Ref.TH.3853-CERN

NON-LEADING CORRECTIONS TO THE DRELL-YAN CROSS-SECTION
AT SMALL TRANSVERSE MOMENTUM

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A B S T R A C T

The production of Drell-Yan lepton pairs at small transverse momentum is analyzed to second order in QCD perturbation theory. By calculating exactly all terms involving powers of the large logarithm $\ln Q^2/q_T^2$, we are able to confront the predictions of "soft gluon" cross-sections in which these logarithms are summed to all orders. Our results confirm the factorization properties of these cross-sections, and yield new information on previously undetermined coefficients.

Ref.TH.3853-CERN
March 1984

1. - INTRODUCTION

The production of large mass lepton pairs by the mechanism of quark-antiquark annihilation into a virtual photon - the Drell-Yan process - continues to be a subject of much theoretical and experimental study. Of particular interest is the lepton pair transverse momentum distribution, which can provide important quantitative tests of perturbative QCD¹⁾. At large transverse momentum, $q_T \sim Q, \sqrt{s}$, the coupling constant $\alpha_s(q_T)$ is small and the cross-section is well described by the standard QCD-improved parton model, i.e., once the collinear mass singularities are absorbed into non-scaling QCD structure functions, the parton cross-section is in principle dominated by the low order terms in the perturbation series. As q_T is made smaller, however, the emission of soft gluons gives rise to large logarithms of Q^2/q_T^2 and the simple perturbative expansion breaks down. One of the major theoretical advances of recent years has been the development of techniques to sum these large logarithms to all orders. The resulting cross-sections allow a new and essentially independent test of the theory in the region $q_T \ll Q$. Of course, at very small q_T values the non-perturbative wave function spread of partons in hadrons ("intrinsic q_T ") also becomes important. For the first generation of Drell-Yan experiments it was difficult to cleanly separate the regions $q_T \ll Q$ and $q_T \sim Q$. However, at the higher energies now available at the $\bar{p}p$ collider, it is likely that the small q_T region can be studied in detail. It is therefore important to establish reliable and accurate QCD predictions in this "soft gluon" region.

The analysis of the small q_T cross-section was pioneered by Dokshitzer, D'yakonov and Troyan²⁾. These original ideas have since been developed and refined and a complete picture - with all large logarithms under control - has begun to emerge³⁾⁻¹⁵⁾. Although the resulting expressions have become rather complex, the basic DDT structure remains valid - the cross-section factors into two structure functions (D) and a form factor (T) which contains the large logarithms from soft gluon emission. In practice, the leading behaviour of the form factor is controlled by coefficients which must be determined from low order calculations. These coefficients then generate a sequence of leading, next-to-leading,... etc., logarithms to all orders.

The purpose of the present study is twofold. By calculating exactly all the logarithmic terms through second order in α_s we are able to extract some

of the previously undetermined coefficients in the all-orders cross-section. We are also able to check the factorization structure at the first non-trivial order. The results confirm the validity of these cross-sections and - using the new coefficients - allow a more complete theoretical prediction.

Our second order analysis is based on the work of Ellis, Martinelli and Petronzio, who have calculated the $O(\alpha_s^2)$ correction at large q_T ¹⁶⁾. We use their exact expressions for the matrix elements (integrated over certain angular variables) to extract the small q_T behaviour, and then compare our results with the all-orders formalism developed by Collins and Soper¹¹⁾.

2. - THE $O(\alpha_s^2)$ CALCULATION

Following Ref. 16), we restrict our attention to the non-singlet cross-section, calculated at the parton level, i.e.,

$$\sigma^{NS} = \sum_{a,b} (\sigma_{\bar{q}_a q_b} - \sigma_{q_a q_b}) \quad (1)$$

where the sum is over quark flavours only. The real $O(\alpha_s^2)$ diagrams fall into four classes

$$q + \bar{q} \rightarrow \gamma^* + G + G \quad (2a)$$

$$(q + \bar{q})_A \rightarrow \gamma^* + q + \bar{q} \quad (2b)$$

$$(q + \bar{q})_S \rightarrow \gamma^* + q + \bar{q} \quad (2c)$$

$$q + q \rightarrow \gamma^* + q + q \quad (2d)$$

where the labels A, S correspond to annihilation and scattering, respectively.

