REALIZATION OF LIE ALGEBRAS AND OF REPRESENTATIONS OF
LIE GROUPS IN TERMS OF HARMONIC OSCILLATORS *)

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ABSTRACT

The algebras and irreducible representations (irreps) of compact Lie groups, including the exceptional groups, are realized in terms of sets of Bose oscillator (sho) creation and annihilation operators. In particular, not only the tensor irreps, but also the spinor irreps of orthogonal groups of all ranks can be constructed using Bose (rather than the customary Fermi) oscillators. A complete set of Casimir operators is also given in a simple closed form for all the classical Lie groups in terms of sho operators. The eigenvalues of the operators are simply related to the Dynkin-Patera indices.

*) Supported in part by the US National Science Foundation.

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I. INTRODUCTION

Realizations of specific simple compact Lie algebras and at least some of the irreducible representations (irreps) of the corresponding groups in terms of Bose and/or Fermi oscillators have been extensively considered in the past, and have been found to be a useful tool in the analysis of a large class of physical problems. In particular, Schwinger¹ has studied the generators and irreps of SU(2) in terms of two Bose oscillators (hereafter abbreviated sho) to deal with angular momentum. These considerations have been extended² to U(n), and have found extensive applications in nuclear physics. Alternatively, the algebras of the classical groups [the unitary, orthogonal and symplectic groups A(n); B(n), D(n); C(n)] have been realized in terms of Fermi oscillators. The spinor irreps of the orthogonal groups have been constructed in this scheme and have had extensive and successful applications in the study of many electron atoms^{3,4,5} and in particle physics^{6,7}.

The oscillator technique has also been fruitfully used in the case of specific non-compact groups, U(1,1) in particular, most recently by Alhassid, Gürsey and Iachello^{8,9}. They study the relationship of bound and continuum states and energy band structures to each other for various one dimensional problems, and to related non-periodic and periodic potentials.

The list of work referred to above is not even an attempt at an exhaustive survey of a very extensive literature. Further references to other work can be found in the papers already quoted. All the references mentioned possess the common characteristic of having bilinear forms $a_i^+ a_j$ for the group generators (in terms of sho creation and annihilation operators¹⁰) and irreps which are constructed by having a suitable number of creation operators operate on an appropriately defined vacuum state. In the case of the sho, elements of a single irrep involve the same numbers of a_i^+ -s.

The spinor irreps of the orthogonal groups (the only ones which have, to our knowledge, been constructed for these groups, using Fermi oscillators) involve various numbers of a_i^+ -s, one a_i^+ for each index i, for different elements of the same spinor irrep. We are not aware of any attempt to treat all simple compact Lie groups on a uniform basis, nor to outline an approach for obtaining

all irreps, since the physical applications which have so far been made in the literature do not require it. In particular, aside from $G(2)$, none of the exceptional group algebras have been realized in terms of sho operators.

A uniform realization of the generators of all classical groups is listed by Gilmore¹¹ in terms of a set of coordinates and corresponding derivatives. Our approach closely parallels that of Gilmore for these groups. Since the operators a_i^+ and a_j for sets of sho-s can be given in the coordinate basis in terms of coordinates and their derivatives, one can reexpress Gilmore's form of the generators in terms of a_i^+ -s and a_j -s. The generators so obtained contain, in addition to the bilinear terms $a_i^+ a_j$, quadratic terms of the form $a_i a_j$ and $a_i^+ a_j^+$ as well as c-numbers. They are therefore less useful than the approach of refs. 1-9 in obtaining simple forms of the irreps. In this connection, we may also mention a non-linear realization of $SU(2)$ generators^{9,12} in terms of sho-s. This realization, like those of Gilmore, is inconvenient for constructing irreps.

The point of departure of the present paper is the approach of refs. 1-9. The work is entirely restricted to simple compact Lie groups. A unified approach is taken with respect to all groups. Generators and irreps are realized entirely in terms of Bose oscillators, even for spinor irreps. Our point of view grows from that of Dynkin^{13,14} and of Patera and coworkers^{15,16}, as further developed in a collaborative effort by the author and two colleagues¹⁷⁻¹⁹. The emphasis of the latter work is to carry out the Dynkin-Patera approach in an orthogonal root and weight space [with small and non-essential modifications required in the case of $A(n)$ and $G(2)$]. Such an approach leads to a concise and simplified treatment, with expressions which are more general and algebraically explicit (for example for the phases of structure constants¹⁸) than those arising from weight and root spaces expressed in terms of a non-orthogonal basis. The practical usefulness of this approach is demonstrated in refs. 17-19 and will be further justified in what follows. The notation of refs. 18 and 19 will be used in the present paper. Extensive use will also be made of the results of refs. 18 and 19, but, in the interests of brevity, they will often only be referred to, rather than repeated here.

The stage for the detailed discussion is set below in Section II, in which much of the notation used later is defined and the group algebras and irreps are written down in terms of it.

Section III is devoted to the realization of the generators, and of all the irreps of each classical group in terms of sho operators. For purposes of illustrating the approach, particular emphasis is placed on the derivation of the results for the groups $D(n) \equiv SO(2n)$; a briefer treatment, emphasizing only the results, is given for the other classical groups: $B(n)$, $C(n)$ and $A(n)$. Casimir operators are expressed in terms of sho operators for all the groups, and their eigenvalues are related to the Dynkin-Patera indices.

In Section IV, the approach is extended to the realization of the generators and irreps of the exceptional groups.

A brief summary and discussion of the salient aspects of the results is presented in Section V.

II. DEFINITIONS AND GENERAL CONSIDERATIONS

We shall begin by expressing all simple Lie algebras in the Dynkin basis^{14,18,19} in the form

$$[\underline{H}, \underline{H}] = 0, \quad (II.1)$$

$$[\underline{\alpha}(p) \cdot \underline{H}, E_{\underline{\alpha}(q)}] = \underline{\alpha}(p) \cdot \underline{\alpha}(q) E_{\underline{\alpha}(q)}, \quad (II.2)$$

$$[E_{\underline{\alpha}(p)}, E_{-\underline{\alpha}(p)}] = \underline{\alpha}(p) \cdot \underline{H}, \quad (II.3)$$

$$[E_{\underline{\alpha}(p)}, E_{\underline{\alpha}(q)}] = N_{\underline{\alpha}(p), \underline{\alpha}(q)} E_{\underline{\alpha}(m)}, \quad (II.4)$$

where the structure constant $N_{\underline{\alpha}(p), \underline{\alpha}(q)}$ is non-vanishing for

$$\underline{\alpha}(m) = \underline{\alpha}(p) + \underline{\alpha}(q) \quad (II.5)$$

a root.

The vector $\underline{H}$ has n components, where n is the rank of the algebra; the operators $E_{\pm, \alpha(p)}$ are step up (step down) operators in weight space.

The magnitude of the structure constants^{5,20} is the same for all groups with roots of the same length $[A(n), D(n), E(n)]$, and also for $B(n)$. There are two magnitudes for the structure constants of $C(n)$, $F(4)$ and $G(2)$. The phase factor of $N_{\underline{\alpha}(p), \underline{\alpha}(q)}$ is defined as

$$e^{i\Phi(p,q)} \equiv \frac{N_{\underline{\alpha}(p), \underline{\alpha}(q)}}{|N_{\underline{\alpha}(p), \underline{\alpha}(q)}|}. \quad (\text{II.6})$$

Detailed commutation relations, taking into account the magnitude of $N_{\underline{\alpha}(p), \underline{\alpha}(q)}$, have been written down for each specific type of simple compact Lie group in ref.²¹ 18. These commutation relations also serve to define various phase factors:

$$\begin{aligned} e^{i\Phi} : & d(p,q,r); b(p,q), \bar{b}(p,q); c(p,q,r), \\ & \sigma(p,q), \bar{\sigma}(p,q); e(p,q;r,s,\dots\omega), \bar{e}(p,q;r,\dots\omega); \\ & f(p,q;r,s), \bar{f}(p,q;r,s), \phi(p,q,r,s), \bar{\phi}(p,q,r,s); \\ & g(p,q), \bar{g}(p,q), \bar{\bar{g}}(p,q). \end{aligned} \quad (\text{II.7})$$

The commutation relations lead to Jacobi identities²⁰ which, in turn, can be expressed as algebraic conditions on the phases. These conditions are displayed in ref. 18. They do not lead to unique phase choices. A particular solution for each phase is obtained²² in ref. 18. All phases in these solutions are 0 or π :

$$e^{i\Phi} = \pm 1. \quad (\text{II.8})$$

We shall adopt these phase choices in what follows and express our results in terms of them. The generators $\underline{H}$ are hermitean. In addition, we impose the physically convenient restriction¹⁸

$$E_{-\underline{\alpha}(p)} = E_{\underline{\alpha}(p)}^\dagger. \quad (\text{II.9})$$

In the orthonormal root and weight space²³ we have a set of basis vectors, $\underline{\mu}_p$,

$$\underline{\mu}_p \cdot \underline{\mu}_q = \delta_{pq}, \quad (\text{II.10})$$

with the notation

$$\underline{\mu}_{|p|} \equiv (|p|), \quad (\text{II.11})$$

$$-\underline{\mu}_{|p|} \equiv \underline{\mu}_{-|p|} \equiv (-|p|), \quad (\text{II.12})$$

$$\frac{1}{2} \underline{\mu}_{|p|} \equiv (\frac{1}{2} |p|). \quad (\text{II.13})$$

For $A(n)$, we will need the set of non-orthogonal vectors $\underline{\chi}_p$:

$$\underline{\chi}_p \equiv \underline{\mu}_p - \frac{1}{n+1} \sum_{q=1}^{n+1} \underline{\mu}_q, \quad p=1, \dots, n+1, \quad (\text{II.14})$$

with

$$\sum_{p=1}^{n+1} \underline{\chi}_p = 0. \quad (\text{II.15})$$

For $G(2)$ we will need $\underline{\chi}_p$, with $p = 1, 2, 3$ as defined in (II.14).

The roots for all compact simple groups²⁴ are given in ref. 18 in terms of $\underline{\lambda}_p$ -s, which have the same properties as the $\underline{\mu}_p$ -s defined in eqn. (II.10). Using this form of the root, we can simplify our notation for the $E_{\underline{\alpha}}$ -s:

$$E_{(p-q)} \equiv \{p-q\} \quad (\text{II.16})$$

for a typical "vector" root and

$$E_{\frac{1}{2}(-p+q+r+s)} \equiv \{\frac{1}{2}(-p+q+r+s)\} \quad (\text{II.17})$$

for a typical "spinor" root. [The example given is for a generator of $F(4)$].

Such roots with "spinor" weights appear²⁴ in the exceptional groups E(8), E(7), E(6) and F(4). In addition, for G(2), we have roots²⁴ of weight $\underline{\chi}_p$, but no roots of weight $\underline{\mu}_p$, so that we define

$$E_{\underline{\chi}(p)} \equiv \{\underline{\chi}_p\}, \text{ for } G(2) \text{ only.} \quad (\text{II.18})$$

In terms of the definitions eqns. (II.16) and (II.17) we have the identity

$$\{p-q\} \equiv \{-q+p\}, \quad (\text{II.19})$$

since

$$\mu_p - \mu_q = -\mu_q + \mu_p, \quad (\text{II.20})$$

and also the identities

$$\{\frac{1}{2}(-p+q+r+s)\} \equiv \{\frac{1}{2}(q-p+r+s)\}, \text{ etc..} \quad (\text{II.21})$$

The notation for irreps is in terms of the Dynkin-Patera indices¹³⁻¹⁶ and the weight vector, $\underline{\Delta}$, of the individual irrep element. For a given irrep element we have

$$H | \underline{\Delta}, [\gamma] \rangle = \underline{\Delta} | \underline{\Delta}, [\gamma] \rangle, \quad (\text{II.22})$$

$$E_{\pm \underline{\alpha}(p)} | \underline{\Delta}, [\gamma] \rangle = C(\underline{\Delta}, \pm \underline{\alpha}(p), [\gamma]) | \underline{\Delta} \pm \underline{\alpha}(p), [\gamma] \rangle, \quad (\text{II.23})$$

and C is a numerical function of its variables. The scalar irrep will be the "vacuum state" and will be denoted by

$$| \underline{0}, [0] \rangle \equiv | 0 \rangle, \quad (\text{II.24})$$

with

$$[0] \equiv (0, 0, \dots, 0).$$

Since most of the Dynkin-Patera symbols we will use will have a large number of zero entries, we will shorten the notation by denoting only the non-zero entries, and indicating their position by a subscript. Thus, we shall take, for example:

$$(1,0,0,0) \equiv [1,],$$

$$(0,1,0,0) \equiv [1_2],$$

(II.25)

$$(2,0,0,0) \equiv [2,],$$

$$(1,0,0,1) \equiv [1, 1_4], \text{ etc. .}$$

The sho creation and annihilation operators will be denoted by

$$a^{(\kappa)\dagger}(\mathcal{L}) \text{ and } a^{(\kappa)}(\mathcal{L}), \quad (\text{II.26})$$

where (κ) and $(\mathcal{L})$ are a single index and a set of labeling indices respectively.

