PERTURBATION THEORY AND ZERO MODES IN $O(N)$ LATTICE σ MODELS

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A B S T R A C T

Standard perturbation theory for $O(N)$ lattice σ models gives incorrect results in a finite volume in $d = 2$ and, even in the large volume limit, in $d = 1$. By introducing appropriate collective co-ordinates in treating the zero modes, the correct Feynman rules are derived and the two-point function is calculated at the one-loop level.

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The two-dimensional $O(N)$ σ model exhibits many of the interesting properties of four-dimensional gauge theories. Its relative simplicity makes it a standard testing ground for new ideas and computational techniques¹⁾⁻⁴⁾.

In this brief note a specific difficulty of perturbation theory for lattice σ models will be discussed and resolved. The problem is related to the existence of zero modes in perturbation theory. These zero modes should be treated specially, for instance, by introducing appropriate collective coordinates. We shall derive the resulting effective action and Feynman rules and the correct one-loop result for the correlation function will also be given.

The intermediate steps of the usual perturbative expansion of the σ model are plagued by infra-red divergences. A standard infra-red regularization is to introduce a small external magnetic field H . It was guessed⁵⁾ and confirmed later⁶⁾ that for $O(N)$ invariant expectation values the infra-red divergences cancel and the $H \rightarrow 0$ limit can be taken at the end.

This procedure interchanges the correct order of two limits ($H \rightarrow 0$ and coupling $\rightarrow 0$) and, as we shall discuss, leads to finite, but incorrect results in a finite volume in $d = 2$ and, even in the large volume limit, in $d = 1$ ⁷⁾.

The simplest example is an $O(3)$ system containing two spins only:

$$Z = \int d\vec{S}_1 d\vec{S}_2 \delta(1-\vec{S}_1^2) \delta(1-\vec{S}_2^2) e^{2\beta \vec{S}_1 \cdot \vec{S}_2} \quad (1)$$

The partition function and $\langle \vec{S}_1 \vec{S}_2 \rangle$ can be calculated exactly, giving:

$$\begin{aligned} \langle \vec{S}_1 \vec{S}_2 \rangle_{\text{exact}} &= \frac{1}{2} \frac{\partial}{\partial \beta} \ln Z = \frac{1}{2} \frac{\partial}{\partial \beta} \ln \left\{ \frac{\pi^2}{\beta} (e^{2\beta} - e^{-2\beta}) \right\} \\ &\xrightarrow{\beta \rightarrow \infty} 1 - \frac{1}{2\beta} + (\text{exponentially small corrections}). \end{aligned} \quad (2)$$

On the other hand, the usual perturbative steps (by introducing a magnetic field regulator) lead to

$$\langle \vec{S}_1 \vec{S}_2 \rangle_{\text{pert.}} = 1 - \frac{1}{2\beta} - \frac{1}{16\beta^2} + \dots, \quad (3)$$

which is incorrect beyond the tree level.

Consider a d dimensional ($d = 1, 2$) $O(N)$ spin model defined on a finite, periodic lattice:

$$Z = \int \prod_n d\vec{S}_n \delta(1 - \vec{S}_n^2) e^{\beta \sum_{n, \mu} \vec{S}_n \vec{S}_{n+\hat{\mu}}} \quad (4)$$

Integrate over the first component of the spin $\vec{S}_n$ with the help of the δ constraints. We get:

$$\int \prod_n d\vec{\pi}_n \frac{1}{(1 - \vec{\pi}_n^2)^{1/2}} e^{\beta \sum_{n, \mu} \vec{S}_n \vec{S}_{n+\hat{\mu}}} \quad (5)$$

where here, and in the following, we use the notation

$$\vec{S}_n = \begin{pmatrix} (1 - \vec{\pi}_n^2)^{1/2} \\ \vec{\pi}_n \end{pmatrix}, \quad (6)$$

where $\vec{\pi}_n$ is an $(N-1)$ component vector.

$\vec{\pi}_n^c = \vec{0}$ is a classical saddle-point. However, there are other equivalent saddle-points due to the fact that both the action and the measure are invariant under the following transformation:

$$\pi_n^i = \Omega_{i1} (1 - \vec{\pi}_n^2)^{1/2} + \Omega_{ik} \pi_n^k, \quad i, k = 2, 3, \dots, N, \quad (7)$$

where Ω_{ab} ($a, b = 1, 2, \dots, N$) is a global $O(N)$ rotation. Consequently, $\Pi_n^i = \Omega_{il}$ ($= \text{constant}$) is also a saddle-point. By expanding the action (and the measure) around the $\vec{\Pi}_n = \vec{0}$ saddle-point one obtains $(N-1)$ zero modes. They will be treated by the usual Faddeev-Popov trick.