There are also virtual gluon corrections to $q+\bar{q} \rightarrow \gamma^*+G$. At the level of approximation to which we work [all logarithms at $O(\alpha_s^2)$], the only real diagrams which contribute are (2a) and (2b). Hereafter, therefore, we will drop the non-singlet label and refer to a single species of quark with unit charge. To regulate the mass singularities which appear at second order we continue the dimension of space-time to $4+2\varepsilon$ dimensions.

Although we are primarily interested in the double differential cross-section $d^2\sigma/dQ^2 dq_T^2$, it proves convenient to take (Q^2/s) moments at fixed q_T^2 . This avoids complicated convolution integrals at intermediate stages of the calculation. However, here we face an ambiguity - in principle, we have the choice of integrating over Q^2 at fixed s or vice versa. Although the former might appear more "physical", it is the latter procedure which we adopt here. This is not just an arbitrary choice, for it can be shown that the factorization of mass singularities is only valid at fixed Q^{2*}). Furthermore, the coefficients of the $\ln q_T^2$ logarithms are also different in the two schemes. It is Q^2 rather than s , therefore, which sets the scale for the logarithms.

In the limit $q_T \rightarrow 0$ the cross-section $d^2\sigma/dQ^2 dq_T^2$ diverges as $1/q_T^2$, modulo powers of $\ln Q^2/q_T^2$. We therefore multiply by q_T^2 and consider the dimensionless quantity

$$\begin{aligned} \Sigma(N; \frac{Q^2}{q_T^2}, \frac{Q^2}{\mu^2}, \alpha_s) &= \int_{Q^2}^{\infty} ds \left(\frac{Q^2}{s}\right)^{N+2} \frac{q_T^2}{\sigma_0} \frac{d^2\sigma}{dQ^2 dq_T^2} \\ &= \int_0^1 d\tau \tau^N \frac{q_T^2 Q^2}{\sigma_0} \frac{d^2\sigma}{dQ^2 dq_T^2} \end{aligned} \quad (3)$$

where

$$\alpha_s \equiv \alpha_s(\mu) \quad \tau = \frac{Q^2}{s} \quad \sigma_0 = \frac{4\pi\alpha^2}{3s}$$

*) With fixed s moments, there is an uncanceled $1/\varepsilon$ pole with coefficient $\int dx x^N (1+x^2) \ln x / (1-x)$.

At first order the calculation is straightforward and the result is^{7),8)}, for $q_T \ll Q$,

$$\Sigma^{(1)} = \frac{\alpha_s}{2\pi} C_F \left\{ 2 \ln \frac{Q^2}{q_T^2} - 3 + 2 \xi_N \right\} + \dots \quad (4a)$$

where

$$\xi_N = \int_0^1 dx (x^{N-1}) \frac{1+x^2}{1-x} \quad (4b)$$

and $C_F = 4/3$. In Eq. (4a) the ... refer to terms which vanish in the limits $q_T \rightarrow 0$, $\epsilon \rightarrow 0$.

The complete second order correction to the transverse momentum distribution has been calculated by Ellis, Martinelli and Petronzio¹⁶⁾. They obtain an exact analytic expression for the differential cross-section $d^3\sigma/dQ^2 dt du$ and extract the q_T distribution by numerical integration. To analyze the small q_T behaviour, we use their expressions for $d^3\sigma/dQ^2 dt du$ but instead perform the integrals analytically in the $q_T \rightarrow 0$ limit. The kinematic variables are related by $ut + Q^2(u+t+s-Q^2)-sq_T^2 = 0$, and so we integrate over u , t and s at fixed Q^2 and q_T^2 . The calculation is then reduced to a series of two-dimensional integrals. The details are too complicated to present here and so instead we list only the results.

