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QUANTUM EQUIVALENCE OF DUAL FIELD THEORIES

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differences in anomalies [5] and also local anomalous finite terms in some amplitudes induced by point transformations on spinors which may accompany duality transformations in supersymmetrical theories [10].

To construct a dual theory one rewrites the initial lagrangian $\mathcal{L}(\varphi)$ in the first order form (linear in derivatives) using the auxiliary (Lagrange multiplier) field $\phi : \mathcal{L}_1(\varphi, \phi)$. Eliminating the original variable φ on its equation of motion one finds ¹⁾ some constraints on ϕ . Solving the latter $\phi = \phi(\tilde{\varphi})$ we end up with a dual theory $\tilde{\mathcal{L}}(\tilde{\varphi})$, which is classically equivalent to the original one. However, there exist theories which are equivalent on the classical equations but are not related through duality transformation (we shall call them "pseudodual"). For example, it is sometimes possible to rewrite the equations of motion for φ in terms of some new set of variables φ' and then to restore the corresponding lagrangian $\mathcal{L}'(\varphi')$. Such pseudodual transformations include, in particular, the reparametrizations $\varphi' = F(\varphi)$. With F being non-local the equivalence theorem cannot be applied and thus pseudodual theories may have different S-matrices. The point that distinguishes true dual theories is the possibility to relate their path integrals by the chain of formal functional transformations (insertions of δ -functions and gaussian integrations). It is this relation (absent for pseudodual models) that is the main reason for on shell quantum equivalence of the dual theories.

Let us now explain what is understood by "on shell equivalence".

1) note that dual theory is local only if this elimination can be carried out algebraically

One, certainly, cannot equate Green's functions for dual variables (though correspondences between averages of some composite operators may exist). It is possible, however, to identify the on shell background functionals²⁾ [11], expressed in terms of the on shell values of all other fields Ψ in the theory (with "dual" fields φ and $\tilde{\varphi}$ being expressed in terms of Ψ using their equations of motion). More explicitly, if

$$e^{-W[\tilde{\varphi}, \tilde{\Psi}]} = \int d\varphi d\psi \exp \left\{ -S(\tilde{\varphi} + \varphi, \tilde{\Psi} + \psi) + \frac{\delta S}{\delta \tilde{\varphi}} \varphi + \left(\frac{\delta S}{\delta \tilde{\varphi}} \right) \psi \right\} \quad (1.1)$$

and $\tilde{W}$ is the analog of W for S substituted by $\tilde{S}$, then³⁾

$$W[\varphi_0(\varphi_0), \varphi_0] = \tilde{W}[\tilde{\varphi}_0(\varphi_0), \varphi_0] \quad , \quad (1.2)$$

where φ_0 , $\tilde{\varphi}_0$ and φ_0 are solutions of the corresponding classical equations. Eq. (1.2) is sufficient for the equivalence of the Ψ -sector S-matrices (and hence, in supersymmetrical case, of the complete S-matrices), equality of β -functions, etc. Note that if we are not assuming Ψ lying on its mass shell then W and $\tilde{W}$ are gauge dependent and (1.2) in general breaks down.

To give a heuristic argument for the validity of (1.2) let us assume that the dual theories have equal vacuum (or temperature) partition functions, i.e.

$$\int d\varphi d\psi \exp(-S(\varphi, \psi)) = \int d\tilde{\varphi} d\psi \exp(-\tilde{S}(\tilde{\varphi}, \psi)) \quad (1.3)$$

This assumption can be justified by translating all steps of the dual transformation into the path integral language. Then the same functional transformations which prove (1.3) can be used to establish the relation

- 2) or "reducible" effective actions, generating reducible connected graphs. Note that at one loop W coincides with the "irreducible" effective action Γ , defined in [12].

- 3) if S is independent of ψ the analog of (1.2) is $W[\varphi_0] = \tilde{W}[\tilde{\varphi}_0(\varphi_0)]$.

$$\int d\varphi d\psi \exp(-S(\varphi+\varphi_0, \psi+\psi_0)) = \int d\tilde{\varphi} d\tilde{\psi} \exp(-\tilde{S}(\tilde{\varphi}+\tilde{\varphi}_0, \tilde{\psi}+\tilde{\psi}_0)), \quad (1.4)$$

where $\varphi_0, \tilde{\varphi}_0, \psi_0$ again are on shell values. The validity of (1.4) is due to translation invariance of the functional integrals involved in the quantum dual transformation. Now we observe that (1.4) simply coincides with (1.2).

Below we shall justify eqs. (1.3) and (1.2) for several examples of dual models. We start (sect. 2) with a discussion of the dual transformation in the principal chiral model. Employing the parametrization in terms of currents we find that the theory of ref. [13] is a pseudodual one, construct (for $d=2$) the true dual model along the lines of ref. [14] and prove the quantum equivalence (1.2) of the latter with the original chiral model. The statement of equivalence provides a general explanation to the observation [14], that both dual theories have equal one-loop β -functions. We also consider the dual transformation for $d > 2$ which leads to a theory of interacting antisymmetric tensors.

In sect. 3 we investigate the duality between a scalar φ and a gauge antisymmetric tensor $A_{\mu\nu}$, which is important for supergravity. We first check the equivalence of the two quantum theories in external gravitational, vector, etc., fields and then

4) it should be noted here that the correct covariant quantization of the free antisymmetric tensor gauge theory (as a theory with the reducible closed gauge algebra) on a gravitational background was first given in [15] and later in [16]. The general quantization schemes for theories with reducible open (or closed) gauge algebras were worked out in refs. [17] and [18], correspondingly, in the lagrangian and hamiltonian frameworks.

generalise the proof to the case when all fields are quantized. Again our statement of equivalence (1.2) explains the result of ref. [19] where it was found that the one-loop on shell infinities in "gravity plus $\mathcal{G}$ " and "gravity plus $A_{\mu\nu}$ " systems are the same. We also remark that (contrary to some claims [20]) no essential complication in quantization of the dual $A_{\mu\nu}$ -theory arises in the case of the scalar - vector coupling through the Chern-Simons current ⁵⁾.

Two more examples, namely, duality transformations for (abelian and non-abelian) gauge vectors and for Einstein gravity with Λ -term are analysed in sect.4. The Einstein theory is dual to the Eddington-Schrodinger theory [23] with dual variable being symmetrical connection. The quantum on shell equivalence of these theories implies that the latter is non-renormalizable just as the former [24] (cf. [25]).

The practical conclusion of this paper is that it is not possible to improve non-trivially ultraviolet behaviour of the theory by changing one dual model for another. This does not, of course, exclude the importance of dual transformations in some technical questions, e.g. in study of different coupling regimes (cf. [26-29]). Also, the cancellation of anomalies in certain dual versions [3,6] may be important in some "off shell" problems [21].