Let us define the functional $\phi(\Pi_n^i)$ by the following equation:

$$1 = \phi(\Pi_n^i) \int d\Omega \prod_{i=2}^N \delta \left[\sum_n \left(\Omega_{in} (1 - \vec{\Pi}_n^2)^{1/2} + \Omega_{ik} \Pi_n^k \right) \right], \quad (8)$$

where $d\Omega$ is the $O(N)$ invariant measure. It is easy to show that $\phi(\Pi_n^i)$ is invariant under the transformation in Eq. (7).

Let us introduce the "1" of Eq. (8) into the functional integral in Eq. (5). By defining new integration variables via Eq. (7) and using the fact that the action, the measure and $\phi(\Pi_n^i)$ are invariant, we get

$$\int \prod_n d\vec{\Pi}_n \frac{1}{(1 - \vec{\Pi}_n^2)^{1/2}} \phi(\vec{\Pi}_n) \cdot \int d\Omega \prod_{i=2}^N \delta \left(\sum \Pi_n^i \right) e^{\beta \sum_{n,\mu} \vec{S}_n' \cdot \vec{S}_{n+\hat{\mu}}'} \quad (9)$$

We need the functional $\phi(\pi)$ on the surface $\sum_n \Pi_n^i = 0$. Therefore we can consider infinitesimal rotations in Eq. (8). The $(N-1)$ infinitesimal rotations in the $l-i$ ($i = 2, \dots, N$) planes give

$$\begin{aligned} 1 &= \phi(\pi) \int d\alpha_{12} d\alpha_{13} \dots d\alpha_{1N} \prod_{i=2}^N \delta \left(\alpha_{1i} \sum_n (1 - \vec{\Pi}_n^2)^{1/2} \right) \\ &= \phi(\pi) \frac{1}{\left(\sum_n (1 - \vec{\Pi}_n^2)^{1/2} \right)^{N-1}}, \end{aligned} \quad (10)$$

which leads to the partition function:

$$Z = \int \prod_n d\vec{\pi}_n \prod_{i=2}^N \delta\left(\sum_n \pi_n^i\right) e^{\beta \sum_{n,\mu} \vec{S}_n \cdot \vec{S}_{n+\hat{\mu}}} - \sum_n \ln(1 - \vec{\pi}_n^2)^{1/2} + (N-1) \ln \left[\sum_n (1 - \vec{\pi}_n^2)^{1/2} \right]} \quad (11)$$

Having treated the zero modes we can now expand the action in terms of $\vec{\pi}_n$ when β is large. Consider first the new term we obtained:

$$(N-1) \ln \left[\sum_n (1 - \vec{\pi}_n^2)^{1/2} \right] = \text{const.} - (N-1) \frac{1}{V} \sum_n \frac{1}{2} \vec{\pi}_n^2 + \dots, \quad (12)$$

where V is the volume of the system. The new term is suppressed by $1/V$; it is a finite volume correction in $d = 2$. (In $d = 2$ the infra-red divergences are mild and this correction goes to zero as $\ln V/V$ for large V .) However, it survives even the large volume limit in $d = 1$, where the term $\sum_n \frac{1}{2} \vec{\pi}_n^2$ creates infra-red divergences proportional to V .

The argument of the δ function in Eq. (11) is just $\Pi^1(k=0)$ in Fourier space implying that the $k = 0$ mode should be left out in the Fourier sums in calculating Feynman graphs. Otherwise, the usual Feynman rules [which include the extra vertices of Eq. (12)] should be applied.

As an example, let us give the two-point function at the one-loop level in an L^d periodic box:

$$\begin{aligned}
 \langle \vec{S}_m \vec{S}_m \rangle &= 1 + \frac{N-1}{\beta} \frac{1}{L^d} \sum_{\ell_\mu=1}^{L-1} \frac{e^{i \frac{2\pi}{L} \ell \cdot (n-m)} - 1}{\sum_{\mu} 4 \sin^2 \frac{\pi}{L} \ell_\mu} \\
 &\cdot \left\{ 1 + \frac{1}{2d\beta} \left(1 - \frac{1}{L^d} \right) + \frac{1}{2\beta} \frac{1}{L^d} \sum_{\ell_\mu=1}^{L-1} \frac{e^{i \frac{2\pi}{L} \ell \cdot (m-n)} - 1}{\sum_{\mu} 4 \sin^2 \frac{\pi}{L} \ell_\mu} \right\} - \\
 &- \frac{(N-1)(N-2)}{\beta^2} \frac{1}{L^d} \cdot \frac{1}{L^d} \sum_{\ell_\mu=1}^{L-1} \frac{e^{i \frac{2\pi}{L} \ell \cdot (m-n)} - 1}{\left(\sum_{\mu} 4 \sin^2 \frac{\pi}{L} \ell_\mu \right)^2},
 \end{aligned} \tag{13}$$

which can be obtained after a long, but straightforward algebra. With the usual magnetic field regularization one would obtain a similar expression, except that in the last term $(N-2)$ is replaced incorrectly by $(N-3)/2$.

Equation (13) complies with the checks I could perform.

a) For $d = 1$, $L = 2$, the exact result can be obtained for arbitrary N , which gives:

$$\langle \vec{S}_1 \vec{S}_2 \rangle_{\beta \rightarrow \infty} = 1 - \frac{N-1}{4\beta} + \frac{(N-1)(N-3)}{32\beta^2} + \dots \tag{14}$$

in agreement with Eq. (13).

b) In the cases of $d = 1$, $N = 3$, $L = 3$ and 4 , the partition function can be reduced to a single one-dimensional integral, giving

$$\begin{aligned} \langle \vec{S}_1 \vec{S}_2 \rangle &= 1 - \frac{2}{3\beta} - \frac{1}{27\beta^2} + \dots, \quad L=3 \\ \langle \vec{S}_1 \vec{S}_2 \rangle &= 1 - \frac{3}{4\beta} - \frac{1}{16\beta^2} + \dots, \quad L=4, \end{aligned} \quad (15)$$

in agreement with Eq. (13).

c) In the $N \rightarrow \infty$, $\beta/N = \lambda$ fixed limit, the model can be solved exactly in a finite volume both in $d = 1$ and $d = 2$. The exact result, when expanded in $1/\lambda$, agrees with the expression obtained from Eq. (13)⁸⁾.

d) In $d = 2$, in the $L \rightarrow \infty$ limit, Eq. (13) is in agreement with Elitzur's result⁵⁾.

Finally, let me remark that in $d = 1$ the large volume and weak coupling limits are not interchangeable - not even in perturbation theory. For a simple example, consider the $N \rightarrow \infty$, $\beta/N = \lambda$ fixed case, where analytic expressions are available. It is easy to show that

$$\langle \vec{S}_i \vec{S}_{i+1} \rangle = 1 - \frac{1}{2\lambda} + \frac{m^2}{2}, \quad (16)$$

where the mass m is the solution of the gap equation

$$\lambda = \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{1}{4 \sin^2 \frac{\pi \ell}{L} + m^2} \quad (17)$$

By taking first $L \rightarrow \infty$, we get³⁾

$$\lambda = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{4 \sin^2 \frac{k}{2} + m^2} = \frac{1}{(4m^2 + m^4)^{1/2}} \quad (18)$$

giving

$$\lim_{\lambda \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \langle \vec{S}_i \vec{S}_{i+1} \rangle \right) = 1 - \frac{1}{2\lambda} + \frac{1}{8\lambda^2} + \dots \quad (19)$$

On the other hand, if $\lambda \rightarrow \infty$ with L fixed, we can separate $\ell = 0$ in the sum in Eq. (17) and expand the rest in m^2 . One obtains:

$$m^2 = \frac{1}{\lambda L} + \frac{1}{\lambda^2} \frac{1}{L^2} \sum_{\ell=1}^{L-1} \frac{1}{4 \sin^2 \frac{\pi}{L} \ell} + \dots$$

$$\xrightarrow{L \rightarrow \infty} \frac{1}{12\lambda^2} + \dots \quad (20)$$

giving

$$\lim_{L \rightarrow \infty} \left(\lim_{\lambda \rightarrow \infty} \langle \vec{S}_i \vec{S}_{i+1} \rangle \right) = 1 - \frac{1}{2\lambda} + \frac{1}{24\lambda^2} + \dots \quad (21)$$

Equation (21) agrees with Eq. (13) as it should, while Eq. (19) agrees with exact results derived in an infinite volume earlier²⁾.

Equation (13) is the basis of the perturbative test of the improved ratio method discussed in Ref. 4). Actually, this work initiated the present investigation.

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