The operators have the usual commutation relations

$$\left. \begin{aligned} [a^{(\kappa)}(\mathcal{L}), a^{(\kappa')\dagger}(\mathcal{L}')] &= \delta(\mathcal{L}, \mathcal{L}') \delta_{\kappa \kappa'}, \\ [a^{(\kappa)}(\mathcal{L}), a^{(\kappa')}(\mathcal{L}')] &= 0, \\ [a^{(\kappa)\dagger}(\mathcal{L}), a^{(\kappa')\dagger}(\mathcal{L}')] &= 0, \end{aligned} \right\} \quad (\text{II.27})$$

where $\delta(\mathcal{L}, \mathcal{L}')$ is unity if all indices of $\mathcal{L}$ match those of $\mathcal{L}'$, and vanishes otherwise. We also define four types of operators, all of form $a^{(\kappa)\dagger}(\mathcal{L}) a^{(\kappa)}(\mathcal{L}')$:

$$N^{(\kappa)}(\mathcal{L}) \equiv a^{(\kappa)\dagger}(\mathcal{L}) a^{(\kappa)}(\mathcal{L}); \quad (\text{II.28})$$

$$N^{(\kappa)}(\mathcal{L}, \mathcal{L}') \equiv a^{(\kappa)\dagger}(\mathcal{L}) a^{(\kappa)}(\mathcal{L}'), \quad (\mathcal{L})=p, (\mathcal{L}')=q,$$

$$\text{or } (\mathcal{L})=p+q+r+s, (\mathcal{L}')=\pm 13, \text{ etc.,}$$

(II.29)

$$N^{(\kappa)}(p-q; r+s+\dots) \equiv a^{(\kappa)\dagger}(p-q+r+s+\dots) a^{(\kappa)}(-p+q+r+s+\dots), \quad (\text{II.30})$$

$$N^{(\kappa)}(p; q+r+\dots) \equiv a^{(\kappa)\dagger}(p+q+r+\dots) a^{(\kappa)}(-p+q+r+\dots). \quad (\text{II.31})$$

The symbols $\kappa, p, q, \dots$ are always single indices. When $(\mathcal{L})$ contains many such indices, each index is to be listed, not just the sum. In all the equations which follow, if more than two indices are listed, it should be understood that their absolute values are not equal, unless otherwise specified. For the case of $B(n)$ and $D(n)$, the superscript (κ) will be suppressed in multi-index operators. On occasion, the subscripts (o) or (e) will appear in multi-index operators or functions. They will signal the presence of odd or even numbers of minus signs in the indices involved.

The operators $N^{(\kappa)}(\mathcal{L})$ of eqn. (II.28) are the number operators for the states labeled by indices (κ) and $(\mathcal{L})$.

The realizations we will seek for a given Lie algebra will be suitably chosen linear superpositions of the $N^{(\kappa)}$ operators appearing in (II.29)-(II.31) (and other operators of similar structure, to be defined later) for the step-up and step-down operators $E_{\pm\alpha(p)}$ and of $N^{(\kappa)}(\mathcal{L})$ -s for the components of $\underline{H}$. The irreps will be realized by homogeneous polynomials (or monomials) of $a^{\pm}(\mathcal{L})$ -s operating on the "vacuum state", $|0\rangle$. We also introduce the summation (multiplication) symbols

$$\Sigma, \Sigma'_{(p)}, \Sigma''_{(p,q)} [\pi, \pi'_{(p)}, \pi''_{(p,q)}, \pi'''_{(p,q,n)}] \quad (\text{II.32})$$

to denote, respectively, summation (multiplication) over all the indices of a set $\mathcal{L}(\Sigma, \Pi)$, all but one of the indices, (p) , of a set $\mathcal{L}(\Sigma'_{(p)}, \Pi'_{(p)})$, all but the two indices, (p, q) , of a set $\mathcal{L}(\Sigma''_{(p,q)}, \Pi''_{(p,q)})$, etc.

Additional new notation will be introduced later, as needed.

III. CLASSICAL GROUPS

1. The groups D(n)

We will carry out the analysis for these groups in some detail, since they present most of the features characteristic of the other classical groups. The latter will be treated subsequently, type by type, but in a more cursory fashion.

a. Realizations of the algebra

We will gradually build up a hierarchy of realizations of the algebra of D(n). At each stage, the algebra will be correctly realized. However, each succeeding stage will allow for the possibility of realizing more and more types of irreps, until, at the last stage, it will be possible to realize all irreps of D(n).

To begin with, it is useful to quote the detailed form of eqn. (II.4) for this case, as given in ref. 18 (together with the specific solution for the phase factor, obtained in that reference):

$$[\{p-q\}, \{q-r\}] = d(p, q, r) \{p-r\}, \quad (\text{III.1.1})$$

where

$$p, q, r = \pm 1, \pm 2, \dots, \pm n, \quad (\text{III.1.2})$$

$$d(p, q, r) = \varepsilon(p+q) \varepsilon(q+r) \varepsilon(r+p), \quad (\text{III.1.3})$$

and

$$\varepsilon(x) = +1, \quad x > 0, \quad (\text{III.1.4})$$

$$\varepsilon(x) = -1, \quad x < 0.$$

We attempt to realize the algebra by letting²⁵

$$\underline{H} = \sum_p \mu_p N(p) \equiv \sum_p \mu_p a^\dagger(p) a(p), \quad (\text{III.1.5})$$

and

$$\{p-q\} = \theta(p, q) \{N(p, q) - N^\dagger(-p, -q)\}, \quad (\text{III.1.6})$$

where $\theta(p, q)$ is a phase factor.

The algebra, defined by eqns. (II.1)-(II.3) and (III.1.1), and the additional conditions (II.9) and (II.19) yield conditions on $\theta(p,q)$. However, the problem is so trivial, because of the factorized form of the phase $d(p,q,r)$, eqn. (III.1.3), that we can immediately guess the solution. It is

$$\theta(p,q) = \varepsilon(p+q). \quad (\text{III.1.7})$$

The algebra of $D(n)$ is fully realized by the generators defined in eqns. (III.1.5) and (III.1.6) [with (III.1.7)]. However, it will be apparent from the discussion of the next section (Section III.1.b) that only the elementary irrep²⁶ (which we will call²⁷ f , for "fundamental") associated with the "unbranched" terminal point of the Dynkin diagram of $D(n)$, and related irreps (built up from the symmetric parts of the Kronecker products of this elementary irrep) can be generated in this scheme. In other words, using our abbreviated version of the Dynkin-Patera notation, only the irreps

$$[\gamma] = [\ell,], \quad \ell = 1, 2, \dots, \quad (\text{III.1.8})$$

$$f \equiv [1,]$$

can be constructed, using the sho operators which appear in eqns. (III.1.5) and (III.1.6).

The scheme can be generalized trivially to allow the realization of all the basic irreps²⁸, with the exception of the two elementary spinor irreps, by introducing another index, κ . (This fact will be demonstrated in Section III.1.b.) We thus realize the algebra by setting²⁵

$$\underline{H} = \sum_p \mu_p N^{(\kappa)}(p) \equiv \sum_{p,\kappa} \mu_p a^{(\kappa)\dagger}(p) a^{(\kappa)}(p), \quad (\text{III.1.9})$$

and

$$\{p-q\} = \varepsilon(p+q) \sum_{(p,q)}'' \{N^{(\kappa)}(p,q) - N^{(\kappa)\dagger}(-p,-q)\}, \quad (\text{III.1.10})$$

with

$$\kappa = 1, 2, \dots, n-2. \quad (\text{III.1.11})$$

In other words, we can now realize all the basic irreps generated from the antisymmetric parts of the Kronecker products of $[1_1]$, or

$$[\gamma] = [1_\kappa]. \quad (\text{III.1.12})$$

The last, and this time non-trivial, step in our hierarchy of realizations of the $D(n)$ algebra is intended to allow for the realization of the two elementary irreps associated with the two "branched" endpoints. These are the elementary spinor irreps, which we will call²⁸ $\sigma_{(o)}$, $\sigma_{(e)}$,

$$\sigma_{(o)} \equiv [1_{n-1}], \quad (\text{III.1.13})$$

$$\sigma_{(e)} \equiv [1_n].$$

We consider imbedding the group $D(n)$ so that it is a non-regular subgroup of a group $D(N)$:

$$D(N) \supset D(n), \quad n \geq 4, \quad (\text{III.1.14})$$

such that the fundamental irrep of $D(N)$, $f_{D(N)}$, goes into the direct sum of the three terminal point irreps of $D(n)$, i.e.

$$f_{D(N)} \rightarrow (f \oplus \sigma_{(o)} \oplus \sigma_{(e)})_{D(n)}. \quad (\text{III.1.15})$$

From the dimensions of the various irreps involved, we get the condition

$$N = n + 2^{n-1}. \quad (\text{III.1.16})$$

We encounter no problems²⁹ for the case $n = 4$, for which the Dynkin diagram is symmetric, so that any of the terminal points can be associated with any one of f , $\sigma_{(e)}$ or $\sigma_{(o)}$.

The problem of obtaining the generators of a group g , as superpositions of the generators of a group of higher rank, $\mathfrak{G}$, where g is a non-regular subgroup of $\mathfrak{G}$, was considered in general and solved for a large number of specific cases in ref. 19. We will employ the methods of this reference as much as possible.

The root and weight space of $D(N)$ can be defined by $2N$ orthonormal vectors

$$2n \quad \lambda_p, \quad p = \pm 1, \pm 2, \dots, \pm n, \quad (\text{III.1.17})$$

and

$$2(N-n) = 2^n \sum (p+q+r+\dots), \quad (\text{III.1.18})$$

$$p, q, r, \dots = \pm 1, \pm 2, \dots, \pm n.$$

There are n labels $p, q, r, \dots$, and each distribution of plus or minus signs defines a unique $\underline{\lambda}$. The many-index $\underline{\lambda}$ -s are symmetric in the index labels. The analogue of (II.10) holds and [see (II.12)]

$$\underline{\lambda}_{-p} = -\underline{\lambda}_p, \quad (\text{III.1.19})$$

$$\underline{\lambda}(-p-q-r-\dots) = -\underline{\lambda}(p+q+r+\dots).$$

We chose the projections

$$\underline{\lambda}_p \rightarrow \underline{\mu}_p \equiv (p), \quad (\text{III.1.20})$$

$$\underline{\lambda}(p+q+r+\dots) \rightarrow \frac{1}{2}(\underline{\mu}_p + \underline{\mu}_q + \underline{\mu}_r + \dots) \equiv \frac{1}{2}(p+q+r+\dots),$$

$$p, q, r, \dots = \pm 1, \pm 2, \dots, \pm n.$$

where the $\underline{\mu}_p$ define the weight and root space of $D(n)$. From (III.1.20), the projection of the roots of $D(N)$ to the roots of $D(n)$ is

$$\left. \begin{aligned} \underline{\lambda}_p - \underline{\lambda}_q \\ \underline{\lambda}(p+q+r+s+\dots) + \underline{\lambda}(p-q-r-s-\dots) \end{aligned} \right\} \rightarrow \underline{\mu}_p - \underline{\mu}_q. \quad (\text{III.1.21})$$

Thus,

$$1 + \frac{1}{2} \cdot 2^{n-2} = 1 + 2^{n-3} \quad (\text{III.1.22})$$

roots of $D(N)$ project into a single root of $D(n)$. The generators, $\{p-q\}$, of $D(n)$ will be a linear superposition of the $1 + 2^{n-3}$ generators of $D(N)$, corresponding

to the $D(N)$ roots listed in (III.1.21). A given component of $\underline{H}$ of $D(n)$ will also be a linear superposition of $1 + 2^{n-1}$ components of the $\underline{H}$ of $D(N)$.

Indeed, we explicitly have

$$\underline{H} = \sum \underline{\lambda}_p \mathcal{H}(p) + \sum \underline{\lambda}(p+q+r\dots) \mathcal{H}(p+q+r\dots), \quad (\text{III.1.23})$$

where there are a total of $2n + 2^n$ terms in the two sums.

If we define $\underline{H}$ as

$$\underline{H} = \sum_{p=-n}^n \underline{\mu}_p H(p) = \sum_{p=1}^n \underline{\mu}_p [H(p) - H(-p)], \quad (\text{III.1.24})$$

we obtain¹⁹ from (III.1.20),

$$H(p) = \mathcal{H}(p) + \frac{1}{2} \sum'_{(p)} \mathcal{H}(p+q+r\dots). \quad (\text{III.1.25})$$

$\underline{H}$ has n independent components, as it must.

So far, we have followed the derivation of ref. 19 step by step. It is at this point that the two approaches diverge, and we gain a considerable advantage by using sho operators. In order to determine the coefficients of the $D(N)$ generators appearing in $\{p-q\}$, we must make use of the equations (II.1)-(II.3) and (III.1.1) which determine both the algebras of $D(N)$ and of $D(n)$. But, in order to make use of (III.1.1) for $D(N)$, i.e. to evaluate $d(p,q,r)$ -s which appear in it, we have to associate a single numerical index, a single "address", with the n -index symbol which is the argument of $\underline{\lambda}(p+q+r\dots)$. This is a daunting task for arbitrary n . On the other hand, if we think in terms of sho realizations of the generators, we have no such problems, since the algebra of sho-s, eqns. (II.27), replaces the algebra of $D(N)$ [in particular (III.1.1)] in our considerations. There is no phase in (II.27) and thus the "address" problem is circumvented. We still have the phases of the superposition coefficients to obtain, but these are determined by the algebra of $D(n)$, for which there is no "address" problem to begin with. We therefore take, as the final version of the $D(n)$ algebra realization, the generators (III.1.24), with

$$H(p) = \sum'_{(p)} N^{(n)}(p) + \frac{1}{2} \sum'_{(p)} N(p+q+r\dots), \quad (\text{III.1.26})$$

and^{23,30}

$$\{p-q\} = \epsilon(p+q) \sum_{(p,q)}'' \{ N^{(\kappa)}(p,q) - N^{(\kappa)\dagger}(-p,q) \} \\ + \sum_{(p,q)}'' \psi(p,q;r,s,\dots) N(p-q;r+s,\dots), \quad (\text{III.1.27})$$

where, as a final reminder of the notation of (II.30),

$$N(p-q;r+s,\dots) \equiv a^\dagger(p-q+r+s,\dots) a(-p+q+r+s,\dots). \quad (\text{III.1.28})$$

The new symbol in (III.1.27), $\psi(p,q;r,s,\dots)$, is a newly introduced phase factor.

To save time, we have postulated a form for $\{p-q\}$ which already takes into account some features of the $D(n)$ algebra: the fact that one phase is $\epsilon(p+q)$ and that $|\psi| = 1$. We impose the subsidiary conditions (II.9) and (II.20) on $\{p-q\}$, as given in (III.1.27). We then demand that H , eqn. (III.1.26), and $\{p-q\}$, eqn. (III.1.27), satisfy the $D(n)$ algebra, eqns. (II.1)-(II.3) and (III.1.1).