At $O(\alpha_s^2)$, then, we find^{*)}

$$\begin{aligned} \Sigma^{(2)} = & \left(\frac{\alpha_s}{2\pi} \right)^2 \left\{ \ln^3 \frac{Q^2}{q_T^2} [-2C_F^2] + \ln^2 \frac{Q^2}{q_T^2} [C_F^2(9 - 6\xi_N) \right. \\ & + \frac{11}{3} C_F C_A - \frac{4}{3} C_F T_R] + \ln \frac{Q^2}{q_T^2} [C_F^2(2\pi^2 \\ & - 23 + \lambda_N) + C_F C_A \left(\frac{35}{18} - \frac{\pi^2}{3} + \frac{11}{3} \xi_N \right) + C_F T_R \left(\right. \\ & \left. \left. - \frac{2}{9} - \frac{4}{3} \xi_N \right) \right] + \kappa_N \left. \right\} \end{aligned}$$

^{*)} The factorization scale M^2 , introduced in Ref. 16), has here been set equal to Q^2 .

$$\begin{aligned}
 & + \sum^{(1)} \frac{\alpha_s}{2\pi} C_F \left(\frac{1}{\epsilon} + \gamma_E - \ln(4\pi) + \ln \frac{Q^2}{\mu^2} \right) 2 \xi_N \\
 & - \sum^{(1)} \frac{\alpha_s}{2\pi} \left(\frac{11}{6} C_A - \frac{2}{3} T_R \right) \ln \frac{Q^2}{\mu^2} + \dots \quad (5a)
 \end{aligned}$$

where

$$\begin{aligned}
 \lambda_N &= 4 \int_0^1 dx (x^N - 1) \left\{ \frac{1+x^2}{1-x} \ln x + 3(1-x) \right. \\
 & \quad \left. - 4 \frac{1+x^2}{1-x} \ln(1-x) + \frac{2x^2}{1-x} \ln x \right\} \quad (5b)
 \end{aligned}$$

and $C_A = 3$, $T_R = \frac{1}{2} \times (n^{\circ} \text{ of quark flavours})$. In the above expression, κ_N represents the constant term which we have not calculated. Note that we have isolated (a) a piece containing the $1/\epsilon$ singularity and the associated finite pieces from the $O(\epsilon)$ term in the expansion of $(Q^2/4\pi\mu^2)^\epsilon (\Gamma(1+\epsilon)/\Gamma(1+2\epsilon))$ and (b) the finite coupling constant renormalization, after using $\overline{\text{MS}}$ subtraction to remove the ultra-violet pole.

Finally, there is another related quantity which can be calculated at first order, and which plays an important role in determining the coefficients in the all-orders cross-section. This is the q_T -integrated cross-section

$$\int_0^{p_T^2} \frac{dq_T^2}{q_T^2} \sum (N; \frac{Q^2}{q_T^2}, \frac{Q^2}{\mu^2}, \alpha_s)$$

with $p_T \ll Q$. The one real gluon contribution contains a double pole at $\epsilon = 0$, but this is exactly cancelled by the virtual correction to $\bar{q}+q \rightarrow \gamma^*$ which contributes at $q_T = 0$. We find, to $O(\alpha_s)$,

$$\begin{aligned}
 \int_0^{p_T^2} \frac{dq_T^2}{q_T^2} \sum (N; \frac{Q^2}{q_T^2}, \frac{Q^2}{\mu^2}, \alpha_s) &= 1 + \\
 \frac{\alpha_s}{2\pi} C_F \left\{ - \ln^2 \frac{Q^2}{p_T^2} + \ln \frac{Q^2}{p_T^2} [3 - 2 \xi_N] \right. \\
 + \pi^2 - 8 + \frac{2}{(N+1)(N+2)} + 2 \xi_N \left(\frac{1}{\epsilon} \right. \\
 \left. \left. + \gamma_E - \ln(4\pi) + \ln \frac{Q^2}{\mu^2} \right) \right\} + \dots
 \end{aligned}$$

(6)

