analogous anticommutators of Eq. (3.49), this will lead to similar results.

In conclusion, the level structure will be the same as that of Fig. 4. The only difference is that now the lowest eigenvalue of R is $\frac{1}{2} \sqrt{f^2 + j^2}$. We notice that the energy gap between the $j = 1$ and $j = 0$, $B = 0$ state is

smaller than the energy gap ($= \frac{1}{2}$) between the closest equal j $B = 0$ and $B = \pm 1$ states. This concludes our discussion of $N = 2$ extended conformal supersymmetry.

4. - CONCLUSIONS

The solution of superconformal quantum mechanics is interesting in its own right. It was shown that for $N = 1$, in one or more dimensions, supersymmetry is broken. Supersymmetry is also broken for a model with an extended $N = 2$ supersymmetry. It was already known that conformal quantum mechanics has no normalizable zero energy solution, that seemed to indicate that supersymmetry would be broken in the superconformal version. If finite theories would suffer from the absence of an $E = 0$ state, similar consequences as to the breaking of the supersymmetry may well follow. In the absence of a ground state, it was prescribed that compact operators should play the role of evolution and observation operators of the system. The spectrum of such operators was found to be discrete. A particular symmetric choice for the modified evolution operator is to diagonalize the compact bosonic operators appearing in the de Sitter subalgebra of the conformal algebra. With such a choice, new conserved supersymmetry operators can be defined. These operators are appropriate combinations of supersymmetry and fermionic conformal operators.

Viewing the system in that way, one found that for $N = 1$ and a system consisting of one particle, the new supersymmetry is unbroken. For the $N = 2$ case, a non-common partial breakdown to $N = 1$ is observed. These features, while demonstrated only in the elementary case of quantum mechanics, may also occur in finite quantum field theories.

The consequence of using compact evolution operators is the breakdown of time translational invariance. The equal spacing in the spectrum is an indication of that property. In view of the necessity of identifying mechanisms for dimensional reduction, this liability may turn into an asset. We believe that these observations merit further investigations of finite field theories in higher dimensions.

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FIGURE CAPTIONS

- Fig. 1: The spectrum of R for $N = 1$ as a function of B . The spectrum has no degeneracy. The minimum value, r_0 of R is $r_0 = \frac{1}{2}(\frac{1}{2}+f)$. The level spacing for a fixed B (bosonic excitation) is one unit. The level spacing for $\Delta B \neq 0$ (fermionic excitation) is half a unit.
- Fig. 2: Lines of fixed T_1 and T_2 as defined in Eq. (2.27). The operators N and $N^\dagger$ connect twins of a fixed T_2 . The degeneracy of the lowest T_2 state reflects the breaking of N supersymmetry and the existence of "Goldstone fermion" (a massless fermionic excitation). The Witten index of T_2 is zero. The operators M and $M^\dagger$ connect twins of a fixed T_1 . The lowest T_1 state is not degenerate (it has $T_1 = 0$). This reflects the unbroken M supersymmetry. Excited T_1 and T_2 states appear in twins. The Witten index of T_1 is one. $L^\pm$ operators create bosonic excitations.
- Fig. 3: The spectrum of R for $N = 2$ as a function of B . $B = -1$ is an empty state. $B = 0$ is a one fermion state (it can be usually either a symmetric or antisymmetric combination of the two fermionic species). $B = 1$ contains two different fermions. The lowest R state is $r_0 = \frac{1}{2}f$ and $x = 0$ (the antisymmetric combination) it is non-degenerate. The spacing in R is half a unit. The excited states are in turn $B = \pm\frac{1}{2}$ states and degenerate $B = 0$ states. The displayed spectrum is in the $O(2)$ singlet sector.
- Fig. 4: a) Typical $N = 2$ supersymmetrical spectrum of a Hamiltonian H . The ground state has a non-zero negative Witten index. It consists of one or more one-fermion states ($B = 0$). The excited states appear in quartets.
b) The spectrum of $\bar{T}_1 = R - B/2 - b/2$ as a function of B . The excited states display an $N = 2$ supersymmetry structure, as in Fig. 5a. The vacuum state is degenerate, one vacuum state is a fermion state ($B = 0$) and the other a boson ($B = 1$, a two Fermi state). The Witten index is zero. $N = 2$ de Sitter extended supersymmetry is broken to $N = 1$. The energy to excite a boson is finite, but the energy to excite a fermion is zero. This reflects the Goldstone fermion resulting from the breaking of one supersymmetry.

FIG. 1

FIG. 2

FIG. 3

FIG. 4a

$$\bar{T}_1 = R - B/2 - f/2$$

FIG. 4b