These requirements result in the following conditions on the phases, ψ :

$$\psi(p,q;r,s,\dots) \text{ is symmetric in } (p,q) \text{ and in } (r,s,\dots), \quad (\text{III.1.29})$$

$$\psi(p,q;r,s,\dots) = \psi(-p,-q;r,s,\dots) = -\psi(p,q;-r,-s,\dots), \quad (\text{III.1.30})$$

and

$$d(p,q,r) = -\psi(p,q;r,s,\dots) \psi(q,r;p,s,\dots) \psi(r,p;q,s,\dots), \quad (\text{III.1.31})$$

as well as other equations which are derivable from (III.1.29)-(III.1.31) and therefore will not be listed. The equations (III.1.29)-(III.1.31) are algebraically identical to the conditions on the structure constant phases derived for the algebras of the groups $E(8)$, $E(7)$, $E(6)$ and $F(4)$ in ref.³¹ 18. We do possess explicit solutions¹⁸ of eqns. (III.1.29)-(III.1.31) for the phases $\psi_{(e)}$ for $n = 4, 6, 8$ and $\psi_{(o)}$ for $n = 4, 5$ (as well as the phase χ for $B(4)$, which is defined below). Since $F(4) \supset B(4) \supset D(4)$, $E(6) \supset D(5) \oplus U(1)$, $E(7) \supset D(6) \oplus A(1)$ and $E(8) \supset D(8)$, these phases appear in the algebras of the larger groups. It is not difficult to generalize these solutions to arbitrary n , ($n \geq 4$).

The results are,

n odd:

$$\psi(p, q; r, s, t \dots)_{(e)} = - \prod_{(p, q)}'' \varepsilon(|p| - |l_-|) \varepsilon(|q| - |l_-|), \quad (\text{III.1.32})$$

$$\psi(p, q; r, s, t \dots)_{(o)} = + \prod_{(p, q)}'' \varepsilon(|p| - |l_+|) \varepsilon(|q| - |l_+|), \quad (\text{III.1.33})$$

$p, q \neq l_+, l_-.$

The subscripts (e) [(o)] indicate, as was stated in Section II, an even [odd] number of minus signs in all of the indices of ψ . The symbols l_+ (l_-) are the values of the positive (negative) indices.

n even:

$$\psi(p, q; r, s, t, \dots, -n) = - \prod_{(p, q, -n)}''' \varepsilon(|p| + l) \varepsilon(|q| + l), \quad (\text{III.1.34})$$

$$\psi(p, q; r, s, t, \dots, +n) = + \prod_{(p, q, +n)}''' \varepsilon(|p| - l) \varepsilon(|q| - l), \quad (\text{III.1.35})$$

$$p, q = \pm 1, \pm 2, \dots, \pm (n-1); \quad p, q \neq l.$$

The list of phases continues with

$$\psi(p, -n; r, s, t \dots) = \prod_{|l| < |m|}''_{(p, -n)} \varepsilon(l + m), \quad p, r, s > 0, \quad (\text{III.1.36})$$

$$\psi(p, n; r, s, t \dots) = (-1)^{p+1} \psi(p, -n; r, s, t \dots), \quad (\text{III.1.37})$$

where r, s, t are numerically ordered:

$$r < s < |t| < \dots,$$

and

$$p, r, s, |t|, \dots = 1, 2, \dots, n-1. \quad (\text{III.1.38})$$

The remaining phases are fixed by the condition

$$\psi(p, -n; r, -s, -t, -u, \dots) \equiv -\psi(p, -n; r, s, t, u, \dots), \quad (III.1.39)$$

$$p, r, s > 0,$$

and one of eqns. (III.1.30),

$$\psi(p, -n; -r, -s, -t, \dots) = -\psi(p, -n; r, s, t, \dots), \quad p, r > 0. \quad (III.1.30)$$

We may note that for

$$n = 4\ell, \quad \ell = 1, 2, 3, \dots, \quad (III.1.40)$$

the numerical ordering, (III.1.38), and the requirement $r > 0, s > 0$ and

eqn. (III.1.39) are not necessary in the phase definitions (III.1.36) and (III.1.37):

eqns. (III.1.30), (III.1.36)-(III.1.39) will in any case be automatically satisfied.

However, for n values

$$n = 4\ell + 2, \quad \ell = 1, 2, 3, \dots, \quad (III.1.41)$$

this is not the case. The conditions as stated in eqns. (III.1.30), (III.1.36)-(III.1.39) cover both of these situations.

We do not wish to stress the tedious details of the expressions (III.1.32)-(III.1.39). Suffice it to say that we have demonstrated the existence of solutions, with real phase factors, to eqns. (III.1.29)-(III.1.31), for arbitrary n . Equation (III.1.31) requires the discovery of a factorized form, other than (III.1.3), for the same phase factor $d(p, q, r)$ as appears in eqn. (III.1.3).

b. Realizations of the irreps

We note the commutation relations

$$[H, a^{(\kappa)\dagger}(p)] = \mu_p a^{(\kappa)\dagger}(p), \quad (III.1.42)$$

$$[H, a^\dagger(p+q+r+\dots)] = \frac{1}{2}(\mu_p + \mu_q + \mu_r + \dots) a^\dagger(p+q+r+\dots), \quad (III.1.43)$$

$$[H, a^{(\kappa)\dagger}(p) a^{(\lambda)\dagger}(q)] = (\mu_p + \mu_q) a^{(\kappa)\dagger}(p) a^{(\lambda)\dagger}(q), \quad (III.1.44)$$

$$[\{p-q\}, a^{(\kappa)\dagger}(q)] = a^{(\kappa)\dagger}(p), \quad (III.1.45)$$

They follow from the commutation relations (II.27), together with the definition of the components of $\underline{H}$, eqn. (III.1.26), and of $\{p-q\}$, eqn. (III.1.27). They, and similar equations, are required in checking that the states we are about to define satisfy equations (II.22) and (II.23).

We now have the irreps:

Scalar irrep

$$|0, [0]\rangle \equiv |0\rangle, \quad (II.24)$$

defined by

$$a(\underline{L})|0\rangle = 0, \quad \text{all } \underline{L}. \quad (III.1.46)$$

The three elementary irreps

They are given by

$$f: |\underline{\mu}_p, [1]\rangle = a^{(\kappa)\dagger}(p) |0\rangle, \quad (III.1.47)$$

$$\sigma_{(o)}: |(\frac{1}{2} \sum_{(o)} (\pm) \underline{\mu}_p), [1_{n-1}]\rangle = a_{(o)}^\dagger(p+q+r+\dots) |0\rangle, \quad (III.1.48)$$

$$\sigma_{(e)}: |(\frac{1}{2} \sum_{(e)} (\pm) \underline{\mu}_p), [1_n]\rangle = a_{(e)}^\dagger(p+q+r+\dots) |0\rangle. \quad (III.1.49)$$

The irrep f is defined redundantly, for any fixed value of κ . The $n-2$ -fold redundancy is trivial: a $2n$ dimensional vector $\underline{x}^{(1)}$ is as good a vector as a $2n$ dimensional vector $\underline{x}^{(2)}$.

The remaining basic irreps

The adjoint, a , is given by

$$a: |(\underline{\mu}_p - \underline{\mu}_q), [1_2]\rangle = \frac{1}{\sqrt{2}} \text{Det}_{(\kappa_1, \kappa_2)} a^{(\kappa_1)\dagger}(p) a^{(\kappa_2)\dagger}(-q) |0\rangle, \quad (III.1.50)$$

where

$$\text{Det}_{(\kappa_1, \kappa_2)} a^{(\kappa_1)\dagger}(p) a^{(\kappa_2)\dagger}(-q) = a^{(\kappa_1)\dagger}(p) a^{(\kappa_2)\dagger}(-q) - (\kappa_1 \leftrightarrow \kappa_2). \quad (III.1.51)$$

As in the case of f , there is a redundancy in the definition of the adjoint. There are $\frac{1}{2}(n-2)(n-3)$ different realizations of it, corresponding to the choice of different (κ_1, κ_2) pairs.

The irrep $[1_3]$ will correspond to a $\text{Det}_{(\kappa_1, \kappa_2, \kappa_3)}$ over three a^+ -s, and so on. The redundancy will disappear for $[1_{n-2}]$. The number of κ indices were chosen so that this last basic irrep could be constructed.

All other irreps

All the other irreps can be constructed from Kronecker products of the basic irreps^{5,14}. We follow a slightly different route. We construct outer products of different a^+ combinations, rather than states, obtaining irreducible subsets of them in a manner completely analogous to the reduction of Kronecker products for states. We then allow the resulting homogeneous polynomial in a^+ -s to operate on the vacuum state.

This procedure will give rise to further redundant irreps. For example, the state

$$\frac{1}{n!} \sum_{(\kappa)} a^{(\kappa)+}(p) a^{(\kappa)+}(-p) |0\rangle \quad (\text{III.1.52})$$

is a possible realization of the scalar irrep. This fact can easily be checked by applying $\underline{H}$ and all the step-up and step-down operators $\{p-q\}$ to this state.

This type of redundancy is not as trivial as that previously discussed, which was related to the (κ) superscripts. Indeed, if it were not eliminated, it would lead to difficulties with our definition of Casimir operators in Section III.5. We therefore append the rule (which is not relevant at the moment, but will be so in Section III.5) that the irreps must be realized in terms of the minimal number of a^+ factors operating on the vacuum and the minimal number of (κ) indices. If more complicated forms arise, say in the reduction of a Kronecker product, they must be replaced by the simpler forms. Thus, for the state considered in eqn. (III.1.52), we must make the replacement

$$\frac{1}{\sqrt{n}} \sum_{(\kappa)} a^{(\kappa)\dagger}(p) a^{(\kappa)\dagger}(-p) |0\rangle \rightarrow |0\rangle. \quad (\text{III.1.53})$$

We will call this process "sho reduction".

A typical non-basic irrep which is an intrinsically two particle state and cannot be sho reduced further is

$$\begin{aligned} |(\underline{\mu}_p - \underline{\mu}_q), [2, 1]\rangle &= [1 + \delta_{p,q} - \frac{1}{n} \delta_{p,q}]^{-1/2} \\ &\times \left[a^{(\kappa)\dagger}(p) a^{(\kappa)\dagger}(-q) - \frac{\delta_{p,q}}{n} \sum_{(\kappa)} a^{(\kappa)\dagger}(\ell) a^{(\kappa)\dagger}(-\ell) \right] |0\rangle. \end{aligned} \quad (\text{III.1.54})$$

The leading factor in eqn. (III.1.54) is a normalization factor which differs from unity for elements of weight $2\underline{\mu}_p$ and $\underline{0}$. The latter elements are defined symmetrically, but redundantly, so that

$$\sum_p | \underline{0}_p, [2, 1] \rangle = 0. \quad (\text{III.1.55})$$

Thus, there are $n-1$ independent elements with weights $\underline{0}$. Since there are, in addition, $\binom{n}{2} \cdot 2^2 = 2n(n-1)$ elements with weights $\underline{\mu}_p - \underline{\mu}_q$, $|p| \neq |q|$, and $2n$ elements with weights $2\underline{\mu}_p$, there are a total of

$$\dim. [2, 1]_{D(n)} = n(2n+1) - 1 \quad (\text{III.1.56})$$

elements in this irrep, the desired result.

We also note that there is only one superscript κ in (III.1.54), rather than two, κ_1 and κ_2 . We could have used two superscripts, but would then have had to specify a reduction of the products, and kept only the symmetric (κ_1, κ_2) combinations. We will always take equal κ limits for such symmetric combinations, using our previously stated sho reduction rules. This will enable us to write Casimir operators in terms of a^+ -s and a -s in Section III.5, as mentioned earlier.

We have thus exhibited, or indicated a method of constructing, each of the irreps at least once.

A final remark on the irreps: Reduction of Kronecker products of spinor irreps can give rise to vector irreps. For example, f can arise from such a Kronecker product. One has in general, for $D(n)$,

$$[1_{n-1}] \otimes [1_{n-1}] = [2_{n-1}] \oplus \sum_{m=1}^{\lfloor n/2 \rfloor} [1_{n-2m}], \quad (\text{III.1.57})$$

$$[1_n] \otimes [1_n] = [2_n] \oplus \sum_{m=1}^{\lfloor n/2 \rfloor} [1_{n-2m}], \quad (\text{III.1.58})$$

and

$$[1_n] \otimes [1_{n-1}] = [1_n, 1_{n-1}] \oplus \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} [1_{n-2m-1}], \quad (\text{III.1.59})$$

$$n \geq 4, \quad [1_0] \equiv [0],$$

with

$$\left\lfloor \frac{n}{2} \right\rfloor \equiv \begin{cases} \frac{1}{2}(n-1), & \text{for } n \text{ odd,} \\ \frac{1}{2}n, & \text{for } n \text{ even.} \end{cases} \quad (\text{III.1.60})$$

Thus, f is generated by (III.1.59) if n is even, and by (III.1.57) or (III.1.58) if n is odd. One could therefore have built a scheme, which, instead of (III.1.15), would have been based on the imbedding

$$f_{D(N')} \longrightarrow (\sigma_{(a)} \oplus \sigma_{(e)})_{D(n)}, \quad (\text{III.1.15}')$$

or

$$D(N') \supset D(n), \quad N' = 2^{n-1}. \quad (\text{III.1.16}')$$

This alternative scheme gives much more complicated irreps than the one we have adopted. For example, an element of the elementary irrep f in this new approach would be a homogeneous quadratic form in the a^+ -s, rather than a single a^+ , as in our present treatment.

2. The Groups B(n)

a. Realizations of the algebra

Since the elementary irrep f has $n-2$ antisymmetrized basic irreps associated with it, rather than $n-3$, as in the case of $D(n)$, (III.1.11) is altered to

$$\kappa = 1, 2, \dots, n-1. \quad (\text{III.2.1})$$

In addition to the $D(n)$ generators, there is a new set of generators, $\{p\}$. Additional sets of commutators of type (II.4) exist³². $\underline{H}$ and $H(p)$ have the same forms as in $D(n)$ [see eqns. (III.1.24) and (III.1.25)], as do the generators $\{p-q\}$ (see eqn. (III.1.27)).