3. - ALL-ORDERS ANALYSIS

The key to the analysis of the small q_T cross-section to all-orders in perturbation theory is to calculate in impact parameter - rather than momentum - space, i.e., to work with the Fourier transform, $\vec{q}_T \rightarrow \vec{b}$. In b-space, exact transverse momentum conservation is implemented automatically³⁾. The general result takes the form^{11),14)}

$$\frac{q_T^2}{\sigma_0} \frac{d^2\sigma^{A+B \rightarrow l^+l^-+X}}{dQ^2 dq_T^2} \approx \sum_q e_q^2 \int_0^1 dx_A dx_B$$

$$\delta(x_A x_B - \frac{Q^2}{S}) \frac{1}{2} \int_0^\infty db b J_0(bq_T) e^{-S(b,Q)}$$

$$\left\{ D_{q/A}(x_A, \frac{1}{b}) D_{\bar{q}/B}(x_B, \frac{1}{b}) + (q \leftrightarrow \bar{q}) \right\}$$

(7)

This approximation is valid at small q_T - all corrections vanish in the limit $q_T \rightarrow 0$. In Eq. (7), $D_{q/A}$ is a non-singlet quark structure function for quark q in hadron A , and e^{-S} is a Sudakov-like form factor which vanishes as $b \rightarrow \infty$.
More precisely,

$$S(b, Q) = \int_{(\frac{b_0}{b})^2}^{Q^2} \frac{dq^2}{q^2} \left\{ \ln \frac{Q^2}{q^2} A(\alpha_s(q)) + B(\alpha_s(q)) \right\}$$

$$A(\alpha_s) = \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{2\pi} \right)^i A^{(i)}$$

$$B(\alpha_s) = \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{2\pi} \right)^i B^{(i)}$$

$$b_0 = 2 e^{-\gamma_E}$$

(8)

In the Collins-Soper result, the limits on the integral in the expression for S are $c_2 Q^2$ and $c_1 b^{-2}$ so our choice here corresponds to $c_2 = 1$, $c_1 = 4e^{-2\gamma_E}$. The constants c_1 and c_2 are arbitrary, corresponding to different factorization and renormalization prescriptions. If all the perturbative coefficients were known, the complete result would of course be independent of c_1 and c_2 . Our choice for these parameters avoids factors of $\ln 2$ and γ_E appearing in the coefficients $A^{(i)}$ and $B^{(i)}$. Note also that all our results are with reference to the $\overline{\text{MS}}$ renormalization prescription.

The structure function can be written as

$$D_{q/A}(x, \frac{1}{b}) = \int_x^1 \frac{dy}{y} d(y; \frac{1}{b}, \mu, \alpha_s(\mu)) f_{q/A}(\frac{x}{y}, \mu) \quad (9)$$

where $f_{q/A}(x, \mu)$ is the probability distribution for the quark in hadron A at renormalization scale μ . (The label non-singlet is to be understood everywhere.) The function $d(y; 1/b, \mu, \alpha_s(\mu))$ satisfies the usual renormalization group equation and, defining moments by

$$d_N(\frac{1}{b}, \mu, \alpha_s(\mu)) = \int_0^1 dx x^N d(x; \frac{1}{b}, \mu, \alpha_s(\mu)) \quad (10)$$

the solution is

$$d_N(\frac{1}{b}, \mu, \alpha_s(\mu)) = C_N(\alpha_s(\frac{b_0}{b})) e^{-\int_{(\frac{b_0}{b})^2}^{\mu^2} \frac{dq^2}{q^2} \gamma_N(\alpha_s(q))}$$

where

$$\begin{aligned} C_N(\alpha_s) &= 1 + \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^i C_N^{(i)} \\ \gamma_N(\alpha_s) &= \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^i \gamma_N^{(i)} \end{aligned} \quad (11)$$

The $\gamma_N^{(i)}$ are the anomalous dimensions for the non-singlet quark structure

function evolution, and C_N is the corresponding coefficient function. Note that the former are universal, i.e., process independent, while the latter is specific to the particular process under consideration. Thus our structure functions, defined by Eq. (7), are related to those measured in other processes (e.g., deep inelastic scattering) by a finite perturbative renormalization (see below).