5) as was noted in [21] such coupling appears in gauged $SU(2) \times SU(2)$ $N=4$ supergravity of ref. [22]. Analogous couplings are present also in $d > 4$ supergravities, e.g. in $d=10$ [4].

2. Duality transformations for chiral model

We shall start with the principal chiral model lagrangian

$$\mathcal{L} = \frac{1}{2\lambda^2} \text{tr} (\partial_\mu U \partial_\mu U^{-1}) \quad (2.1)$$

where $U = e^{\varphi} \in G$ and G is some compact group with the algebra $\mathfrak{g}$ ($\varphi \in \mathfrak{g}$). The corresponding classical equations

$$\partial_\mu (U \partial_\mu U^{-1}) = 0 \quad (2.2)$$

can be rewritten as a set of two equations for the current

$$\partial_\mu J_\mu = 0 \quad (2.3)$$

$$F_{\mu\nu}(J) \equiv \partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \quad (2.4)$$

Solving (2.4) we get

$$J_\mu = U \partial_\mu U^{-1} \quad (2.5)$$

and thus establish the equivalence of (2.2) and (2.3). If, following [13], we solve first the eq. (2.3) (taking $d=2$)

$$J_\mu = \epsilon_{\mu\nu} \partial_\nu \xi, \quad \xi \in \mathfrak{g}, \quad (2.6)$$

and then substitute the result in (2.4) we find an equation for ξ which can be obtained from the lagrangian [13]

$$\mathcal{L}' = -\frac{1}{2\lambda'^2} \text{tr} (\partial_\mu \xi \partial_\mu \xi + \frac{1}{3} \epsilon_{\mu\nu} \xi \partial_\mu \xi \partial_\nu \xi) \quad (2.7)$$

As was shown in [13], the quantum theory, based on (2.8), is inequivalent to the initial one, corresponding to (2.1): one-loop β -functions for λ and λ' have opposite signs and there is particle creation in the theory (2.7), while it is absent for (2.1). Thus (2.7) is "pseudodual" to (2.1).

The reason for inequivalence lies in the impossibility to carry out the transformation from (2.1) to (2.7) at the quantum

level. To see that eq. (2.3) never arises in the ^{chiral}/model quantum theory (and thus φ cannot be introduced) let us rewrite the formal path integral for (2.1) in terms of the current

$$\int d\varphi \exp \left\{ -\frac{1}{2\lambda^2} \int d^d x \operatorname{tr} (\partial_\mu e^\varphi \partial_\mu e^{-\varphi}) \right\} \quad (2.8)$$

$$= \int dJ_\mu d\varphi \delta(J_\mu - e^\varphi \partial_\mu e^{-\varphi}) \exp \left\{ \frac{1}{2\lambda^2} \operatorname{tr} J_\mu^2 \right\} \quad (2.9)$$

$$= C \int dJ_\mu \delta(\varepsilon_{\nu\lambda} F_{\nu\lambda}(J)) \exp \left\{ \frac{1}{2\lambda^2} \operatorname{tr} J_\mu^2 \right\} \quad (2.10)$$

(throughout this paper we neglect all local measure factors, irrelevant in dimensional regularization; also, we often do not indicate explicitly $\int d^d x$ in the exponents). In order to prove (2.10) one is to use the following general formula

$$\begin{aligned} \int d^n x_i \delta(f_\alpha(x_i)) A(x) &= \\ = \int d^{n-m} y_\lambda \left[\det K_{\lambda\mu}(y) / \det M_{\alpha\beta}(y) \right]^{1/2} A(\bar{x}(y)) \end{aligned} \quad (2.11)$$

where $i=1, \dots, n$, $\alpha=1, \dots, m$, $\lambda=1, \dots, n-m$, $f_\alpha(x) = 0$ has the solution $x^i = \bar{x}^i(y)$ (if there are several solutions, one is to sum over them in the r.h.s. of eq. (2.11)) and

$$K_{\lambda\mu} = \frac{\partial \bar{x}^i}{\partial y^\lambda} \frac{\partial \bar{x}^i}{\partial y^\mu}, \quad M_{\alpha\beta} = \left(\frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_i} \right)_{x=\bar{x}(y)} \quad (2.12)$$

In the case of our interest

$$x \rightarrow J_\mu, \quad y \rightarrow \varphi, \quad f_\alpha \rightarrow \varepsilon_{\mu\nu} F_{\mu\nu}(J), \quad \bar{x} \rightarrow e^\varphi \partial_\mu e^{-\varphi} \quad (2.13)$$

Taking into account the "pure gauge" nature of J_μ , one finds ($a, b = 1, \dots, \dim g$).

$$\det K_{\lambda\mu} \rightarrow \det(-\delta_{ab} \square), \quad \det M_{\alpha\beta} \rightarrow \det(-\delta_{ab} \square) \quad (2.14)$$

and thus $(\det K / \det M) = 1$. This proves the equivalence of (2.9) and (2.10).

As was already claimed, eq. (2.10) suggests no connection with the pseudodual model (2.7). Rather it makes possible to define the true dual model for (2.1) (this was first noted in ref. [14], where, however, the equivalence of (2.10) and (2.8) was not proved). Introducing an auxiliary field $\phi = \phi^a t^a$ (t^a is a basis in the algebra $\mathfrak{g}$, $\text{tr}(t_a t_b) = -2\delta_{ab}$, $[t_a, t_b] = f_{abc} t_c$) one can rewrite (2.10) as

$$\tilde{Z} = \int dJ_\mu^a d\phi^a \exp \left\{ \frac{1}{\lambda^2} \left(i \phi^a \varepsilon_{\mu\nu} F_{\mu\nu}^a(J) - (J_\mu^a)^2 \right) \right\} \quad (2.15)$$

Next, one is to note that the (gaussian) integration over J_μ is easy to carry out exactly with the result

$$\tilde{Z} = \int d\phi \exp(-\tilde{S}(\phi)) \quad , \quad (2.16)$$

$$\tilde{\mathcal{L}} = \frac{1}{\lambda^2} (\varepsilon_{\mu\rho} \partial_\rho \phi^a) N_{\mu\nu}^{-1 a b}(\phi) (\varepsilon_{\nu\sigma} \partial_\sigma \phi^b) \quad ,$$

$$N_{\mu\nu}^{ab} = \delta^{ab} \delta_{\mu\nu} - i f^{abc} \phi^c \varepsilon_{\mu\nu} \quad , \quad N N^{-1} = 1 \quad (2.17)$$