The new generators are given by

$$\{p\} = \sum_{(p)} \{ N^{(\kappa)}(p,0) + N^{(\kappa)\dagger}(-p,0) + \chi(p;q,r,\dots) N(p;q+r,\dots) \}, \quad (\text{III.2.2})$$

where $\chi(p;q,r,\dots)$ are new phase factors, and we have also introduced another set of $n-1$ Bose oscillators $[a^{(\kappa)\dagger}(0), a^{(\kappa)}(0)]$, associated with zero weight vectors. The commutators of the algebra and the subsidiary conditions (II.9) and (II.19) yield the new phase conditions

$$\chi(p;q,r,\dots) \text{ is symmetric in } (q,r,\dots), \quad (\text{III.2.3})$$

$$\chi(p;q,r,\dots) = \chi(-p;q,r,\dots) = -\chi(p;-q,-r,\dots), \quad (\text{III.2.4})$$

$$\psi(p,q;r,s,\dots) = -\varepsilon(p+q) \chi(p;q,r,s,\dots) \chi(q;p,r,s,\dots). \quad (\text{III.2.5})$$

A set of solutions for the new phases is

n odd:

$$\chi(p;q,r,\dots)_{(e)} = -\prod_{(p)} \varepsilon(|p|-1), \quad (\text{III.2.6})$$

$$\chi(p; q, r, \dots)_{(o)} = \prod_{(p)} \varepsilon(|p| - |l_+|), \quad (III.2.7)$$

$$p \neq l_+, l_-.$$

For the definitions of l_+ and l_- , see the text following eqn. (III.1.33).
Of course in the present case, because of (III.2.4), (e) and (o) refer to the indices other than p.

n even:

$$\chi(p; q, r, \dots, -n) = -\prod_{(p, -n)}'' \varepsilon(|p| + l), \quad (III.2.8)$$

$$\chi(p; q, r, \dots, +n) = \prod_{(p, n)}'' \varepsilon(|p| - l), \quad (III.2.9)$$

$$p = \pm 1, \pm 2, \dots, \pm(n-1); \quad p \neq l.$$

The remaining phases are

$$\chi(\pm n; p, q, r, s, \dots) = -\prod_{|l| < |m|}'' (\pm n) \varepsilon(l + m), \quad p, q, r > 0, \quad (III.2.10)$$

$$p, q, r, |s|, \dots = 1, 2, \dots, n-1; \quad p < q < r < |s| < \dots,$$

with

$$\chi(\pm n; p, q, -r, -s, \dots) \equiv -\chi(\pm n; p, q, r, s, \dots), \quad p, q, r > 0,$$

$$\chi(\pm n; p, -q, -r, \dots) \equiv -\chi(\pm n; p, q, r, \dots), \quad p, q > 0,$$

$$\chi(\pm n; -p, -q, -r, \dots) \equiv -\chi(\pm n; p, q, r, \dots), \quad p > 0.$$

(III.2.11)

Note that, in analogy with (III.1.36) and (III.1.37), p, q, r, ... are taken in numerical order in eqns. (III.2.10) and (III.2.11). As in the case of D(n), these

requirements, eqns. (III.2.11) and the positivity conditions on the indices p, q, r are only needed for the cases of $n = 6, 10, 14, \dots$. For the cases $n = 4, 8, 12, \dots$ (III.2.11) is automatically satisfied, even if these conditions are not imposed.

The χ -s essentially are obtained from the ψ -s by factorizing the latter [see eqn. (III.2.5)].

b. Realizations of the irreps

The changes from the $D(n)$ case are trivial. We note

$$[\underline{H}, a^{(\kappa)\dagger}(0)] = 0. \quad (\text{III.2.12})$$

Scalar irrep

Equations (II.24) and (III.1.46) hold. Note that the set $\mathcal{L}$ includes 0, and

$$\kappa = 1, 2, \dots, n-1, \quad p = \pm 1, \pm 2, \dots, \pm n. \quad (\text{III.2.13})$$

The two elementary irreps

f: (III.1.47), and an additional element

$$|0, [1, 1]\rangle = a^{(\kappa)\dagger}(0)|0\rangle, \quad (\text{III.2.14})$$

$$\sigma: \quad \sigma = \sigma_{(e)} \oplus \sigma_{(o)}, \quad (\text{III.2.15})$$

where $\sigma_{(e)}$ and $\sigma_{(o)}$ are given in (III.1.48) and (III.1.49). These results were to be expected, since $B(n) \supset D(n)$.

The other irreps

They are identical to those defined in connection with $D(n)$, provided the set of labels is extended, as indicated in (III.2.13).

3. The Groups $C(n)$

a. Realizations of the algebra

This is the simplest of all classical groups to treat. There are no spinors, and all irreps can be constructed from f, with the other basic irreps being a total of $n-1$ associated antisymmetrized irreps (with suitable vanishing trace conditions). We thus require

$$\kappa = 1, 2, \dots, n. \quad (\text{III.3.1})$$

The algebra in this case leads to phase solutions³³ which are entirely different from those of $D(n)$ and $B(n)$. The generators $\underline{H}$ are given by eqn. (III.1.9) [noting (III.3.1)]. The structure of the $\{p-q\}$ generators is the same as in eqn. (III.1.10). They, and the generators $\{2p\}$ are given in detail by

$$\begin{aligned} \{p-q\} = & \frac{1}{2} (1 + \varepsilon(p) - \varepsilon(q) + \varepsilon(p)\varepsilon(q)) \\ & \times \sum''_{(p,q)} \{ \varepsilon(p) N^{(\kappa)}(p,q) - \varepsilon(q) N^{(\kappa)\dagger}(-p,-q) \}, \end{aligned} \quad (\text{III.3.2})$$

$$\begin{aligned} \{2p\} = & \sqrt{2} \sum'_{(p)} N^{(\kappa)}(p,-p), \\ & p = \pm 1, \pm 2, \dots, \pm n. \end{aligned} \quad (\text{III.3.3})$$

b. Realizations of the irreps

Scalar and elementary irreps

The scalar and f irreps are those given for $D(n)$ [but note (III.3.1)].

The remaining basic irreps

The only minor difficulty for $C(n)$ arises here, since the antisymmetric parts of Kronecker products are reducible. We illustrate for two irreps, which make the general pattern clear. The irrep constructed from the antisymmetric product of two a^+ -s is

$$\left\{ \begin{aligned} |(\underline{\mu}_p + \underline{\mu}_q), [l_2]\rangle &= \frac{1}{\sqrt{2}} \text{Det}_{(\kappa_1, \kappa_2)} a^{(\kappa_1)\dagger}(p) a^{(\kappa_2)\dagger}(q) |0\rangle, \\ & |p| \neq |q|, \end{aligned} \right. \quad (\text{III.3.4.a})$$

$$\left\{ \begin{aligned} |0_p, [l_2]\rangle &= \frac{1}{\sqrt{2}} \left(\frac{n}{n-1} \right)^{1/2} \text{Det}_{(\kappa_1, \kappa_2)} [a^{(\kappa_1)\dagger}(p) a^{(\kappa_2)\dagger}(p) - \frac{1}{n} \sum''_{(\kappa_1, \kappa_2)} a^{(\kappa_1)\dagger}(e) a^{(\kappa_2)\dagger}(-e)] |0\rangle. \end{aligned} \right. \quad (\text{III.3.4.b})$$

The zero weight elements are defined redundantly, with the single condition

$$\sum_p |0_p, [l_2]\rangle = 0, \quad (\text{III.3.4.c})$$

so that, since $p = q$ is excluded, we get the expected number of elements for the irrep:

$$\dim. [1_2]_{C(n)} = \binom{n}{2} \cdot 2^2 + n - 1 = 2n^2 - n - 1. \quad (\text{III.3.5})$$

The irrep constructed from the antisymmetric product of three a^+ -s is

$$\left\{ \begin{aligned} |(\underline{\mu}_p + \underline{\mu}_q + \underline{\mu}_r), [1_3]\rangle &= \frac{1}{\sqrt{3!}} \text{Det}_{(\kappa_1, \kappa_2, \kappa_3)} a^{(\kappa_1)\dagger}(p) a^{(\kappa_2)\dagger}(q) a^{(\kappa_3)\dagger}(r) |0\rangle, \\ |(\underline{\mu}_p + \underline{0}_q), [1_3]\rangle &= \left(\frac{n-1}{n-2}\right)^{1/2} \frac{1}{\sqrt{3!}} \text{Det}_{(\kappa_1, \kappa_2, \kappa_3)} a^{(\kappa_1)\dagger}(p) \\ &\quad \times [a^{(\kappa_2)\dagger}(q) a^{(\kappa_3)\dagger}(-q) - \frac{1}{n-1} \sum_{(\kappa_2, \kappa_3, p)} a^{(\kappa_2)\dagger}(l) a^{(\kappa_3)\dagger}(-l)] |0\rangle, \end{aligned} \right. \quad (\text{III.3.6.a})$$

$$|p| \neq |q|. \quad (\text{III.3.6.b})$$

The $\underline{\mu}_p$ weight elements are defined redundantly, with one condition for each p ,

$$\sum_q |(\underline{\mu}_p + \underline{0}_q), [1_3]\rangle = 0, \quad |p| \neq |q|, \quad (\text{III.3.6.c})$$

or $2n$ conditions in all. We again get the expected number of elements for this irrep:

$$\dim. [1_3]_{C(n)} = \binom{n}{3} \cdot 2^3 + 2n(n-2) = \frac{2}{3} n(n-2)(2n+1). \quad (\text{III.3.7})$$

The method for constructing all other basic irreps is now apparent.

The adjoint irrep

We illustrate only this single non-basic irrep. It is

$$|(\underline{\mu}_p - \underline{\mu}_q), [2,]\rangle = (1 + \delta_{p,-q})^{-1/2} a^{(\kappa)\dagger}(p) a^{(\kappa)\dagger}(-q) |0\rangle, \quad (\text{III.3.8})$$

with $\binom{n}{2} \cdot 2^2$ elements $|p| \neq |q|$, $2n$ elements $p = -q$ and n zero weight elements, or, as expected

$$\dim. [2,]_{C(n)} = n(2n+1). \quad (\text{III.3.9})$$

4. The Groups A(n)

The added complication here is that the basis space of A(n) is defined in terms of $n + 1$ orthonormal basis vectors $\underline{\mu}_p$. There are again no spinor irreps and all irreps can be constructed from a single f , with weights $\underline{\chi}_p$, where $\underline{\chi}_p$ is defined in eqn. (II.14). However, we prefer to discuss this case using two elementary irreps, f with weights $\underline{\chi}_p$, and $\bar{f}$ with weights $-\underline{\chi}_p$. The treatment of the adjoint irrep as well as of G(2) generators is more natural in such an approach, than in one based on the construction of the irreps from a single f .

a. Realizations of the algebra

The generators in $\underline{H}$, where

$$\underline{H} = \sum \underline{\chi}_p H(p), \quad (\text{III.4.1})$$

are

$$H(p) = \sum_{(p)} N^{(\kappa)}(p) - \frac{1}{n+1} \sum N^{(\kappa)}(q) N^{(\kappa)}(-q), \quad (\text{III.4.2})$$

so that $\underline{H}$ has the expected n , rather than $n + 1$, independent components.

The step up or step down generators are

$$\{p, q\} = \sum_{(p, q)} [N^{(\kappa)}(p, q) - N^{(\kappa)\dagger}(-p, -q)], \quad (\text{III.4.3})$$

with

$$\left. \begin{aligned} p, q &= 1, 2, \dots, n+1, \text{ or } -1, -2, \dots, -(n+1), \\ \kappa &= 1, 2, \dots, \left[\frac{n-1}{2}\right] + 1, \end{aligned} \right\} \quad (\text{III.4.4})$$

where the symbol $[(n-1)/2]$ is defined in eqn. (III.1.62).

b. Realizations of the irreps

We note that (III.1.42)-(III.1.44) are modified by the replacement

$$\underline{\mu}_p \rightarrow \underline{\chi}_p: \quad [\underline{H}, a^{(\kappa)\dagger}(p)] = \underline{\chi}_p a^{(\kappa)\dagger}(p), \text{ etc..} \quad (\text{III.4.5})$$

Scalar irrep

It is defined as for D(n), but with index ranges as given in (III.4.4).

The two elementary irreps

We have

$$F: \quad |\underline{\chi}_p, [1, 1]\rangle = a^{(\kappa)\dagger}(p) |0\rangle, \quad (\text{III.4.6})$$

$$\bar{F}: \quad |-\underline{\chi}_p, [1, n]\rangle = a^{(\kappa)\dagger}(-p) |0\rangle, \quad (\text{III.4.7})$$

$$p = 1, 2, \dots, n+1.$$

The other basic irreps

They are obtained by taking the antisymmetric parts of the Kronecker products of $a^{(\kappa)\dagger}(p)$ -s or $a^{(\kappa)\dagger}(-p)$ -s. For n odd, $\kappa = \kappa_{\max} = [(n-1)/2] + 1$, and both $a^{(\kappa)\dagger}(p)$ and $a^{(\kappa)\dagger}(-p)$ products lead to the same irrep.

The adjoint irrep

All other irreps can now be constructed. However, because of the special role it plays, we once more specifically give the adjoint irrep. It is

$$\left\{ \begin{aligned} & |(\underline{\chi}_p - \underline{\chi}_q), [1, 1, n]\rangle \equiv |(\underline{\mu}_p - \underline{\mu}_q), [1, 1, n]\rangle = a^{(\kappa)\dagger}(p) a^{(\kappa)\dagger}(-q) |0\rangle, \\ & \qquad \qquad \qquad p \neq q, \qquad \qquad \qquad (\text{III.4.8.a}) \\ & |0_p, [1, 1, n]\rangle = \left(\frac{n+1}{n}\right)^{1/2} \left[a^{(\kappa)\dagger}(p) a^{(\kappa)\dagger}(-p) - \frac{1}{n+1} \sum_{(\kappa)} a^{(\kappa)\dagger}(l) a^{(\kappa)\dagger}(-l) \right] |0\rangle, \end{aligned} \right.$$

(III.4.8.b)

$$p, q, l = 1, 2, \dots, n+1,$$

with the redundancy

$$\sum_{p=1}^{n+1} |O_p, [l_1, l_n]\rangle = 0. \quad (\text{III.4.8.c})$$

5. Casimir Operators in Terms of sho Number Operators

Considerable simplification results if the sho number operators, rather than the generators of the algebras, are used to construct Casimir operators. Moreover, the eigenvalues of the Casimir invariants so constructed have a simple and direct relationship to the Dynkin-Patera labeling scheme for irreps. A hint of this simplification is given by the example³⁴ of $A(1) \equiv SU(2)$. The only Casimir operator in this case is C_2 , which is second order in the generators. However, it can also be expressed as a quadratic operator function of the total number operator. In terms of sho operators (a^+ -s and a -s), C_2 is biquadratic in a^+ and a (i.e. of form a^+aa^+a) while the number operator is bilinear in a^+ and a (i.e. of form a^+a). While C_2 has the eigenvalues

$$j(j+1), \quad j = 0, \frac{1}{2}, 1, \dots \quad (\text{III.5.1})$$

the number operator has the eigenvalues

$$n, \quad n = 0, 1, 2, \dots \quad (\text{III.5.2})$$

and the two sets of eigenvalues are related by

$$n = 2j. \quad (\text{III.5.3})$$

Moreover, n is precisely the Dynkin-Patera index for $A(1)$. Clearly, the number operator can play the role of a Casimir invariant in this case, and is much simpler than C_2 .