Since we wish to compare the above result with an explicit parton level calculation we must replace the functions $f_{q/A}$ with the corresponding quark distribution $q \equiv f_{q/q}$. In the $\overline{MS}$ scheme, this is given by^{*)}:

$$q(x, \alpha_s) = \delta(x-1) + \frac{\alpha_s}{2\pi} P^{qq}(x) \left\{ \frac{1}{\epsilon} + \gamma_E - \ln(4\pi) \right\} + \dots$$

$$P^{qq}(x) = C_F \left(\frac{1+x^2}{1-x} \right)_+ \quad (12)$$

or in terms of moments

$$q_N(\alpha_s) = 1 + \frac{\alpha_s}{2\pi} \gamma_N^{(1)} \left\{ \frac{1}{\epsilon} + \gamma_E - \ln(4\pi) \right\} + \dots$$

$$\gamma_N^{(1)} = \int_0^1 dx x^N P^{qq}(x) = C_F \xi_N \quad (13)$$

We can now combine Eqs. (7)-(13) to obtain the complete prediction for the moments of the cross-section, Σ_N , defined in Eq. (3):

^{*)} The + prescription is, as usual, defined by

$$\int_0^1 dx f(x) [g(x)]_+ = \int_0^1 dx g(x) [f(x) - f(1)]$$

$$\begin{aligned}
 \Sigma(N; \frac{Q^2}{q_T^2}, \frac{Q^2}{\mu^2}, \alpha_s \equiv \alpha_s(\mu)) \\
 = \frac{1}{2} \int_0^\infty db \, b \, J_0(bq_T) \left\{ C_N(\alpha_s(\frac{b_0}{b})) \right\}^2 q_N^2(\alpha_s) \\
 \exp \left[- \int_{(\frac{b_0}{b})^2}^{Q^2} \frac{dq^2}{q^2} \left\{ \ln \frac{Q^2}{q^2} A(\alpha_s(q)) + B(\alpha_s(q)) \right\} \right. \\
 \left. - 2 \int_{(\frac{b_0}{b})^2}^{\mu^2} \frac{dq^2}{q^2} \gamma_N(\alpha_s(q)) \right] + \dots \quad (14)
 \end{aligned}$$

The next step is to expand the integrand in Eq. (14) as a power series in α_s , collecting together all large logarithms, $\ln(Q^2 b^2/b_0^2)$, which can then be transformed into logarithms, $\ln(Q^2/q_T^2)$, in momentum space. Thus, in a notation similar to that of Ref. 17)^{*}, we write

$$\begin{aligned}
 \Sigma(N; \frac{Q^2}{q_T^2}, \frac{Q^2}{\mu^2}, \alpha_s) = \frac{1}{2} \int_0^\infty db \, b \, J_0(bq_T) q_N^2(\alpha_s) \\
 \exp \left[\sum_{n=1}^{\infty} \sum_{m=0}^{n+1} \left(\frac{\alpha_s}{2\pi} \right)^n \left(\ln \frac{Q^2 b^2}{b_0^2} \right)^m \mathcal{D}_m(N; t) \right] \quad (15)
 \end{aligned}$$

^{*}) Note, however, that in Ref. 17) the expansion parameter is α_s/π rather than $\alpha_s/2\pi$ which we use here.

with $t = \ln Q^2/\mu^2$. Then explicit calculation gives

$$\begin{aligned}
 {}_1 D_2 &= -\frac{1}{2} A^{(1)} \\
 {}_1 D_1 &= -B^{(1)} - 2\gamma_N^{(1)} \\
 {}_1 D_0 &= 2\gamma_N^{(1)} t + 2C_N^{(1)}
 \end{aligned} \tag{16a}$$

$$\begin{aligned}
 {}_2 D_3 &= -\frac{1}{3} \beta_0 A^{(1)} \\
 {}_2 D_2 &= -\frac{1}{2} A^{(2)} + \beta_0 \left(\frac{1}{2} t A^{(1)} - \frac{1}{2} B^{(1)} - \gamma_N^{(1)} \right) \\
 {}_2 D_1 &= -B^{(2)} - 2\gamma_N^{(2)} + \beta_0 (t B^{(1)} + 2t \gamma_N^{(1)} + 2C_N^{(1)}) \\
 \beta_0 &= \frac{11}{6} C_A - \frac{2}{3} T_R
 \end{aligned} \tag{16b}$$