More explicitly, $N_{\mu\nu}^{-1} = C \delta_{\mu\nu} + i \varepsilon_{\mu\nu} \mathcal{D}$, $C^{ab} = \delta^{ab} + \dots$,
 $\mathcal{D}^{ab} = -f^{abc} \phi^c + \dots$, we get

$$\tilde{\mathcal{L}} = \frac{1}{\lambda^2} \left\{ C^{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b + i \varepsilon_{\mu\nu} \mathcal{D}^{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b \right\} \quad (2.18)$$

The classical equivalence of (2.1) and (2.18) follows from the observation that the action in (2.15) is the first order action for (2.18) and at the same time, when calculated on shell ($J_\mu = U \partial_\mu U^{-1}$), coincides with the chiral model action $(\hat{S}(\phi, J)_{J(\varphi)} = S(\varphi)$, $\hat{S}(\phi, J)_{J(\phi)} = \tilde{S}(\phi)$). Varying the former with respect to ϕ and J one finds

$$\varepsilon_{\mu\nu} F_{\mu\nu}(J) = 0 \quad , \quad J_\mu = e^\varphi \partial_\mu e^{-\varphi} \quad , \quad (2.19)$$

$$J_\mu^a = i \varepsilon_{\mu\nu} \partial_\nu^{ab}(\varphi) \phi^b, \text{ or } J_\mu^a = i N_{\mu\nu}^{-1 ab}(\phi) \varepsilon_{\nu\rho} \partial_\rho \phi^b \quad (2.20)$$

where $\partial_\nu^{ab} = \delta^{ab} \partial_\nu + f^{abc} J_\nu^c$. Eq. (2.20) defines the classical correspondence between the dual variables φ^a and ϕ^a . Being non-local (like the one given by (2.6)), this transformation (in contrast to $\varphi \rightarrow \xi$) can be carried out in the path integral.

Now let us prove the quantum equivalence of (2.1) and (2.18). As was already stressed in the introduction, it is inadequate to try to compare the standard Greens functions for φ and ϕ . In fact, the dual transformation is impossible if one adds a source term $i j \varphi$ in the exponent in (2.8). If the source is introduced for the current, $i j_\mu (e^\varphi \partial_\mu e^{-\varphi})$, then the same chain of transformations which led from (2.8) to (2.16) gives

$$Z[j] = \int d\phi e^{-\tilde{S}(\phi, j)}, \quad \tilde{S}(\phi, j) = \tilde{S}(\phi) \Big|_{\partial_\mu \phi \rightarrow \partial_\mu \phi + \frac{1}{2} \varepsilon_{\mu\nu} j_\nu} \quad (2.21)$$

Thus the correlation functions for the currents in the φ -theory can be expressed in terms of expectation values of some non-linear composite operators in the ϕ -theory. This off shell correspondence is obviously inconvenient. More direct, one advocated in this paper, is possible if one goes on shell and compares the background functionals (1.1) in both theories. It is easy to prove the analog of (1.2), namely

$$W[\varphi_0] = \tilde{W}[\phi_0(\varphi_0)] \quad , \quad (2.22)$$

where φ_0 is a solution of (2.2) and ϕ_0 is the corresponding solution of (2.19), (2.20). We have already proved ^{this} equality in the vacuum sector ($\varphi_0 = 0$). To switch on non-trivial backgrounds (φ_0 and ϕ_0) one is to use the translational invariance ($\varphi \rightarrow \varphi + \varphi_0$,

$\phi \rightarrow \phi + \phi_0$) of the integrals involved. Then all functional transformations go just in the same way as in the vacuum sector (one introduces the background values for the auxiliary fields J_μ and ϕ , which for consistency must be taken to be their classical values, etc). Eq. (2.22) implies the identity of $\beta(1)$ functions and correspondence of S-matrices in both dual theories. Thus any explicit checks of equivalence (cf. [14]) are now unnecessary.

We shall end this section with several remarks about generalizations of the duality transformation described above. Let us first discuss the generalization to higher dimensional ($d > 2$) case. Solving (2.3),

$$J_\mu^a = \partial_\rho B_{\rho\mu}^a, \quad B_{\mu\nu} = -B_{\nu\mu}, \quad (2.23)$$

and substituting this in (2.4)

$$\partial_\mu \partial_\rho B_{\rho\nu}^a - \partial_\nu \partial_\rho B_{\rho\mu}^a + f^{abc} \partial_\rho B_{\rho\mu}^b \partial_\sigma B_{\sigma\nu}^c = 0, \quad (2.24)$$

we get a pseudodual theory in terms of a set of antisymmetric tensors $B_{\mu\nu}^a$

$$\mathcal{L}' = \partial_\mu B_{\mu\nu}^a \partial_\rho B_{\rho\nu}^a + f^{abc} (B_{\mu\nu}^a \partial_\rho B_{\rho\mu}^b \partial_\sigma B_{\sigma\nu}^c + \text{terms of analogous structure}) \quad (2.25)$$

Note that (2.24) is invariant under

$$\delta B_{\mu\nu}^a = \partial_\rho \Lambda_{\rho\mu\nu}^a, \quad \Lambda_{\rho\mu\nu} = \Lambda_{[\rho\mu\nu]}, \quad (2.26)$$

while (2.25) is not. This is an interesting example of a theory with gauge-invariant equations of motion but non-invariant lagrangian. Note that it is the interaction term in (2.25) that is not

invariant under (2.26)⁶⁾. This may imply inconsistency of the theory at the quantum level (even if we neglect its power counting non-renormalizability).

To find the dual theory we start again with (2.8) and repeat the argument for equivalence of (2.9) and (2.10) (now with $\delta(\epsilon_{\mu\nu} F_{\mu\nu}) \rightarrow \prod_{\mu < \nu} \delta(F_{\mu\nu})$). The simplest reasoning for the triviality of the jacobian factor $(\det K / \det M)^{1/2}$ (see (2.11)-(2.14)) is based on the invariance of the measure $dJ_\mu \prod_{\mu < \nu} \delta(F_{\mu\nu}(J))$ under the gauge transformations of J_μ , implying that no non-trivial ($\mathcal{G}$ -dependent) terms can appear in (2.9) if one starts with (2.10). It should be noted that the whole argument is at least formal since the number of δ -functions in (2.10) is greater than the number of J_μ -integrals for $d > 3$. Now instead of (2.15) we get

$$Z = \int dJ_\mu^a dA_{\mu\nu}^a \exp \left\{ \frac{1}{\lambda^2} \left[i A_{\mu\nu}^a F_{\mu\nu}^a(J) - (J_\mu^a)^2 \right] \right\} \quad (2.27)$$