We again discuss each type of classical group separately and provide the most details for $D(n)$.

a. $D(n)$

Define, first, the total number operator, $N_{(1)}$,

$$N_{(1)} \equiv \sum N^{(\kappa)}(p), \quad p = \pm 1, \pm 2, \dots, \pm n, \quad (\text{III.5.4})$$

$$\kappa = 1, 2, \dots, \kappa_{\max}, \quad \kappa_{\max} = n-2.$$

For all the irreps we have constructed and can construct, $N_{(1)}$ will have zero or positive integer eigenvalues, which we shall label $n_{(1)}$.

Next, we construct the "unmatched pair" operator $N_{(2)}$, where

$$N_{(2)} \equiv \sum \{ N^{(\kappa_1)}(p) N^{(\kappa_2)}(q) + (\kappa_1 \longleftrightarrow \kappa_2) \}$$

$$\equiv \sum \text{Sym}_{(\kappa_1, \kappa_2)} N^{(\kappa_1)}(p) N^{(\kappa_2)}(q), \quad (\text{III.5.5})$$

$$\kappa_1 \neq \kappa_2, \quad |p| \neq |q|,$$

and eqn. (III.5.5) serves to define the symbol $\text{Sym}_{(\kappa_1, \kappa_2)}$: $\text{Sym}_{(\kappa_1, \kappa_2)}$ is obtained from $\text{Det}_{(\kappa_1, \kappa_2)}$, defined in (III.1.51) by setting all negative signs arising from the determinant positive. $N_{(2)}$ will have eigenvalues $[n_{(2)}]^2$ for all irreps, where $n_{(2)}$ will be zero or a positive integer³⁵.

In a similar way, we can define $\text{Sym}_{(\kappa_1, \kappa_2, \kappa_3)}$ from $\text{Det}_{(\kappa_1, \kappa_2, \kappa_3)}$, with eigenvalues $[n_{(3)}]^3$, and so on. In other words, we can construct

$$N_{(\omega)}, \quad \omega = 1, 2, \dots, \kappa_{\max}, \quad (\text{III.5.6})$$

$$\kappa_{\max} = n-2.$$

Finally, we define the total spinor number operators,

$$N_{(\pi)} \equiv \sum_{(\pi)} N(p+q+r\dots), \quad \pi = o, e, \quad (\text{III.5.7})$$

where $o(e)$ stands for summations over index sets $(p, q, r \dots)$ involving an odd (even) number of negative indices. The operators $\mathbb{N}_{(\pi)}$ have eigenvalues $n_{(\pi)}$.

The operators $\mathbb{N}_{(\omega)}$ and $\mathbb{N}_{(\pi)}$ defined in eqns. (III.5.6) and (III.5.7) can play the role of the n Casimir invariants of the group $D(n)$. Their eigenvalues, $n_{(\omega)}$ and $n_{(\pi)}$, can be used to label a given irrep. It can be shown that they are simply related to the Dynkin-Patera indices $\ell^{(\omega)}$, where

$$(\ell^{(1)}, \dots, \ell^{(t)}, 0, 0, \ell^{(t+3)}, \dots) \equiv [\ell_1^{(1)}, \dots, \ell_t^{(t)}, \ell_{t+3}^{(t+3)}, \dots]. \quad (\text{III.5.8})$$

The l.h.s of (III.5.8) is the usual Dynkin-Patera notation, and its r.h.s. is the notation of the present paper. The usual Dynkin-Patera indices $\ell^{(\omega)}$, $\ell^{(\pi)}$ are related to the eigenvalues $n_{(\beta)}$, $n_{(\pi)}$ by

$$\ell^{(\omega)} = \sum_{\beta=\omega}^{\kappa_{\max}} (-1)^{\omega-\beta} \binom{\beta}{\beta-\omega} n_{(\beta)}, \quad (\text{III.5.9})$$

$$\omega = 1, 2, \dots, \kappa_{\max}; \quad \kappa_{\max} = n-2,$$

where $\binom{\beta}{\beta-\omega}$ are the binomial coefficients, and

$$\ell^{(\pi)} = n_{(\pi)}, \quad \pi = o, e. \quad (\text{III.5.10})$$

The inverse of (III.5.9) is

$$n_{(\beta)} = \sum_{\omega=\beta}^{\kappa_{\max}} \binom{\omega}{\omega-\beta} \ell^{(\omega)}. \quad (\text{III.5.11})$$

b. $B(n)$

The operators $\mathbb{N}_{(\omega)}$ and $\mathbb{N}_{(\pi)}$ are defined the same way as for $D(n)$, except the argument, p , of $N^{(\kappa)}(p)$ in $\mathbb{N}_{(\omega)}$ (but not in $\mathbb{N}_{(\pi)}$) has

$$p = 0, \pm 1, \pm 2, \dots \pm n, \quad (\text{III.5.12})$$

and

$$\kappa_{\max} = n-1. \quad (\text{III.5.13})$$

There is only one total spinor number operator:

$$N_{\sigma} \equiv N_{(o)} + N_{(e)}. \quad (\text{III.5.14})$$

c. C(n)

This is the simplest of all cases. There are no spinor number operators.

The $N_{(\omega)}$ are defined as in D(n), but with

$$\kappa_{\max} = n. \quad (\text{III.5.15})$$

d. A(n)

There are no spinor number operators. The results are slightly complicated by the fact that irreps are constructed from both f and $\bar{f}$.

We thus have

$$N_{(1)} \equiv \sum N^{(\kappa)}(p), \quad (\text{III.5.16})$$

and

$$\bar{N}_{(1)} \equiv \sum N^{(\kappa)}(-p), \quad (\text{III.5.17})$$

$$p = 1, 2, \dots, n+1,$$

$$\kappa = 1, 2, \dots, \kappa_{\max}, \quad \kappa_{\max} = \left[\frac{n-1}{2} \right] + 1,$$

with $N_{(\omega)}$ and $\bar{N}_{(\omega)}$, $\omega = 2, 3, \dots, \kappa_{\max}$, similarly defined. The eigenvalues $n_{(\omega)}$ and $\bar{n}_{(\omega)}$ also constitute obvious generalizations from the D(n) case. The Dynkin-Patera labeling is

$$\left. \begin{aligned} & (l^{(1)}, l^{(2)}, \dots, l^{(\kappa_{\max})}, \bar{l}^{(\kappa_{\max})}, \dots, \bar{l}^{(2)}, \bar{l}^{(1)}), \text{ for } n \text{ even,} \\ & (l^{(1)}, l^{(2)}, \dots, l^{(\kappa_{\max})} = \bar{l}^{(\kappa_{\max})}, \dots, \bar{l}^{(2)}, \bar{l}^{(1)}), \text{ for } n \text{ odd.} \end{aligned} \right\} \quad (\text{III.5.18})$$

Equations (III.5.9) and their inverses (III.5.11) hold separately for $[l^{(\omega)}, n_{(\beta)}]$ and for $[\bar{l}^{(\omega)}, \bar{n}_{(\beta)}]$.

It is understood that all irreps [including the irreps with spinor content in $B(n)$ and $D(n)$, and those with f and $\bar{f}$ content in $A(n)$] are "sho reduced", as discussed at the end of Section III.1. That is, whether the irreps appear directly, or as a consequence of reductions of Kronecker products, each of them must be realized in terms of a minimal monomial or polynomial of a^+ -s operating on the vacuum, and a minimal number of κ superscripts must be used.

IV. EXCEPTIONAL GROUPS

Our emphasis in this section will be on the realization of the algebras. The treatment of the irreps will be much sketchier than for the classical groups. We will give explicit expressions at most for the elementary irreps, often not even for all of these. However, for every exceptional group, the irreps given will be sufficient to generate all other irreps through Kronecker products.

The strategy we follow is the same as that which led to the realization of the $D(n)$ and $B(n)$ algebras, yielding spinor irreps in terms of Bose oscillators: we embed each exceptional group of rank n , g_n , in an orthogonal group $\mathfrak{G}_N$ of higher rank, such that N is as small as possible and g_n is a non-regular subgroup of $\mathfrak{G}_N$. It must be emphasized in this connection that, in obtaining the phases which appear, those algebraic conditions in (II.4) for which the structure constants vanish can play an important role. In general they do not lead to additional independent constraints on the phases. However, in the case of $F(4)$ and $E(8)$, where the lowest dimensional elementary irreps contain more than one vanishing weight, such relations³⁶ play a crucial role and lead to complex phase factors.

We consider each of the exceptional groups in order of increasing rank, and emphasize results, rather than the details of the ways in which these results have been obtained. We do this both for the sake of brevity, and because the approach we follow has been adequately illustrated in ref. 19 and in our present Section III.1.

1. G(2)

a. Realization of the algebra

We note that

$$B(3) \supset G(2), \quad (IV.1.1)$$

and that the relation

$$\begin{array}{ccc} [1_1] & \longrightarrow & [1_2] \\ f & \longrightarrow & f_1 \\ 7 & & 7 \end{array} \quad (IV.1.2)$$

holds³⁷, where f_1 is one of the two elementary irreps of $G(2)$. The other is f_2 , where

$$f_2 \equiv a \equiv [1_1], \quad \dim. f_2 = 14. \quad (IV.1.3)$$

We project from the $\underline{\lambda}$ weight space of $B(3)$ to the $\underline{\mu}$ weight space of $G(2)$:

$$\left. \begin{array}{l} \underline{0} \longrightarrow \underline{0} \\ \underline{\lambda}_p \longrightarrow \underline{\chi}_p = \underline{\mu}_p - \frac{1}{3} \sum_{q=1}^3 \underline{\mu}_q \end{array} \right\}, \quad (IV.1.4)$$

$p = \pm 1, \pm 2, \pm 3.$

Because of (II.15), the actual weight space of $G(2)$ is two-, and not three-dimensional.

The projection³⁸ of the roots of $B(3)$ into roots of $G(2)$ is

$$\underline{\lambda}_p - \underline{\lambda}_q \longrightarrow \underline{\mu}_p - \underline{\mu}_q, \quad (IV.1.5)$$

$$\left. \begin{array}{l} \underline{\lambda}_p \\ -\underline{\lambda}_q - \underline{\lambda}_r \end{array} \right\} \longrightarrow \underline{\chi}_p, \quad 2 \text{ roots}, \quad (IV.1.6)$$

$p \neq q \neq r,$

and p, q, r all positive, or all negative.

The resulting generators, with the appropriate phase solutions, are

$$\underline{H} = \sum \underline{\chi}_p H(p), \quad (IV.1.7)$$

with

$$H(p) = \sum_{(p)}' N^{(\kappa)}(p) - \frac{1}{3} \sum N^{(\kappa)}(q), \quad (IV.1.8)$$

and

$$\{p-q\} = \varepsilon(p+q) \sum_{(\kappa, q)} [N^{(\kappa)}(p, q) - N^{(\kappa)\dagger}(p, q)], \quad (\text{IV.1.9})$$

$$\begin{aligned} \{\chi_p\} = \frac{1}{\sqrt{3}} \sum_{\kappa=1}^2 & \left[\sqrt{2} (N^{(\kappa)}(p, 0) + N^{(\kappa)\dagger}(-p, 0)) \right. \\ & \left. + (-1)^{p+1} \varepsilon(-q+r) (N^{(\kappa)}(-q, r) - N^{(\kappa)\dagger}(q, -r)) \right], \end{aligned} \quad (\text{IV.1.10})$$

with p, q, r in (IV.1.9) and (IV.1.10) all positive or all negative, $p \neq q \neq r$ and p, q, r in (IV.1.10) cyclic.

As indicated in some of the explicitly written sums above, κ has the values

$$\kappa = 1, 2. \quad (\text{IV.1.11})$$

b. Realization of the irreps

The scalar and $[1_2] \equiv f$, irrep have the, by now, familiar forms, in analogy with the classical groups. The adjoint irrep is generated from the Kronecker product³⁹ of two a^+ -s:

$$\begin{aligned} [1_2] \otimes [1_2] &= [0] \oplus [1_2] \oplus [1_1] \oplus [2_2] \\ f_1 \otimes f_1 &= s \oplus f_1 \oplus f_2 \oplus S(2f_1) \\ 49 &= 1 + 7 + 14 + 27. \end{aligned} \quad (\text{IV.1.12})$$

The explicit form of f_2 is

$$\left\{ \begin{aligned} |(\chi_p - \chi_q), [1_1]\rangle &= \frac{1}{\sqrt{2}} \text{Det}_{(1,2)} a^{(1)\dagger}(p) a^{(2)\dagger}(-q) |0\rangle, \\ & p, q = \begin{pmatrix} + \\ - \end{pmatrix}, (1, 2, 3), \\ | \chi_p, [1_1]\rangle &= \frac{1}{\sqrt{6}} \text{Det}_{(1,2)} \left[\sqrt{2} a^{(1)\dagger}(p) a^{(2)\dagger}(0) \right. \\ & \quad \left. + (-1)^{p+1} \varepsilon(-q+r) a^{(1)\dagger}(-q) a^{(2)\dagger}(-r) \right] |0\rangle, \\ |0_p, [1_1]\rangle &= \frac{\sqrt{3}}{2} \text{Det}_{(1,2)} \left[a^{(1)\dagger}(p) a^{(2)\dagger}(-p) - \frac{1}{3} \sum_{q=1}^3 a^{(1)\dagger}(q) a^{(2)\dagger}(-q) \right] |0\rangle. \end{aligned} \right. \quad (\text{IV.1.13.a})$$

$$\quad (\text{IV.1.13.b})$$

$$\quad (\text{IV.1.13.c})$$

We call attention to the appearance of relative phase factors in (IV.1.13.b).