If we also define

$$\begin{aligned}
 &\sum (N; \frac{Q^2}{q_T^2}, e^t, \alpha_s) \\
 &= q_N^2(\alpha_s) \left\{ \sum_{n=1}^{\infty} \sum_{m=0}^{2n-1} \left(\frac{\alpha_s}{2\pi} \right)^n \left(\ln \frac{Q^2}{q_T^2} \right)^m {}_n C_m (N; t) \right\}
 \end{aligned} \tag{17}$$

we can use the results of Refs. 17) and 18) to relate the ${}_nC_m$ to the ${}_nD_m$. Thus

$$\begin{aligned}
 {}_1C_1 &= -2 {}_1D_2 & {}_1C_0 &= -{}_1D_1 \\
 {}_2C_3 &= -2 {}_1D_2^2 \\
 {}_2C_2 &= -3 [{}_1D_1 {}_1D_2 + {}_2D_3] \\
 {}_2C_1 &= -2 [{}_2D_2 + \frac{1}{2} {}_1D_1^2 + {}_1D_2 {}_1D_0] \\
 {}_2C_0 &= -[{}_2D_1 + {}_1D_1 {}_1D_0 - 8\zeta(3) {}_1D_2^2] \quad (18)
 \end{aligned}$$

Notice that at the level of our exact second-order calculation (i.e., ${}_2C_3$, ${}_2C_2$, ${}_2C_1$), the only second-order coefficient is $\Lambda^{(2)}$, which appears in ${}_2C_1$ via ${}_2D_2$. The first-order coefficients are most easily read off from the small q_T -integrated cross-section, Eq. (6). It is straightforward to show from Eq. (15) that

$$\begin{aligned}
 &\int_0^{p_T^2} \frac{dq_T^2}{q_T^2} \sum (N; \frac{Q^2}{q_T^2}, e^t, \alpha_s) \\
 &= q_{N}^2(\alpha_s) \left\{ 1 + \frac{\alpha_s}{2\pi} \sum_{m=0}^2 \left(\ln \frac{Q^2}{p_T^2} \right)^m {}_1D_m(N; t) + \dots \right\} \quad (19)
 \end{aligned}$$

which in turn implies

$$\begin{aligned}
 {}_1D_2 &= -C_F \\
 {}_1D_1 &= C_F (3 - 2\zeta_N) \\
 {}_1D_0 &= C_F \left(\pi^2 - 8 + \frac{2}{(N+1)(N+2)} \right. \\
 &\quad \left. + 2t\zeta_N \right) \quad (20)
 \end{aligned}$$

Then, using Eq. (16a), we confirm that $\gamma_N^{(1)} = C_F \xi_N$ and determine

$$\begin{aligned} A^{(1)} &= 2 C_F \\ B^{(1)} &= -3 C_F \\ C_N^{(1)} &= \frac{1}{2} C_F \left(\pi^2 - 8 + \frac{2}{(N+1)(N+2)} \right) \end{aligned} \quad (21)$$

With this result for $C_N^{(1)}$, it is straightforward to relate our structure functions to those measured in other processes. For example, if we define structure functions using the Drell-Yan "total" cross-section:

$$\begin{aligned} \frac{1}{\sigma_0} Q^2 \frac{d\sigma}{dQ^2} &= \sum_q e_q^2 \int_0^1 dx_A dx_B \delta(x_A x_B - \frac{Q^2}{s}) \\ &\quad \left\{ \tilde{D}_{q/A}(x_A, Q) \tilde{D}_{\bar{q}/B}(x_B, Q) + (q \leftrightarrow \bar{q}) \right\} \end{aligned} \quad (22)$$

we find, with the coefficient function from Ref. 16)

$$\begin{aligned} \tilde{D}_{q/A}(x, Q) &= \int_x^1 \frac{dy}{y} D_{q/A}\left(\frac{x}{y}, Q\right) \left\{ \delta(1-y) + \frac{\alpha_s}{2\pi} f(y) + \dots \right\} \\ f(y) &= C_F \left[2(1+y^2) \left(\frac{\ln(1-y)}{1-y} \right)_+ - \frac{1+y^2}{1-y} \ln y \right. \\ &\quad \left. - 1 + y - \frac{\pi^2}{6} \delta(1-y) \right] \end{aligned} \quad (23)$$