Integrating out J_μ^a we are left with the second order dual lagrangian (cf. (2.16)) with the dual variables being antisymmetric tensors $A_{\mu\nu}^a$

$$\tilde{\mathcal{L}} = \frac{1}{\lambda^2} \left\{ \partial_\rho A_{\rho\mu}^a N^{-1ab}_{\mu\nu}(A) \partial_\nu A_{\mu\nu}^b \right\}, \quad (2.28)$$

$$N_{\mu\nu}^{ab} = \delta^{ab} \epsilon_{\mu\nu} - i f^{abc} A_{\mu\nu}^c$$

The classical equivalence of this model and the principal chiral model (2.1) was first demonstrated in [30]. Here we assert that the equivalence (cf. (2.22)) holds also at the quantum level. It is

6) this is easy to understand from the absence of the corresponding Noether identity which would follow from (2.24) if (2.25) were invariant.

interesting to note that because of $\mathcal{Q}_{\mu} F_{\mu\alpha} = 0$ eq.(2.27) possesses a new gauge invariance absent in d=2 case (cf. (2.26))

$$\delta A_{\mu\nu}^a = \mathcal{Q}_{\mu}^a(J) \Lambda_{\nu}^b \quad (2.29)$$

It leads to a non-linear gauge invariance of (2.28), namely, (2.29) with $J_{\mu}^a \rightarrow i N_{\mu\nu}^{-1a\beta} \mathcal{Q}_{\mu} A_{\nu\beta}^{\beta}$. However, the corresponding algebra closes only on the mass shell [30] producing complications in straightforward quantization of the theory (cf. [30, 31]). These complications are probably irrelevant if we are interested not in the Green's functions for $A_{\mu\nu}$ but only in on shell background functional (which is gauge-independent).

Our next remark is about generalization on the G/H -chiral model case. Let h be an invariant subalgebra of g and $H_{\mu} \in h$. Then instead of (2.27) we have

$$Z = \int dJ_{\mu}^a dH_{\mu}^i \exp \left\{ \frac{1}{\lambda^2} \left[i A_{\mu\nu}^a F_{\mu\nu}^a(J) - (J_{\mu}^i + H_{\mu}^i)^2 - (J_{\mu}^{\alpha})^2 \right] \right\} \quad (2.30)$$

where $i = 1, \dots, \dim h$, $\alpha = 1, \dots, \dim g - \dim h$ (we have splitted the basis t^a on t^i and t^{α}). Integrations over H_{μ}^i and J_{μ}^a are easily carried out. The resulting dual action depends on $A_{\mu\nu}^a$ and is analogous to (2.28). It generates the quantum theory which is on shell equivalent to the G/H -chiral model theory.

3. Duality transformations for antisymmetric gauge tensors

Here for simplicity we shall consider only the example of scalar - second rank antisymmetric tensor duality in d=4 (generalizations to other cases are straightforward).

The construction of classically equivalent dual model is based

on the first order formulation. Suppose that the lagrangian for the scalar field φ depends only on its derivatives, $\mathcal{L}(\partial_\mu \varphi, \psi)$ (ψ stands for all other fields). Then introducing two auxiliary fields we have (imaginary unit is inserted in order to keep the analogy with quantum path integral case)

$$\mathcal{L}_1 = \mathcal{L}(f_\mu, \psi) + i \lambda_\mu (f_\mu - \partial_\mu \varphi) \quad (3.1)$$

Eliminating φ on its classical equations

$$\partial_\mu \lambda_\mu = 0, \quad \lambda_\mu = \epsilon_{\mu\alpha\beta\gamma} \partial_\alpha A_{\beta\gamma} \quad (3.2)$$

we get

$$\tilde{\mathcal{L}}_1 = \mathcal{L}(f_\mu, \psi) + i \epsilon_{\mu\alpha\beta\gamma} \partial_\alpha A_{\beta\gamma} f_\mu \quad (3.3)$$

"Integrating" over f_μ (substituting its classical value) we finish with the dual lagrangian

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(F_{\mu\nu\rho}, \psi), \quad F_{\mu\nu\rho} = \partial_{[\mu} A_{\nu\rho]} \quad (3.4)$$

Note that in order for $\tilde{\mathcal{L}}$ to be local, $\mathcal{L}(f_\mu, \psi)$ must be an algebraic function of f_μ , i.e. $\mathcal{L}$ must depend only on first derivatives of φ .

Let us now specialize the form of $\mathcal{L}$ to

$$\mathcal{L} = \left[\frac{1}{2} (\partial_\mu \varphi)^2 + \partial_\mu \varphi J_\mu(\psi) \right] \sqrt{g} \quad (3.5)$$

Such a (pseudo)scalar lagrangian is found, e.g., in $SU(4)$ (or gauged $SU(2) \times SU(2)$) supergravity [1,22] (any additional factors in the kinetic term can be absorbed in redefinition of the metric). Here J_μ depends on all other fields in the theory, e.g. vectors, spinors, etc. Following the steps (3.1)-(3.3) we get the dual lagrangian [3]

$$\mathcal{L}_1 = \left[\frac{1}{2} f_\mu^2 + f_\mu J_\mu + i \lambda_\mu (f_\mu - \partial_\mu \varphi) \right] \sqrt{g} , \quad (3.6)$$

$$\tilde{\mathcal{L}} = \frac{1}{2} (F_\mu^* - i J_\mu)^2 \sqrt{g} , \quad F_\mu^* \equiv \varepsilon_{\mu\alpha\beta\gamma} \partial_\alpha A_{\beta\gamma} \quad (3.7)$$

An apparent paradox with (3.7) is that it depends on the full "current" J_μ , while (3.5) depends only on its the longitudinal part

$$J_\mu^\parallel = \nabla_\mu \nabla^{-2} \nabla_\nu J_\nu \equiv J_\mu - J_\mu^\perp , \quad \nabla_\mu = \partial_\mu + \{ \frac{1}{2} \}$$

The resolution of this "paradox" is provided by the observation that in view of the trivial invariance of (3.6) under $\delta J_\mu = a_\mu^\perp$, $\delta \lambda_\mu = i a_\mu^\perp$, we can always rewrite (3.7) as

$$\tilde{\mathcal{L}} = \frac{1}{2} (\bar{F}_\mu^* - i J_\mu^\parallel)^2 \sqrt{g} , \quad (3.8)$$

$$\bar{A}_{\mu\nu} = A_{\mu\nu} + q_{\mu\nu}(J) , \quad J_\mu^\perp \equiv \varepsilon_{\mu\alpha\beta\gamma} \nabla_\alpha A_{\beta\gamma}$$

Thus any of the equivalent models (3.7) and (3.8) can be called dual to (3.5). The only advantage of (3.7) is its explicite locality. At the same time, eq. (3.8) can be simplified to

$$\tilde{\mathcal{L}} = \left[\frac{1}{2} (\bar{F}_\mu^*)^2 - \frac{1}{2} (J_\mu^\parallel)^2 \right] \sqrt{g} \quad (3.9)$$

The analogous factorized form of (3.5) is

$$\mathcal{L} = \left[\frac{1}{2} (\partial_\mu \bar{\varphi})^2 - \frac{1}{2} (J_\mu^\parallel)^2 \right] \sqrt{g} , \quad (3.10)$$

$$\bar{\varphi} = \varphi + \nabla^{-2} \nabla_\nu J_\nu$$

The above observation helps to clarify also the following point.