The elements of vanishing weight are, as in (III.4.8.b), defined redundantly [see eqn. (III.4.8.c)]. There are only two independent ones.

All other irreps can be constructed from f_1 by Kronecker products.

2. $F(4)$

From this point on, all the exceptional groups will contain roots with spinor weights. The trick of associating Bose oscillators with such weights, whether in irreps, such as those of $D(n)$ and $B(n)$, or in the generators themselves, is the same one: one must find a group $D(N)$ in which the exceptional group can be non-regularly embedded. In the present case, we have

$$D(13) \supset F(4), \quad (IV.2.1)$$

and the relation³⁷

$$\begin{array}{ccc} [1_1] & \longrightarrow & [1_4] \\ f & \longrightarrow & f_1 \\ 26 & & 26 \end{array} \quad (IV.2.2)$$

The other elementary irrep of $F(4)$ is f_2 ,

$$f_2 \equiv \alpha \equiv [1_1]. \quad (IV.2.3)$$

The projection from $\underline{\lambda}$ -space $[D(13)]$ to $\underline{\mu}$ -space $[F(4)]$ is

$$\left. \begin{array}{l} \underline{\lambda}_p \longrightarrow \underline{\mu}_p \\ \underline{\lambda}_{(p+q+r+s)} \longrightarrow \frac{1}{2}(\underline{\mu}_p + \underline{\mu}_q + \underline{\mu}_r + \underline{\mu}_s) \\ \underline{\lambda}_{13} \longrightarrow \underline{0} \end{array} \right\}, \quad (IV.2.4)$$

$$p, q, r, s = \pm 1, \pm 2, \pm 3, \pm 4.$$

The resulting projection^{40,41} of roots into roots is

$$\left. \begin{array}{l} \underline{\lambda}_p \pm \underline{\lambda}_{13} \\ \underline{\lambda}(p+q+r+s) + \underline{\lambda}(p-q-r-s) \end{array} \right\} \rightarrow \underline{\mu}_p, \quad (\text{IV.2.5})$$

$$\left. \begin{array}{l} \underline{\lambda}_p - \underline{\lambda}_q \\ \underline{\lambda}(p-q+r+s) + \underline{\lambda}(p-q-r-s) \end{array} \right\} \rightarrow \underline{\mu}_p - \underline{\mu}_q, \quad |p| \neq |q| \quad (\text{IV.2.6})$$

and

$$\left. \begin{array}{l} \underline{\lambda}(p+q+r+s) \pm \underline{\lambda}_{13} \\ \underline{\lambda}(-\ell+r_1+r_2+r_3) + \underline{\lambda}_\ell \end{array} \right\} \rightarrow \frac{1}{2}(\underline{\mu}_p + \underline{\mu}_q + \underline{\mu}_r + \underline{\mu}_s), \quad (\text{IV.2.7})$$

where ℓ is an element of the set p, q, r, s ; r_1, r_2, r_3 , are the remaining elements, and $p, q, r, s = \pm 1, \pm 2, \pm 3, \pm 4$. We note that the following numbers of $D(13)$ roots project to a single root of $F(4)$ of each type⁴²:

$$\left. \begin{array}{ll} 6 \text{ of type} & (\text{IV.2.5}) \\ 3 \text{ of type} & (\text{IV.2.6}) \\ 6 \text{ of type} & (\text{IV.2.7}) \end{array} \right\} \cdot \quad (\text{IV.2.8})$$

The $F(4)$ generators in terms of sho operators are⁴³ as follows:

The generators in $\underline{H}$, where

$$\underline{H} = \sum \underline{\mu}_p H(p), \quad (\text{IV.2.9})$$

are

$$H(p) = \sum'_{(p)} N^{(\kappa)}(p) + \frac{1}{2} \sum'_{(p)} N^{(\kappa)}(p+q+r+s). \quad (\text{IV.2.10})$$

The generators corresponding to roots of non-vanishing weight, with phase solutions explicitly given, are³⁰

$$\{p-q\} = \varepsilon(p+q) \sum_{(p,q)}'' [N^{(\kappa)}(p,q) - N^{(\kappa)\dagger}(-p,-q)] \\ + \sum_{(p,q)}'' \psi(p,q;r,s) N^{(\kappa)}(p-q;r+s), \quad (\text{IV.2.11})$$

$$\{p\} = \frac{1}{\sqrt{2}} \left(\sum_{(p)}' [N^{(\kappa)}(p,x) + N^{(\kappa)\dagger}(-p,-x)] \right. \\ \left. + \sum_{(p)}' \chi(p;q,r,s) [N^{(\kappa)}(p;q+r+s) - N^{(\kappa)\dagger}(-p;-q-r-s)] \right), \\ x = \pm 13, \quad (\text{IV.2.12})$$

$$\{\frac{1}{2}(p+q+r+s)\} = \frac{1}{\sqrt{2}} \sum_{\kappa=1}^4 \left(\sum_{x=\pm 13} e^{i\varepsilon(x)p} [N^{(\kappa)}(p+q+r+s,x) \right. \\ \left. + N^{(\kappa)\dagger}(-p-q-r-s,-x)] + \sum_{|l|=1}^4 \chi(l;r_1,r_2,r_3) \right. \\ \left. \times [N^{(\kappa)}(-l|r_1+r_2+r_3) - N^{(\kappa)\dagger}(l|-r_1-r_2-r_3)] \right). \quad (\text{IV.2.13})$$

In the equations above,

$$\kappa = 1, 2, 3, 4, \quad (\text{IV.2.14})$$

$$p, q, r, s = \pm 1, \pm 2, \pm 3, \pm 4,$$

l is an element of the set p, q, r, s and r_1, r_2, r_3 are the remaining elements.

The new expressions which appear in (IV.2.13) are

$$e^{i\varphi} \equiv \frac{1}{2}(1 + i\sqrt{3}), \quad (\text{IV.2.15})$$

and

$$N^{(\kappa)}(-l|r_1+r_2+r_3) \equiv a^{(\kappa)\dagger}(-l+r_1+r_2+r_3) a^{(\kappa)}(-l). \quad (\text{IV.2.16})$$

The remaining phase factors $\varepsilon(p+q)$, $\psi(p,q;r,s)$ and $\chi(p;q,r,s)$ are identical to the phase factors ε , ψ and χ defined in connection with our previous discussions of $D(4)$ and $B(4)$ sho realizations⁴⁴.

We note the appearance, for the first time, of the complex phase factor $e^{i\rho}$, associated with the occurrence of two zero weight elements in the irrep f_1 of $F(4)$.

The scalar and f_1 irreps have the familiar forms, in analogy with the classical groups. The irrep $f_2 \equiv a$ is generated from the Kronecker product³⁹ of two f_1 -s:

$$[1_4] \otimes [1_4] = [1_6] \oplus [1_4] \oplus [1_1] \oplus [1_3] \oplus [2_4]$$

$$f_1 \otimes f_1 = s \oplus f_1 \oplus f_2 \oplus A(2f_1) \oplus S(2f_1). \quad (\text{IV.2.17})$$

$$676 = 1 + 26 + 52 + 273 + 324$$

All the other irreps can also be constructed from Kronecker products of f_1 , but we will not explicitly display any of them.

3. E(6)

$E(6)$ and $E(7)$ are the simplest of the exceptional groups to treat. All of their non-vanishing roots are of the same length, and their lowest dimensional elementary irreps have no zero weight elements.

a. Realization of the algebra

The simplest embedding of $E(6)$ is to choose

$$D(27) \supset E(6), \quad (\text{IV.3.1})$$

with the relationships³⁷

$$\begin{aligned} [1_1] &\rightarrow [1_1] \oplus [1_5] \\ f &\rightarrow f_1 \oplus f_2, \quad f_2 \equiv \bar{f}_1. \\ 54 &\quad 27 \quad \quad \overline{27} \end{aligned} \quad (\text{IV.3.2})$$

It is convenient to introduce an unnormalized $\underline{\mu}_6$ basis vector¹⁸, which is still orthogonal to the other, normalized, $\underline{\mu}_p$ -s:

$$\underline{\mu}_6 \cdot \underline{\mu}_6 = \frac{1}{3}; \quad \underline{\mu}_p \cdot \underline{\mu}_p = 1, \quad p=1, \dots, 5. \quad (\text{IV.3.3})$$

The projection from $\underline{\lambda}$ -space $[D(27)]$ to $\underline{\mu}$ -space $[E(6)]$ is

$$\left. \begin{aligned} \underline{\lambda}(\pm p-6) &\rightarrow \pm \underline{\mu}_p - \underline{\mu}_6 \\ \underline{\lambda}(p+q+r+s+t+6) &\rightarrow \frac{1}{2} \left[\sum_{(e)} (\pm) \underline{\mu}_p + \underline{\mu}_6 \right] \\ \underline{\lambda}_{27} \equiv \underline{\lambda}(2(6)) &\rightarrow 2 \underline{\mu}_6 \end{aligned} \right\}, \quad (\text{IV.3.4})$$

where the number of negative signs of $p, \dots, t$ on the left and right of the second expression above is even.

We note that

$$-\underline{\lambda}(p+\dots+t+6) = \underline{\lambda}(-p-\dots-t-6) \rightarrow \frac{1}{2} \left[\sum_{(o)} (\pm) \underline{\mu}_p - \underline{\mu}_6 \right], \quad (\text{IV.3.5})$$

where the number of negative signs of $p, \dots, t$ is odd, i.e. the number of negative signs in $\underline{\lambda}$ indices and in front of $\underline{\mu}$ -s is always odd, if the index 6 is included.

The projection of roots into roots^{40,45} is

$$\left. \begin{aligned} \underline{\lambda}(p-6) - \underline{\lambda}(q-6) \\ \underline{\lambda}(p+6) + \underline{\lambda}(q+6) \\ \underline{\lambda}(p-q+r+s+t+6) - \underline{\lambda}(-p+q+r+s+t+6) \end{aligned} \right\} \rightarrow \underline{\mu}_p - \underline{\mu}_q, \quad (\text{IV.3.6})$$

$$p, q, \dots, t = \pm 1, \dots, \pm 5,$$

where the number of negative signs in $p, -q, r, \dots, t$ is even,

$$\left. \begin{aligned} \underline{\lambda}(p+q+r+s+t-6) + \underline{\lambda}(2(6)) \\ \underline{\lambda}(-\ell+r_1+r_2+r_3+r_4+6) + \underline{\lambda}(\ell+6) \end{aligned} \right\} \rightarrow \frac{1}{2} \left[\sum_{(o)} (\pm) \underline{\mu}_p + 3 \underline{\mu}_6 \right], \quad (\text{IV.3.7})$$

$$p, q, \dots, t = \pm 1, \dots, \pm 5,$$

where the number of negative signs in p, q, r, s, t is odd, ℓ is an element of the set p, q, r, s, t and r_1, r_2, r_3, r_4 are the remaining elements,

$$\left. \begin{aligned} \underline{\lambda}(p+q+r+s+t+6) - \underline{\lambda}(2(6)) \\ \underline{\lambda}(-\ell+r_1+r_2+r_3+r_4-6) - \underline{\lambda}(2(6)) \end{aligned} \right\} \rightarrow \frac{1}{2} \left[\sum_{(e)} (\pm) \mu_p - 3\mu_6 \right], \quad (\text{IV.3.8})$$

$$p, q, \dots, t = \pm 1, \dots, \pm 5,$$

where the number of negative signs in p, q, r, s, t is odd.

Six $D(27)$ roots project into one $E(6)$ root of each type.

The $E(6)$ generators in terms of sho operators are as follows:

The generators in $\underline{H}$, where

$$\underline{H} = \sum \mu_p H(p) + \mu_6 [H(6) - H(-6)] \quad (\text{IV.3.9})$$

are

$$H(p) = \sum'_{(p)} N^{(\kappa)}(p+x) + \frac{1}{2} \sum'_{(p)} N^{(\kappa)}(p+q+r+s+t+x), \quad (\text{IV.3.10})$$

$$x = \pm 6,$$

and

$$H(6) = \sum'_{(6)} N^{(\kappa)}(p+6) + \sum'_{(6)} N^{(\kappa)}(p+q+r+s+t+6) + 2 \sum'_{(6)} N^{(\kappa)}(2(6)). \quad (\text{IV.3.11})$$

The generators corresponding to roots of non-vanishing weight, with real phase factors, and explicit solutions given, are³⁰

$$\begin{aligned} \{p-q\} = \varepsilon(p+q) \sum''_{(p,q)} \varepsilon(x) [N^{(\kappa)}(p+x, q+x) - N^{(\kappa)\dagger}(-p-x, -q-x)] \\ + \sum''_{(p,q)} \psi(p, q; r, s, t) [N^{(\kappa)}(p-q; r+s+t+6) - N^{(\kappa)}(p-q; -r-s-t-6)], \\ x = \pm 6, \end{aligned} \quad (\text{IV.3.12})$$

$$\begin{aligned} \left\{ \frac{1}{2}(p+q+r+s+t)_{(\pm)} \pm \frac{3}{2}(6) \right\} = \sum_{\kappa=1}^5 [N^{(\kappa)}(p+q+r+s+t_{(\pm)}(6)) + N^{(\kappa)\dagger}(-p-q-r-s-t_{(\pm)}(6))] \\ + \sum_{\kappa=1}^5 \sum_{\ell=1}^5 \chi(\ell; r_1, r_2, r_3, r_4) [N^{(\kappa)}(-\ell+r_1+r_2+r_3+r_4_{(\pm)}(6)) - N^{(\kappa)\dagger}(\ell-r_1-r_2-r_3-r_4_{(\pm)}(6))], \end{aligned} \quad (\text{IV.3.13})$$

with the upper (lower) signs, taken together, and with the following minus sign parities of the indices:

$$\left. \begin{array}{l} \{\frac{1}{2}(p+q+r+s+t)_{\pm}^{\pm} \frac{3}{2}(6)\} \quad \text{overall odd, including 6} \\ N^{(\kappa)}(\dots), a^{(\kappa)\dagger}(\dots) \quad \text{overall even, including 6} \end{array} \right\} \quad (\text{IV.3.14})$$

The following new $N^{(\kappa)}$ -s and their definitions are

$$N^{(\kappa)}(p+q+r+s+t_{\pm}^{\pm} 6) \equiv a^{(\kappa)\dagger}(p+q+r+s+t_{\pm}^{\pm} 6) a^{(\kappa)}(\pm 2(6)), \quad (\text{IV.3.15})$$

and

$$N^{(\kappa)}(-l|r_1+r_2+r_3+r_4_{\pm}^{\pm} 6) \equiv a^{(\kappa)\dagger}(-l+r_1+r_2+r_3+r_4_{\pm}^{\pm} 6) a^{(\kappa)}(-l_{\pm}^{\pm} 6). \quad (\text{IV.3.16})$$

We note that, in order to construct all irreps, we must take

$$\kappa = 1, \dots, 5. \quad (\text{IV.3.17})$$

In many ways, μ_6 plays the role of the zero weights in $B(n)$.