We can now confront the predictions of the all-orders cross-section with the second-order coefficients calculated in Section 2. Combining the results of Eqs. (16)-(21) we obtain

$$C_1 = 2 C_F \quad (24a)$$

$${}_1C_0 = C_F (2 \xi_N - 3) \quad (24b)$$

$${}_2C_3 = -2 C_F^2 \quad (24c)$$

$${}_2C_2 = C_F^2 (9 - 6 \xi_N) + \frac{11}{3} C_F C_A - \frac{4}{3} C_F T_R \quad (24d)$$

$$\begin{aligned} {}_2C_1 &= A^{(2)} + C_F^2 \left(2\pi^2 - 25 + \frac{4}{(N+1)(N+2)} + 12 \xi_N \right. \\ &\quad \left. - 4 \xi_N^2 + 4t \xi_N \right) + C_F C_A \frac{11}{6} (2 \xi_N - 2t - 3) \\ &\quad - \frac{2}{3} C_F T_R (2 \xi_N - 2t - 3) \\ &= A^{(2)} + C_F^2 (2\pi^2 - 23 + 4t \xi_N + \lambda_N) \\ &\quad + C_F \left(\frac{11}{6} C_A - \frac{2}{3} T_R \right) (2 \xi_N - 2t - 3) \end{aligned} \quad (24e)$$

$$\begin{aligned} {}_2C_0 &= B^{(2)} + 2 \gamma_N^{(2)} + C_F^2 \left[8 \zeta(3) - 3\pi^2 + 24 \right. \\ &\quad \left. - \frac{6}{(N+1)(N+2)} + \xi_N \left(2\pi^2 - 16 + \frac{4}{(N+1)(N+2)} - 6t \right) + 4 \xi_N^2 t \right] \\ &\quad - C_F \left(\frac{11}{6} C_A - \frac{2}{3} T_R \right) \left(\pi^2 - 8 + \frac{2}{(N+1)(N+2)} - 3t + 2t \xi_N \right) \end{aligned} \quad (24f)$$

Note that in simplifying Eq. (24e), we have used the result

$$\lambda_N = -4 \xi_N^2 + 12 \xi_N - 2 + \frac{4}{(N+1)(N+2)} \quad (25)$$

Substituting Eqs. (24) into Eq. (17), and comparing with the result of the explicit calculation of ${}_2C_3$, ${}_2C_2$ and ${}_2C_1$ given in Eqs. (4) and (5), we find that:

- (i) the singular quark structure functions q_N do factor out as predicted by Eq. (17);
- (ii) the predicted N-dependence of all the calculated coefficients is correct;
- (iii) the N-independent terms are also in exact agreement, provided that

$$A^{(2)} = 2C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R \right] \quad (26)$$

In summary, we have verified that, at the parton level, the all-orders result of Eq. (7) - with the quoted parameter values - correctly accounts for all the $\ln Q^2/q_T^2$ terms in the small q_T cross-section to $O(\alpha_s^2)$.

4. - DISCUSSION

The value for $A^{(2)}$ which we have determined is precisely that given by Trentadue and Kodaira for the analogue process $e^+e^- \rightarrow q\bar{q}+X$ ^{14),*)}. In the notation of Ref. 14),

$$K = \frac{A^{(2)}}{A^{(1)}} = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R \quad (27)$$

In their approach, the leading behaviour as $q_T \rightarrow 0$ is determined by the $x \rightarrow 1$ limit of the Altarelli-Parisi functions. Thus K is the coefficient of the leading $(1-x)^{-1}$ singularity in the second-order non-singlet function, $\tilde{P}^{qq}(x)$. We take our agreement with this value of $A^{(2)}$ as evidence that not only is the approach of Trentadue and Kodaira compatible with that of Collins and Soper, but also that there is an intimate relationship between the coefficients $A^{(1)}$, $B^{(1)}$ and the corresponding i^{th} order Altarelli-Parisi functions.