If φ interacts with a gauge vector through the $\varphi F_{\mu\nu} F_{\mu\nu}^*$ -term, then J_μ in (3.5) contains the Chern-Simons current

$$J_\mu = i \varepsilon_{\mu\alpha\beta\gamma} \left(A_\alpha^a \partial_\beta A_\gamma^a + \frac{1}{3} f^{abc} A_\alpha^a A_\beta^b A_\gamma^c \right) \quad (3.11)$$

$$\nabla_\mu J_\mu = \frac{i}{2} F_{\mu\nu}^a F_{\mu\nu}^{*a}, \quad J_\mu'' = \frac{i}{2} \nabla_\mu \nabla^{-2} (F_{\mu\nu}^a F_{\mu\nu}^{*a}). \quad (3.12)$$

It is J_μ'' that is invariant under vector gauge transformations. Hence the lagrangian (3.8) or (3.9) is explicitly gauge invariant under the independent transformations of A_μ^a and $\bar{A}_{\mu\nu}$

$$\delta A_\mu^a = \partial_\mu^{ab} (A_\lambda) \varepsilon^b, \quad \delta \bar{A}_{\mu\nu} = 0, \quad (3.13)$$

$$\delta \bar{A}_{\mu\nu} = \nabla_{[\mu} \Lambda_{\nu]} \quad , \quad \delta A_\mu^a = 0 \quad (3.14)$$

To provide the invariance of (3.7) one is to note that under (3.13)

$\delta J_\mu = a_\mu^\perp$, $a_\mu^\perp = \varepsilon_{\mu\alpha\beta\gamma} \partial_\alpha \Omega_{\beta\gamma}$ and thus non-invariance of J_μ can be compensated by the additional transformation of $A_{\mu\nu}$

$$\delta A_{\mu\nu} = i \Omega_{\mu\nu}, \quad \Omega_{\mu\nu} = 2i F_{\mu\nu}^a \varepsilon^a \quad (3.15)$$

Note that $\bar{A}_{\mu\nu}$ is invariant under (3.15) and thus is a proper gauge invariant variable. Using (3.8) or (3.9) we never confront the problem of the modification (3.15) of the gauge algebra and thus anticipate no specific problems in quantization of the theory. It should be stressed that the possibility to introduce the gauge invariant variable here is due to a trivial (bilinear) dependence of the action on $A_{\mu\nu}$. Thus the situation is similar to that in the theory of quantum abelian vector in external gravitational field where one can easily split the variables on the "longitudinal" (which drops from the action) and gauge invariant "transverse" ones. Just the same separation of variables applies to (3.7) ⁷⁾. Having

7) that is why the complicated quantization scheme, developed for (3.7) in ref. [20], seems to be unnecessary and artificial.

said all this, in what follows we shall not specify the particular structure of J_μ in (3.5) since it is irrelevant for the proof of quantum equivalence of dual models.

Before turning to the quantum theory we want to add a remark on the classical equivalence of (3.5) and (3.7). Writing the field equations for (3.6)

$$f_\mu + J_\mu + i\lambda_\mu = 0, \quad f_\mu - \partial_\mu \varphi = 0, \quad \nabla_\mu \lambda_\mu = 0, \quad (3.16)$$

we find the on shell "interpolation relation" between the dual variables φ and $A_{\mu\nu}$

$$\partial_\mu \varphi + J_\mu + i E_{\mu\alpha\beta\gamma} \partial_\alpha A_{\beta\gamma} = 0. \quad (3.17)$$

The two consequences of (3.17)

$$\nabla^2 \varphi + \nabla_\mu J_\mu = 0, \quad E_{\mu\nu\rho\sigma} \nabla_\rho (F_\sigma^* - iJ_\sigma) = 0 \quad (3.18)$$

are recognized as the classical equations for (3.5) and (3.7). It should be understood that (3.17) is senseless off the mass shell defined by (3.18) (in fact, it states that an exact one-form is equal to a closed one, but such an equality is non-trivial only for harmonic forms).

Let us now discuss the quantum equivalence the theories (3.5) and (3.7) starting with the simplest case when gravity and fields ψ in J_μ are external. Thus we are to compare the "partition functions"

$$Z[g, J] = \int d\varphi \exp \left\{ - \left[\frac{1}{2} (\partial\varphi)^2 + \partial_\mu \varphi J_\mu \right] \sqrt{g} \right\}, \quad (3.19)$$

$$\tilde{Z}[g, J] = \int dA_{\mu\nu} \exp \left\{ - \frac{1}{2} (E_{\mu\alpha\beta\gamma} \partial_\alpha A_{\beta\gamma} - iJ_\mu)^2 \sqrt{g} \right\}. \quad (3.20)$$

Integration over φ in (3.19) gives

$$Z = \exp\left[\frac{1}{2}(J_\mu^H)^2\sqrt{g}\right] (\det \Delta_0)^{-1/2}, \quad \Delta_0 = -\nabla^2 \quad (3.21)$$

To integrate over $A_{\mu\nu}$ in (3.20) it is not necessary to use the general procedures for quantization of gauge antisymmetric tensors (cf. [15 - 18]). It is sufficient to split $A_{\mu\nu}$ on gauge invariant and pure gauge parts with respect to the background metric

$$A_{\mu\nu} = A_{\mu\nu}^{(H)} + A_{\mu\nu}^\perp + \nabla_\mu \xi_\nu^\perp - \nabla_\nu \xi_\mu^\perp, \quad \nabla_\mu A_{\mu\nu}^\perp = 0 \quad (3.22)$$

($A^{(H)}$ is a harmonic two-form; we assume the space-time to be compact), to integrate over $A^\perp$ and to define the integral over pure gauge variables. The corresponding splitting of the measure is

$$dA = dA^{(H)} dA^\perp d\xi^\perp (\det' \Delta_{1\perp})^{1/2}, \quad \Delta_{1\perp} = -\nabla_{\mu\nu}^2 + R_{\mu\nu} \quad (3.23)$$