The phase factors ψ and χ are identical to those defined in connection with our previous discussion of $D(5)$ and $B(5)$ sho realizations⁴⁶.

b. Realizations of the irreps

The irreps have the familiar forms, in analogy with the classical groups. However, because of reality requirements, it was necessary to embed $E(6)$ in a larger group in such a way that f_1 of the larger group went into both f_1 and $\bar{f}_1$ of $E(6)$ [see eqn. (IV.3.2)]. We stress this feature by explicitly exhibiting the f_1 and $\bar{f}_1$ irreps:

$$\left\{ \begin{array}{l} |(\frac{1}{2}, \mu_p - \mu_c), [1,1]\rangle = a^{(\kappa)\dagger}(\frac{1}{2}, p-6) |0\rangle \\ |(\frac{1}{2} \sum_{(e)} (\pm) \mu_p + \mu_c), [1,1]\rangle = a^{(\kappa)\dagger}(p+q+r+s+t+6) |0\rangle \\ |2\mu_c, [1,1]\rangle = a^{(\kappa)\dagger}(2(6)) |0\rangle, \end{array} \right. \quad (\text{IV.3.18})$$

and

$$\left\{ \begin{array}{l} |(\frac{1}{2}, \underline{\mu}_p + \underline{\mu}_6), [1_5]\rangle = \alpha^{(\kappa)\dagger}(\frac{1}{2}, p+6) |0\rangle \\ |(\frac{1}{2} \sum_{(6)} (\pm) \underline{\mu}_p - \underline{\mu}_6), [1_5]\rangle = \alpha^{(\kappa)\dagger}(p+q+r+s+t-6) |0\rangle \\ | -2 \underline{\mu}_6, [1_5]\rangle = \alpha^{(\kappa)\dagger}(-2(6)) |0\rangle . \end{array} \right. \quad (\text{IV.3.19})$$

The irrep $f_3 \equiv a$ is generated from the Kronecker product³⁹ of f_1 and $\bar{f}_1 \equiv f_2$

$$\begin{aligned} [1_1] \otimes [1_5] &= [0] \oplus [1_6] \oplus [1_1, 1_5] \\ f_1 \otimes f_2 &= s \oplus f_3 \oplus (f_1 f_2) \\ 726 &= 1 + 78 + 650 . \end{aligned} \quad (\text{IV.3.20})$$

All other irreps can also be generated from Kronecker products.

4. E(7)

The simplest embedding of E(7) is

$$D(28) \supset E(7), \quad (\text{IV.4.1})$$

with the relations³⁷

$$\begin{aligned} [1_1] &\rightarrow [1_6] \\ f &\rightarrow f_1 \\ 56 & \quad 56 . \end{aligned} \quad (\text{IV.4.2})$$

As in the case of E(6), it is convenient to introduce an unnormalized basis vector¹⁸, this time $\underline{\mu}_7$:

$$\begin{aligned} \underline{\mu}_7 \cdot \underline{\mu}_7 &= \frac{1}{2} , \\ \underline{\mu}_p \cdot \underline{\mu}_q &= \delta_{pq} , \quad \underline{\mu}_p \cdot \underline{\mu}_7 = 0, \quad p, q = 1, \dots, 6 . \end{aligned} \quad (\text{IV.4.3})$$

The projection from $\underline{\lambda}$ space $[D(28)]$ to $\underline{\mu}$ space $[E(7)]$ is

$$\left. \begin{aligned} \underline{\lambda}(\pm p \pm 7) &\rightarrow \pm \underline{\mu}_p \pm \underline{\mu}_7 \\ \underline{\lambda}_{(0)}(p+q+r+s+t+u) &\rightarrow \frac{1}{2} \sum_{(0)} (\pm) \underline{\mu}_p \end{aligned} \right\} , \quad (IV.4.4)$$

$$p, q, \dots, u = \pm 1, \dots, \pm 6.$$

The projection of roots into roots^{40,45} is

$$\left. \begin{aligned} \underline{\lambda}(p+7) - \underline{\lambda}(q+7) \\ \underline{\lambda}(p-7) - \underline{\lambda}(q-7) \\ \underline{\lambda}_{(0)}(p-q+r+s+t+u) + \underline{\lambda}_{(0)}(p-q-r-s-t-u) \end{aligned} \right\} \rightarrow \underline{\mu}_p - \underline{\mu}_q, \quad (IV.4.5)$$

$$\underline{\lambda}(p \pm 7) + \underline{\lambda}(-p \pm 7) \rightarrow \pm 2 \underline{\mu}_7, \quad (IV.4.6)$$

and

$$\underline{\lambda}(-\ell + r_1 + r_2 + r_3 + r_4 + r_5) + \underline{\lambda}(\ell \pm 7) \rightarrow \frac{1}{2} \left(\sum_{(e)} (\pm) \underline{\mu}_p \pm 2 \underline{\mu}_7 \right), \quad (IV.4.7)$$

where ℓ is one element of the set $p, q, \dots, u$ and $r_1, \dots, r_5$ are the remaining elements.

Six $D(28)$ roots project into one $E(7)$ root of each type.

The $E(7)$ generators in terms of sho operators are given as follows:

The generators in $\underline{H}$, where

$$\underline{H} = \sum \underline{\mu}_p H(p) + \underline{\mu}_7 (H(7) - H(-7)) \quad (IV.4.8)$$

are:

$$H(p) = \sum'_{(p)} N^{(\kappa)}(p+x) + \frac{1}{2} \sum'_{(p)} N^{(\kappa)}_{(0)}(p+q+r+s+t+u), \quad (IV.4.9)$$

$$x = \pm 7,$$

$$H(7) = \sum'_{(7)} N^{(\kappa)}(p+7). \quad (IV.4.10)$$

The generators corresponding to roots of non-vanishing weight, with real phase factors, and explicit solutions given³⁰, are

$$\begin{aligned} \{p-q\} = \varepsilon(p+q) \sum_{(p,q)}'' [N^{(\kappa)}(p+x, q+x) - N^{(\kappa)\dagger}(-p-x, -q-x)] \\ + \sum_{(p,q)}'' \psi_{(e)}(p, q; r, s, t, u) N_{(o)}^{(\kappa)}(p-q; r+s+t+u), \\ x = \pm 7, \end{aligned} \quad (IV.4.11)$$

$$\{2(7)\} = \sum_{(7)}' [N^{(\kappa)}(p+7, p-7) - N^{(\kappa)\dagger}(-p-7, -p+7)], \quad (IV.4.12)$$

with $\{-2(7)\}$ the hermitean conjugate of (IV.4.12), and

$$\begin{aligned} \{\frac{1}{2}(p+q+r+s+t+u)_{(e)}, \xi_{(e)}(\frac{1}{2}7|\ell; r_1, \dots, r_5)\} = \sum_{\kappa=1}^6 \sum_{\ell=1}^6 \xi_{(e)}(\frac{1}{2}7|\ell; r_1, \dots, r_5) \\ \times [N_{(o)}^{(\kappa)}(-\ell|\frac{1}{2}7|r_1+\dots+r_5) - N^{(\kappa)\dagger}(\ell|\frac{1}{2}7|-r_1-\dots-r_5)]. \end{aligned} \quad (IV.4.13)$$

We note that only even (odd) numbers of negative indices p, q, r, s, t, u appear in $\frac{1}{2}(p+q+r+s+t+u)_{(e)}$, $\xi_{(e)}(\frac{1}{2}7|\ell; r_1+\dots+r_5)$, $(N_{(o)}^{(\kappa)}(p-q; r+s+t+u), a^{(\kappa)\dagger}(p-q+r \dots))$.

The new $N^{(\kappa)}$ which appears in (IV.4.13) is defined as

$$N_{(o)}^{(\kappa)}(-\ell|\frac{1}{2}7|r_1+\dots+r_5) \equiv a^{(\kappa)\dagger}(-\ell+r_1+\dots+r_5) a^{(\kappa)}(-\ell|\frac{1}{2}7). \quad (IV.4.14)$$

We have not succeeded in obtaining simple algebraic expressions for the real phase factors $\psi_{(e)}$ and $\xi_{(e)}$ which appear in eqns. (IV.4.11) and (IV.4.13), respectively. However, they can easily be generated in tabular form from the following algebraic conditions, generated from the $E(7)$ algebra, with $E(7)$ phase solutions as given in ref. 18:

$$\xi_{(e)}(7|l; r_1, \dots, r_5) = -\xi_{(e)}(-7|-l; -r_1, \dots, -r_5), \quad (\text{IV.4.15})$$

$$\xi_{(e)}(7|-p; q, \dots, u) \xi_{(e)}(7|-q; p, \dots, u) = \prod_{(p,7)}'' \epsilon(1|p|+l) \prod_{(q,7)}'' \epsilon(1|q|+l), \quad (\text{IV.4.16})$$

$$\xi_{(e)}(7|l; r_1, \dots, r_5) \xi_{(e)}(-7|l; r_1, \dots, r_5) = (-1)^{\sum l_-}, \quad (\text{IV.4.17})$$

$$\psi_{(e)}(p, q; r, s, t, u) = -\epsilon(p+q) \xi_{(e)}(7|p; q, \dots, u) \xi_{(e)}(7|q; p, \dots, u), \quad (\text{IV.4.18})$$

where l_- are all possible values of negative indices. The tables can be generated, starting with the choice of a suitable set of $\xi_{(e)}$ -s, and using (IV.4.15)-(IV.4.18) to obtain the remaining $\xi_{(e)}$ -s and the $\psi_{(e)}$ -s. [The latter are the phases $\psi_{(e)}$ which appear in the appropriate expressions for D(6) and B(6)]. We have done so, but will not further encumber the present work by listing our specific results.

The scalar and f_1 irreps have the familiar forms, in analogy with the classical groups. The irreps $f_2 \equiv a$ and f_3 can be generated, as follows³⁹:

$$\begin{aligned} f_2: \quad [1_6] \otimes [1_6] &= [0] \oplus [1_7] \oplus [1_5] \oplus [2_6] \\ f_1 \otimes f_1 &= s \oplus f_2 \oplus A(2f_1) \oplus S(2f_1) \quad (\text{IV.4.19}) \\ 3,136 &= 1 + 133 + 1,539 + 1,463, \end{aligned}$$

$$\begin{aligned} f_3: \quad [1_6] \otimes [1_7] &= [1_6] \oplus [1_7] \oplus [1_1, 1_6] \\ f_1 \otimes f_2 &= f_1 \oplus f_3 \oplus (f_1 f_2) \quad (\text{IV.4.20}) \\ 7,448 &= 56 + 912 + 6,480. \end{aligned}$$

All other irreps can be generated from appropriate Kronecker products.

5. E(8)

The consideration of E(8) is straightforward, but algebraically tedious, largely because of the eight irrep elements of vanishing weight in $f_1 \equiv a$.

Because of this fact, the solutions for the complex phase factors need to be given in tabular form. In the interests of brevity, we have only given the algebraic constraint equations for these phase factors.

The simplest embedding of $E(8)$ is to choose

$$D(124) \supset E(8), \quad (IV.5.1)$$

with the relations³⁷

$$\begin{aligned} [1_i] &\rightarrow [1_i] \\ f_{248} &\rightarrow f_{i248}, \quad f_i \equiv a. \end{aligned} \quad (IV.5.2)$$

The projection from $\underline{\lambda}$ -space $[D(124)]$ to $\underline{\mu}$ -space $[E(18)]$ is

$$\left. \begin{aligned} \underline{\lambda}(p-q) &\rightarrow \underline{\mu}_p - \underline{\mu}_q \\ \underline{\lambda}_{(0)}(p+q+r+s+t+u+v+w) &\rightarrow \frac{1}{2} \sum_{(0)} (\pm) \underline{\mu}_p \\ \underline{\lambda}_z &\rightarrow 0 \end{aligned} \right\}, \quad (IV.5.3)$$

$z = 121, 122, 123, 124, \quad p, q, \dots, w = \pm 1, \dots, \pm 8.$

The projection of roots into roots^{40,47} is

$$\left. \begin{aligned} \underline{\lambda}(p-q) \pm \underline{\lambda}_z, \quad z = 121, \dots, 124 \\ \underline{\lambda}(p-r) + \underline{\lambda}(r-q), \quad |p| \neq |q| \neq |r| \\ \underline{\lambda}_{(0)}(p-q+r+\dots+w) + \underline{\lambda}_{(0)}(p-q-r-\dots-w) \end{aligned} \right\} \rightarrow \underline{\mu}_p - \underline{\mu}_q, \quad (IV.5.4)$$

and

$$\left. \begin{aligned} \underline{\lambda}_{(0)}(p+q+\dots+w) \pm \underline{\lambda}_z, \quad z = 121, \dots, 124 \\ \underline{\lambda}_{(0)}(-\ell_1 - \ell_2 + r_1 + \dots + r_6) + \underline{\lambda}(\ell_1 + \ell_2) \end{aligned} \right\} \rightarrow \frac{1}{2} \sum_{(0)} (\pm) \underline{\mu}_p, \quad (IV.5.5)$$

where ℓ_1 and ℓ_2 are two elements of the set $p, q, \dots, w$ and $r_1, \dots, r_6$ are the remaining elements of the set.