In this respect it would be interesting to have the exact result for the coefficient $2C_0$. The calculation of this is more difficult - the set of

*) The same value has been used for the Drell-Yan cross-section in Ref. 15).

contributing diagrams is much larger - but work is in progress and will be reported elsewhere¹⁹⁾. From Eq. (24f), we see that knowledge of ${}_2C_0$ will serve to check the N-dependence generated by $\gamma_N^{(2)}$, and will determine the non-leading second-order coefficient $B^{(2)}$. Also of interest is the term proportional to $\zeta(3)$ - this is the first non-trivial "footprint" of the b-space transform, or equivalently, of the exact conservation of transverse momentum^{9),10)}.

Given the agreement discussed above, it might at first sight appear surprising that a similar (numerical) exercise for the energy-energy correlation function in e^+e^- annihilation revealed a rather large discrepancy between the expected ${}_2C_1$ coefficient (using the value for K determined by Trentadue and Kodaira) and that obtained by fitting an appropriate parametric form to the exact $O(\alpha_s^2)$ result¹⁷⁾. However, the comparison was made using non-singlet coefficients, as for the present analysis, whereas the e^+e^- process is manifestly singlet as all final state partons are summed over. This may be the source of the discrepancy - a more detailed investigation is in progress.

In conclusion, our analysis provides the first non-trivial verification that the all-orders "soft gluon" cross-sections do indeed account for all the large logarithms at small transverse momentum, at least to second order in perturbation theory. A numerical analysis which incorporates these non-leading corrections is in preparation¹⁹⁾.

ACKNOWLEDGEMENTS

We are grateful to B. Webber, D. Soper, K. Ellis, S. Ellis and L. Trentadue for useful discussions.

REFERENCES

- 1) For a recent review, see the Proceedings of the Drell-Yan Workshop, Fermilab (October 1982).
- 2) Yu.L. Dokshitzer, D.I. D'yakonov and S.I. Troyan - Physics Reports 58 (1980) 269.
- 3) G. Parisi and R. Petronzio - Nucl.Phys. B154 (1979) 427.
- 4) G. Curci, M. Greco and Y. Srivastava - Phys.Rev.Lett. 43 (1979) 834; Nucl.Phys. B159 (1979) 451.
- 5) C.Y. Lo and J.D. Sullivan - Phys.Lett. 86B (1979) 327.
- 6) A. Bassetto, M. Ciafaloni and G. Marchesini - Phys.Lett. 86B (1979) 366.
- 7) S.D. Ellis and W.J. Stirling - Phys.Rev. D23 (1981) 214.
- 8) H.F. Jones and J. Wyndham - Nucl.Phys. B176 (1980) 466.
- 9) P.E.L. Rakow and B.R. Webber - Nucl.Phys. B187 (1981) 254.
- 10) S.D. Ellis, N. Fleishon and W.J. Stirling - Phys.Rev. D24 (1981) 1386.
- 11) J.C. Collins and D.E. Soper - Nucl.Phys. B193 (1981) 381; B194 (1982) 445; B197 (1982) 446.
- 12) P. Chiappetta and M. Greco - Phys.Lett. 106B (1981) 219; Nucl.Phys. B199 (1982) 77.
- 13) M. Pennington - Nucl.Phys. B204 (1982) 189.
- 14) J. Kodaira and L. Trentadue - Phys.Lett. 112B (1982) 66; 123B (1983) 335.
- 15) P. Chiappetta and M. Greco - Nucl.Phys. B221 (1983) 269.
- 16) R.K. Ellis, G. Martinelli and R. Petronzio - Phys.Lett. 104B (1981) 45; Nucl.Phys. B211 (1983) 106.
- 17) S.D. Ellis, D.G. Richards and W.J. Stirling - Cambridge University Preprint DAMTP 82/34 (1982), to be published in Phys.Lett.
- 18) S.C. Chao, D.E. Soper and J.C. Collins - Nucl.Phys. B214 (1983) 513.
- 19) C.T.H. Davies, W.J. Stirling and B.R. Webber - in preparation.