(the prime means that zero modes are omitted). Then (cf. (3.8), (3.9))

$$\tilde{Z} = (\det' \Delta_{1\perp})^{1/2} \int dA^{(H)} d\xi^\perp dA^\perp \exp\left\{\frac{1}{2}(J_\mu^H)^2 - \frac{1}{2}(\varepsilon_{\mu\nu\rho\sigma} \partial_\alpha \bar{A}_{\beta\delta}^\perp)^2 \sqrt{g}\right\} \quad (3.24)$$

Shifting the variable $A^\perp \rightarrow \bar{A}^\perp$ (observing that $u_{\mu\nu}^\perp$ drops from (3.8)) and integrating over $\bar{A}^\perp$ we get the additional factor

$$(\det' \Delta_{2\perp})^{-1/2}, \quad \Delta_{2\perp}^{\mu\nu} = -\nabla^2 \delta^{\mu\nu} + 2R_{\mu\alpha}^\perp \delta_{\beta\gamma}^\perp - 2R_{\alpha\beta}^{\mu\nu} \quad (3.25)$$

The integral $\int d\xi^\perp$ is defined using the relation

$$d\xi_\mu = d\xi_\mu^{(H)} d\xi_\mu^\perp d\phi (\det' \Delta_0)^{1/2}, \quad \xi_\mu = \xi_\mu^{(H)} + \xi_\mu^\perp + \partial_\mu \phi, \quad \int d\xi^\perp \sim \left\{ \int d\xi^{(H)} (\det' \Delta_0)^{1/2} \right\}^{-1} \times \text{"group volume"} \quad (3.26)$$

Then

$$\tilde{Z} = \left\{ \exp\left[\frac{1}{2}(J_\mu^H)^2\sqrt{g}\right] (\det \Delta_0)^{-1/2} \right\} \times \quad (3.27)$$

$$\{[\det' \Delta_{11} / \det' \Delta_{21}]^{1/2}\} \times \left\{ \int dA_{\mu\nu}^{(n)} \int d\phi^{(n)} / \int d\xi_{\mu}^{(n)} \right\}$$

The first bracket here is identical to $\tilde{Z}$ in (3.21). It is easy to prove that the second bracket is equal to one (we substitute $A_{\mu\nu}^{\perp}$ by $\varepsilon_{\mu\nu\alpha\beta} \partial_{\alpha} B_{\beta}^{\perp}$ and check that $dA_{\mu\nu}^{\perp} = dB_{\mu}^{\perp} (\det' \Delta_{11})^{1/2}$, $A^{\perp} \Delta_2 A^{\perp} = B^{\perp} (\Delta_1)^2 B^{\perp}$). The third bracket is the contribution of harmonic zero modes which produce additional anomalous scale dependence of $\tilde{Z}$ (cf. [5])

$$\tilde{Z} \sim \mu^{1/2 \chi} Z, \quad \chi = 2(b_2 - b_1 + b_0) \quad (3.28)$$

(χ is the Euler number and b_n are the Bettin numbers). The conclusion is that (3.19) is equivalent to (3.20) up to topological anomaly (3.28) (which is absent for asymptotically flat boundary conditions).

The same conclusion can be reached by direct functional transformation of (3.19). Rewriting the action in the first^{order} form and integrating over φ one finds

$$Z = \int d\lambda_{\mu} \delta(\nabla_{\mu} \lambda_{\mu}) \exp \left\{ -\frac{1}{2} (\lambda_{\mu} - iJ_{\mu})^2 \bar{g} \right\} \quad (3.29)$$

Splitting then λ_{μ} on harmonic, exact and coexact parts

$$\lambda_{\mu} = \lambda_{\mu}^{(H)} + \partial_{\mu} \phi + \varepsilon_{\mu\alpha\beta\gamma} \partial_{\alpha} A_{\beta\gamma}^{\perp} \quad (3.30)$$

we obtain

$$Z = \int d\lambda^{(H)} \exp \left\{ -\frac{1}{2} (\lambda_{\mu}^{(H)} - iJ_{\mu}^{(H)})^2 \bar{g} \right\} \int d\phi (\det' \Delta_0)^{1/2} \delta(\Delta_0 \phi) \\ \times \int dA_{\mu\nu}^{\perp} (\det' \Delta_{21})^{1/2} \exp \left\{ -\frac{1}{2} (\varepsilon_{\mu\nu\alpha\beta} \partial_{\alpha} A_{\beta\gamma}^{\perp} - iJ_{\mu})^2 \bar{g} \right\} \quad (3.31)$$

Integration over $\lambda^{(H)}$ and ϕ brings us back to (3.20), (3.23) ⁸⁾ (with $d\xi^{\perp} \sim (\det' \Delta_0)^{-1/2}$) but now with zero modes contribution being

⁸⁾ we again make use of the relation $\det' \Delta_{11} = \det' \Delta_{21}$

subtracted.

Now let us extend the proof of equivalence to the case when gravity is quantized (but $J_\mu = 0$). According to (1.2) we are to prove that

$$W[g_0, \varphi_0(g_0)] = \tilde{W}[g_0, A_0(g_0)] \quad (3.32)$$

where $g_{0\mu\nu}$, φ_0 and $A_{0\mu\nu}$ are solutions of the classical equations (here we neglect harmonic zero mode contributions assuming space-time to be topologically trivial). We have already discussed the case of $\varphi_0 = 0$ and $A_{0\mu\nu} = 0$. To switch on the backgrounds one makes the shift $\varphi \rightarrow \varphi + \varphi_0$, $g \rightarrow g_0 + h$, $A \rightarrow A + A_0$ and proceeds in the same manner as in sects. 1 and 2. Namely, under the on shell condition all terms linear in quantum fields cancel and thus simple functional transformations of path integrals go just as in the vacuum case (cf. (3.6), (3.29)). We have symbolically

$$e^{-W[g_0, \varphi_0]} = \int [dh] d\varphi \exp \left\{ -S_G(g_0 + h) - \frac{1}{2} (\partial_\mu \varphi + \partial_\mu \varphi_0)^2 G(g_0 + h) \right\} \quad (3.33)$$

where S_G is an action for gravity and G stands for $g^{\mu\nu} \sqrt{g}$. Inserting $1 = \int d f_\mu d \lambda_\mu \exp(i \lambda_\mu (f_\mu - \partial_\mu \varphi - \partial_\mu \varphi_0))$ and integrating over φ one at the same time is to assume a non-trivial background for λ_μ (satisfying (3.17) in order not to spoil the consistency, i.e. the cancellation of all terms linear in quantum fields). Treating $g_0 + h$ as an effective background metric, finally we are left with

$$e^{-W[g_0, \varphi_0]} = \int [dh] e^{-S_G(g_0 + h)} \int [dA_{\mu\nu}] \exp \left\{ -\frac{1}{2} [\epsilon_{\mu\alpha\beta\gamma} \partial_\alpha (A_{\beta\gamma} + A_{0\beta\gamma})]^2 \tilde{G}(g_0 + h) \right\} = e^{-\tilde{W}[g_0, A_0]} \quad (3.34)$$

where $\tilde{G}$ stands for the metric factor and $[dA]$ is a proper gauge fixing measure, depending on g_0 and $g_0 + h$ (we assume that the gravitational gauge is independent of $A_{\mu\nu}$)⁹⁾. The correctness of all formal manipulations with the path integral can be directly checked in the one-loop approximation. Thus our result (3.32) provides the general explanation for the equivalence of on shell infinities for gravity plus scalar and gravity plus antisymmetric tensor theories, first found in explicite one-loop calculation in [19].

If $J_\mu \neq 0$ all reasoning goes analogously (for simplicity here we ignore gravity)

$$e^{-W[\varphi_0(\psi_0), \psi_0]} = \int d\psi d\varphi \exp \left\{ -S_\psi(\psi + \psi_0) - \frac{1}{2} [(\partial_\mu \psi + \partial_\mu \psi_0)^2 + (\partial_\mu \varphi + \partial_\mu \varphi_0) J_\mu(\psi + \psi_0)] \right\} = \int d\psi [dA_{\mu\nu}] \exp \left\{ -S_\psi(\psi + \psi_0) - \frac{1}{2} [F_\mu^*(A + A_0) - i J_\mu(\psi + \psi_0)]^2 \right\} \quad (3.35)$$

where φ_0 and A_0 satisfy (3.17), (3.18), while ψ_0 is a solution of the classical equation $\delta S_\psi / \delta \psi + \frac{1}{2} \partial_\mu \varphi_0 \delta J_\mu / \delta \psi = 0$ (S_ψ is an action for ψ).

Finally, let us comment on an alternative attempt to relate the generating functionals for Green's functions in both theories. Starting, for example, with

$$Z[j_\mu] = \int d\varphi d\psi \exp \left\{ -S_\psi - \left[\frac{1}{2} (\partial_\mu \varphi)^2 + (J_\mu(\psi) + j_\mu) \partial_\mu \varphi \right] \right\} \quad (3.36)$$

and repeating the chain of functional transformations, we end with

$$Z[j_\mu] = \int d\psi \exp \left\{ -S_\psi - \frac{1}{2} \partial_\mu (J_\mu(\psi) + j_\mu) \square^{-1} \partial_\rho (J_\rho(\psi) + j_\rho) \right\} = \int d\psi [dA_{\mu\nu}] \exp \left\{ -S_\psi - \frac{1}{2} [E_{\mu\alpha\rho\beta} (\partial_\alpha A_{\beta\delta}) - i J_\mu - i j_\mu]^2 \right\} \quad (3.37)$$

) this assumption is of purely technical nature since W is gauge independent on shell

Changing the variable $A \rightarrow \bar{A}$ as in (3.8), (3.9) we see that j_μ decouples from $\bar{A}_{\mu\nu}$, i.e. that $Z[j_\mu]$ does not generate Green's functions for $\bar{A}_\mu$ (cf. [32]). We conclude once more that the comparison of on shell background functionals seems the most natural way of demonstrating the equivalence of dual theories.

4. More examples of duality transformations

We start with duality transformations for abelian vector theories. Let us consider a prototypical example of vector lagrangian which appears in extended supergravities

$$\mathcal{L} = \frac{1}{2} [f_1(\varphi) F_{\mu\nu} F_{\mu\nu} + i f_2(\varphi) F_{\mu\nu} F_{\mu\nu}^*] \quad (4.1)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{\mu\nu}^* = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$, and f_1 and f_2 are functions of the scalar field φ (in a realistic context there are several vector and scalar fields, f_i are matrices and there are also terms linear in $F_{\mu\nu}$ and bilinear in fermions). The first order lagrangian for (4.1) is

$$\mathcal{L}_1 = \frac{1}{2} [f_1 F_{\mu\nu}^2 + i f_2 F_{\mu\nu} F_{\mu\nu}^*] + i G^{\mu\nu} (F_{\mu\nu} - 2 \partial_\mu A_\nu) \quad (4.2)$$

Integrating out A_μ we get a constraint

$$\partial_\mu G^{\mu\nu} = 0 \quad (4.3)$$

with a solution

$$G_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} B_{\alpha\beta}, \quad B_{\alpha\beta} = \partial_\alpha B_\beta - \partial_\beta B_\alpha \quad (4.4)$$

Elimination of $F_{\mu\nu}$ then gives the lagrangian for the dual variable, i.e. abelian gauge vector B_μ

$$\tilde{\mathcal{L}} = \frac{1}{2} [\tilde{f}_1 B_{\mu\nu}^2 + i \tilde{f}_2 B_{\mu\nu} B_{\mu\nu}^*] \quad (4.5)$$

$$\tilde{f}_1 = - \frac{f_1}{f_1^2 + f_2^2}, \quad \tilde{f}_2 = \frac{f_2}{f_1^2 + f_2^2}$$

The quantum equivalence of (4.1) and (4.5) (understood again as in (1.2), i.e. $W[\varphi_0, A_\mu(\varphi_0)] = \tilde{W}[\varphi_0, B_\mu(\varphi_0)]$) can be demonstrated in complete analogy with the examples discussed above in sect. 3 and 4. As a by product, we conclude that inequivalence of the finite parts of S-matrices in $SO(4)$ and $SU(4)$ supergravities¹⁰⁾, observed in [10], is completely due to anomalous redefinitions of spinors and not to ^{the} vector duality transformation. It is worth noting also that the abelian vector duality transformations were discussed at the classical level in ref. [33], where a remark was made that these transformations should leave invariant the S-matrix because they leave invariant the equations of motion and the energy - momentum tensor (and hence the hamiltonian).

Next we consider the case of non-abelian vector gauge field.

The analog of (4.2) now looks like

$$\mathcal{L}_1 = \frac{1}{2} \left[\tilde{f}_1 F_{\mu\nu}^a F_{\mu\nu}^a + i \tilde{f}_2 F_{\mu\nu}^a F_{\mu\nu}^{*a} \right] + \quad (4.6)$$

$$+ i A_{\mu\nu}^a (F_{\mu\nu}^a - 2 \partial_\mu A_\nu^a - g f^{abc} A_\mu^b A_\nu^c)$$

Integrating over A_μ^a (formally defining the corresponding gaussian path integral by an analytic continuation) we obtain the dual lagrangian in terms of the dual variable - now the antisymmetric tensor $A_{\mu\nu}^a$

$$\tilde{\mathcal{L}} = \frac{1}{2} \left[\tilde{f}_1 A_{\mu\nu}^a A_{\mu\nu}^a + i \tilde{f}_2 A_{\mu\nu}^a A_{\mu\nu}^{*a} \right] +$$

$$+ \frac{i}{g} (\partial_\mu A_\nu^a) M^{-1}_{\mu\nu}{}^{ab} (\partial_\nu A_\mu^b) , \quad (4.7)$$

$$M_{\mu\nu}{}^{ab} = f^{abc} A_\mu^c , \quad M \cdot M^{-1} = 1$$

10) which differ on the vector duality transformation plus point transformation of spinors and scalars [1].

where $\tilde{f}_1$ and $\tilde{f}_2$ are the same as in (4.5). The limit $\tilde{g} \rightarrow 0$ brings us back to the abelian case with $A_{\mu\nu}^q$ having a potential. Analogous duality transformation for a non-abelian gauge theory was several times considered in the literature [27-29]. Here we want to stress the on shell quantum equivalence of the dual theory (4.7) and the original Yang-Mills one (for example, both theories are to have the same β -functions for the gauge coupling g). It is also interesting to compare (4.7) with the lagrangian (2.28) of the theory, dual to the chiral model (2.1). The invariance of (4.7) which is the direct analog of (2.29) is generated by

$$\delta A_{\mu\nu}^q = (\delta^{ab} \partial_\rho - M_{\rho\sigma}^{-1}{}^{dc} (A) f^{adb} \partial_\nu A_{\rho\sigma}^c) \Lambda_{\rho\mu\nu}^b. \quad (4.8)$$

This gauge algebra should be taken into account in an attempt to calculate Green's functions, corresponding to (4.7).

Our last example is provided by the Einstein gravity with a cosmological constant

$$\mathcal{L} = - \frac{1}{\kappa^2} (R - 2\Lambda) \sqrt{g}, \quad (4.9)$$

which is classically equivalent to the Eddington-Schrodinger theory [23]

$$\tilde{\mathcal{L}} = \frac{1}{4} \sqrt{\det R_{\mu\nu}(\Gamma)}, \quad \Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda, \quad (4.10)$$

$$R_{\mu\nu}(\Gamma) = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\rho\lambda}^\lambda \Gamma_{\mu\nu}^\rho - \Gamma_{\rho\nu}^\lambda \Gamma_{\mu\lambda}^\rho.$$

To demonstrate the classical equivalence of (4.9) and (4.10) we rewrite (4.9) in the first order form, introducing $R_{\mu\nu}$ and $\Gamma_{\mu\nu}^\lambda$ as independent fields

$$\mathcal{L}_1 = - \frac{1}{\kappa^2} (g^{\mu\nu} R_{\mu\nu} - 2\Lambda) \sqrt{g} + i N_1^{\mu\nu} (\Gamma_{\mu\nu}^\lambda - \{\Gamma_{\mu\nu}^\lambda\}) + i G^{\mu\nu} (R_{\mu\nu} - R_{\mu\nu}(\Gamma)) \quad (4.11)$$

(N and G are Lagrange multipliers). The corresponding equations of motion can be rewritten as follows

$$G^{\mu\nu} = -\frac{i}{\kappa^2} \sqrt{g} g^{\mu\nu}, \quad N_{,1}{}^{\mu\nu} = 0, \quad \Gamma_{\mu\nu}^{\lambda} = \{\}_{\mu\nu}^{\lambda}, \quad R_{\mu\nu} = R_{\mu\nu}(\Gamma) \quad (4.12)$$

$$R_{\mu\nu}(\Gamma) = \Lambda g_{\mu\nu}, \quad \nabla_{\lambda}(\Gamma) g^{\mu\nu} = 0 \quad (4.13)$$

(note that $N = 0$ is the origin of the Palatini principle). Here we recognize (4.13) as the equations of motion for (4.10) written in the first order form (with $g_{\mu\nu}$ being auxiliary field). The corresponding first order lagrangian is a natural starting point in the proof of quantum equivalence

$$Z = \int dR dG d\Gamma \exp(-\tilde{S}_1), \quad (4.14)$$

$$\tilde{\mathcal{L}}_1 = \frac{1}{\lambda} \sqrt{\det R_{\mu\nu}} + i G^{\mu\nu} (R_{\mu\nu} - R_{\mu\nu}(\Gamma)) \quad (4.15)$$

Trading $G^{\mu\nu}$ for $g_{\mu\nu}$ by the point transformation

$$G^{\mu\nu} = -\frac{i}{\kappa^2} \sqrt{g} g^{\mu\nu}, \quad \lambda = -\Lambda \kappa^2, \quad (4.15)$$

we identify the $G^{\mu\nu} R_{\mu\nu}(\Gamma)$ term in (4.15) as being the Einstein lagrangian in the first order (Palatini) form. Therefore, the result of the gaussian integration over $\Gamma_{\mu\nu}^{\lambda}$ is well known (see, e.g. [34]): it reproduces the Einstein action in the metric representation,

$-\frac{1}{\kappa^2} \int R_{\mu\nu}(\{ \}) g^{\mu\nu} \sqrt{g} d^4x$. To carry out the integration over $R_{\mu\nu}$ we expand it near the classical extremum

$$R_{\mu\nu} = R_{\mu\nu}^{(0)} + \epsilon_{\mu\nu}, \quad R_{\mu\nu}^{(0)} = \Lambda g_{\mu\nu},$$

$$Z = \int d g_{\mu\nu} \exp \left\{ -\frac{1}{\kappa^2} \int (R(g) - 2\Lambda) \sqrt{g} d^4x \right\} \times \int d \epsilon_{\mu\nu} \exp \left\{ -\frac{1}{8\lambda} \int \sqrt{g} d^4x [(\epsilon_{\mu}^{\mu} \epsilon_{\rho}^{\rho} - 2 \epsilon_{\mu\nu} \epsilon^{\mu\nu}) + O(\epsilon^3)] \right\}, \quad (4.17)$$

where the integral over $\phi_{\mu\nu}$ contributes only in a local measure.

The proof of equivalence of the on shell background functionals

$$W[g_0] = \tilde{W}[\Gamma_{\mu\nu}^\lambda = \{\lambda_{\mu\nu}\}_{g_0}] \quad (4.18)$$

goes analogously. A natural consequence of on shell equivalence of (4.9) and (4.10) is non-renormalizability of the latter theory [24]. This conclusion contradicts the claim made in ref. [25].

Finally, we note that analogous quantum equivalence considerations can be applied to various other kinds of dual theories (for example, to those recently discussed in [35]).

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