Thirty-six D(124) roots project into one E(8) root of each type.

The E(8) generators in terms of sho operators are given as follows:

The generators in $\underline{H}$, where

$$\underline{H} = \sum \mu_p H(p) \quad (\text{IV.5.6})$$

are

$$H(p) = \sum_{(p)}' N^{(\kappa)}(p-r) + \frac{1}{2} \sum_{(p)}' N_{(0)}^{(\kappa)}(p+q+\dots+w). \quad (\text{IV.5.7})$$

The generators corresponding to roots of non-vanishing weight, with most of the phase solutions explicitly given³⁰, are

$$\begin{aligned} \{p-q\} = & \frac{1}{2} \sum_{(p,q)}'' \omega(p-q; z) [N^{(\kappa)}(p-q; z) + N^{(\kappa)\dagger}(-p+q; -z)] \\ & + \sum_{(p,q)}'' d(p, q, r) [N^{(\kappa)}(p-r, -r+q) - N^{(\kappa)\dagger}(-p+r, r-q)] \\ & + \sum_{(p,q)}'' \psi_{(e)}(p, q; r, \dots, w) N_{(0)}^{(\kappa)}(p-q; r+s+\dots+w), \end{aligned} \quad (\text{IV.5.8})$$

and

$$\begin{aligned} \{\frac{1}{2}(p+q+\dots+w)\} = & \sum_{\kappa=1}^g \left(\frac{1}{2} \sum_{z=\pm 121}^{\pm 124} e^{i \zeta_{(0)}(p, \dots, w; z)} \right. \\ & \times [N_{(0)}^{(\kappa)}(p+q+\dots+w; -z) + N_{(0)}^{(\kappa)\dagger}(-p-q-\dots-w; z)] + \sum_{\ell_1, \ell_2} \psi_{(e)}(\ell_1, \ell_2; r_1, \dots, r_6) \\ & \left. \times [N_{(0)}^{(\kappa)}(-\ell_1 - \ell_2 | r_1 + \dots + r_6) - N_{(0)}^{(\kappa)\dagger}(\ell_1 + \ell_2 | -r_1 - \dots - r_6)] \right), \end{aligned} \quad (\text{IV.5.9})$$

with

$$\zeta_{(0)}(p, \dots, w; z) = \varepsilon(n_- - n_+) (-1)^z \frac{\pi}{6}. \quad (\text{IV.5.10})$$

In eqn. (IV.5.10), $n_+(n_-)$ are the number of positive (negative) indices $p, q, \dots, w$. The factors $\frac{1}{2}$ in eqns. (IV.5.8) and (IV.5.9) are introduced so that the remaining coefficients are phase factors⁴⁸. The definition of $N_{(0)}^{(K)}(-\ell_1 - \ell_2 | r_1 + \dots + r_6)$ is given in eqn. (IV.2.16). The phase factor $d(p, q, r)$ is defined in eqn. (III.1.3) and $\psi_{(e)}$ first makes its appearance above in connection⁴⁹ with $D(8)$.

We have not been able to obtain a general form for the remaining set of phase factors, $\omega(p-q; z)$. One can obtain a tabular form of these terms by generating them from the solutions for the set of simple roots. Since we are principally concerned in this paper with demonstrating the existence of the realizations of algebras and irreps, we will not give the details of the results, but will provide the equations constraining the ω -s for simple roots.

A set of eight simple roots of $E(8)$ can be taken to be

$$\begin{aligned} \beta_{\bar{1}} &= \beta_p^q, \quad \beta_{\bar{2}} = \beta_q^r, \quad \beta_{\bar{3}} = \beta_r^s, \quad \beta_{\bar{4}} = \beta_s^t, \quad \beta_{\bar{5}} = \beta_t^u, \\ \beta_{\bar{6}} &= \beta_{uvw}^{pqrst}, \quad \beta_{\bar{7}} = \beta_{pqrstuvw}^w, \quad \beta_{\bar{8}} = \beta_u^v, \end{aligned} \quad (IV.5.11)$$

$$p, q, \dots, w = \pm 1, \dots, \pm 8, \quad |p| \neq |q| \neq \dots \neq |w|$$

with the Dynkin-Patera ordering, $\bar{j}, \bar{j} = 1, \dots, 8$, of simple roots^{14,15} and the definitions¹⁹

$$\beta_p^q \equiv \mu_p - \mu_q, \quad \beta_{uvw}^{pqrst} \equiv \frac{1}{2}(-\mu_p - \dots - \mu_t + \mu_u + \mu_v + \mu_w), \text{ etc..} \quad (IV.5.12)$$

The conditions on the ω -s associated with $\beta_{\bar{1}}, \dots, \beta_{\bar{5}}$ and $\beta_{\bar{8}}$ are

$$\begin{aligned} \sum_{\bar{z}} \omega(\bar{j}; z) \omega^*(\bar{k}; z) &= 4 \beta_{\bar{j}} \cdot \beta_{\bar{k}}, \\ \sum_{\bar{z}} \omega(\bar{j}; z) e^{-i(-1)^{\bar{z}} \pi/6} &= 4 \delta_{\bar{j}5}, \\ \sum_{\bar{z}} \omega(\bar{j}; z) e^{i(-1)^{\bar{z}} \pi/6} &= 0, \end{aligned} \quad (IV.5.13)$$

$$\bar{j}, \bar{k} = \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{8}.$$

The scalar and $f_1 \equiv a$ irreps have the familiar form. The irreps f_2 and f_3 can be generated in a fashion analogous to the method used in E(7):

$$\begin{aligned} f_2: [1,] \otimes [1,] &= [0] \oplus [1,] \oplus [1,] \oplus [1,] \oplus [2,] \\ f_1 \otimes f_1 &= s \oplus f_1 \oplus f_2 \oplus A(2f_1) \oplus S(2f_1), \\ 61,504 &= 1 + 248 + 3,875 + 30,380 + 27,000 \end{aligned} \quad (\text{IV.5.14})$$

$$\begin{aligned} f_3: [1,] \otimes [1,] &= [1,] \oplus [1,] \oplus [1,] \oplus [1,] \oplus [1,1,] \\ f_1 \otimes f_2 &= f_1 \oplus f_2 \oplus f_3 \oplus A(2f_1) \oplus (f_1 f_2), \\ 961,000 &= 248 + 3,875 + 147,250 + 30,380 + 779,247 \end{aligned} \quad (\text{IV.5.15})$$

All other irreps can also be generated from appropriate Kronecker products.

V. SUMMARY AND DISCUSSION

Our aim in the present work was to realize systematically and simply both the algebras and all irreps of all simple compact Lie groups in terms of Bose oscillators (sho). In our uniform approach to all of these groups, the generators are bilinear expressions of sums of a^+ -s and a -s (sho creation and annihilation operators) and the irreps are homogeneous polynomials of a^+ -s of fixed order, operating on a vacuum state, of zero weight. In particular, the elementary spinor irreps of the orthonormal groups can be written in terms of a single sho creation operator acting on the vacuum state. This is to be contrasted with the customary treatment of spinors in terms of Fermi oscillators³⁻⁷ (Clifford variables). In such a treatment the "vacuum state" does not have zero weight, but is an element of a spinor irrep. The other elements of the irreps are obtained by having different powers of Fermi oscillator creation operators act on this "vacuum state". For $D(n)$, the "vacuum state", together with all states formed by even powers of

a^+ -s operating on this state, up to the maximum possible such power, constitute the elements of one of the elementary spinor irreps; all possible odd powers of a^+ -s, operating on the "vacuum" constitute the set of all elements of the other elementary spinor irrep.

Two features of our results, the possibility of realizing all classical Lie algebras and their non-spinor irreps in terms of sho-s, and the need for the introduction of phase factors in the realization of the algebras¹¹, are at least implicit in previous work. The novel features of the current work-- the treatment of spinor irreps of orthogonal groups and of the exceptional groups-- stem from a single ansatz: the systematic embedding of the group under consideration, g , in a larger group, $\mathfrak{G}$, such that g is a non-regular subgroup of $\mathfrak{G}$. $\mathfrak{G}$ is always an orthogonal group. It is $B(N)$ in the case of $B(n)$ and $G(2)$, and is $D(N)$ in the case of all the other compact Lie groups, with appropriately chosen N -s. Given the rank, n , of g , the rank, N , of $\mathfrak{G}$, is fixed so that the elementary non-spinor irrep of $\mathfrak{G}$ (called f) goes into those of the elementary irreps of g , from which all other irreps can be conveniently constructed through Kronecker products. When g is an orthogonal group, this requirement is met by having the f irrep of $\mathfrak{G}$ go into the sum of all the elementary irreps of g ; for the exceptional groups, the f irrep of $\mathfrak{G}$ goes into the lowest dimensional irrep of g . In $E(6)$, there are two such irreps, complex conjugates of each other, and the f irrep of $\mathfrak{G}$, which is real, goes into the sum of them.

The explicit solution of the embedding problem just outlined is made possible by two elements in our approach. The first is the expression of root and weight space of all groups in terms of a set of mutually orthogonal basis vectors¹⁷⁻¹⁹. While such a choice of basis is a natural one in the case of the orthogonal groups, other, alternative, choices are frequently made for exceptional groups^{5,14}. The second element is the avoidance of the "address" problem, by the use of sho operators, an issue which is discussed in connection with the presentation of the $D(n)$ results, above, but which we will now summarize.

In the embedding $\mathfrak{g} \supset \mathfrak{g}$, with $\mathfrak{g}$ a non-regular subgroup, generators of type e_{β} in $\mathfrak{g}$ are sums of generators E_{α} of $\mathfrak{g}$. The coefficients in these sums (both magnitudes and phases), depend on the structure constants of both $\mathfrak{g}$ and $\mathfrak{g}$. This, in turn, requires that every element, λ , of the root and weight space basis in $\mathfrak{g}$ be given a single numerical "address". This requirement is obviated by the use of sho operators. To be sure, such operators must be suitably labeled, and we label them in a one-to-one relation to the weights of the $\mathfrak{g}$ irreps into which the $\mathfrak{f}$ irrep of $\mathfrak{g}$ goes. However, we know that the commutator $[a, a^+]$ is unity only if the pair a^+, a has the same label; otherwise it vanishes. An exact "address" becomes irrelevant.

This feature of sho operators enables us to exhibit explicit algebraic solutions for the embedding problem for orthogonal groups of arbitrary rank, and also for the exceptional groups. The embedding problem would, in principle, be soluble without the use of sho operators, but would, in practice, be tedious and unmanageable. Indeed, an unforeseen consequence of the present work is that when maximal non-regular subgroups of a group are to be constructed, as is often the case when chains of symmetry breaking are considered in particle physics, the construction can be most simply carried out in the sho picture.

The phase factors of the structure constants of $\mathfrak{g}$ are, of course, still required to be explicitly known. There is, however, no "address" problem in this case, and we can make use of the simple algebraic results for consistent solutions of these phase factors, recently obtained by the author and colleagues¹⁸. These are necessary to get the results presented above, as are the algebraic techniques elucidated for carrying out the embedding of non-regular subgroups in a given group¹⁹. A curious, and as yet not completely understood, feature of our results is the reappearance of the phase solutions (and their obvious generalizations) obtained in reference 18, whenever phase factors with the same algebraic structure occur in the present work.

All the phase complications occur in the realization of the generators. The elementary irreps of the classical groups are trivial in the sho picture we present,

and are not much more complicated for the exceptional groups. The only additional prerequisite for constructing all basic irreps is the trivial introduction of an additional label (superscript κ) so that antisymmetric products of a^+ -s can be generated.

No general expressions are presented for all irreps (not even for all basic irreps), but the tools for constructing all irreps are provided. We also present a sufficient number of specific illustrations, so that the reader can obtain a particular irrep as needed.

There is considerable redundancy in the irreps we construct. We have to eliminate some of them by means of "sho reduction", as discussed at the end of Section III.1, in order to be able to develop sho realizations of Casimir operators in Section III.5. For example, there is a redundant scalar irrep, which arises from a reduction of a product of two a^+ -s. It is in the form of a homogeneous quadratic polynomial of a^+ -s, operating on the vacuum, and is "sho reduced" to the standard scalar, the vacuum state.

Other, more trivial, redundancies, associated with different sets of (κ) superscripts, remain, but could be eliminated by a simple ordering ansatz. There also exist alternative realizations of the algebras and therefore of the irreps. We have pointed out one such alternative in connection with $D(n)$. Two other examples are provided by the groups $E(7)$ and $E(6)$, because of the regular maximal subgroup relations $E(8) \supset E(7) \otimes A(1)$ and $E(7) \supset E(6) \otimes U(1)$. Realizations of the $E(7)$ and $E(6)$ generators, alternative to the ones we give above, can be gotten by taking appropriate subsets of the $E(8)$ generators exhibited in Section IV.5. Correspondingly, we obtain alternative realizations of the irreps of these groups. In particular, the adjoint irreps are linear, rather than quadratic in the a^+ -s in this approach. However, we feel that the realizations of the algebras and irreps of $E(7)$ and $E(6)$ we present are simpler than these alternatives.

In any case, there is at least one way of realizing each irrep in terms of sho operators, including irreps with spinor content. Since the sho operators a^+ and a , as well as the sho ground state (our vacuum state, $|0\rangle$), can easily be

expressed in the coordinate basis, in terms of suitable coordinates and their derivatives, the realizations of the algebra and the other irreps can also be so expressed.

Finally, we have shown for the groups $A(n)$, $B(n)$, $C(n)$ and $D(n)$ that a set of operators, which are invariants for a given irrep and therefore play the role of Casimir operators, can be simply realized in terms of polynomials of the operators, homogeneous and separately of equal degree in the a^+ -s and a -s. The eigenvalues of these operators are simply related to the Dynkin-Patera indices, and are an alternative set of labels for a given irrep.

Possible extensions and applications of this work to non-compact groups associated with the groups considered, and to questions of eigenvalues, eigenfunctions^{8,9}, matrix elements and the Wigner-Eckart problem^{2,3,4}, exist. We hope to pursue some of these questions in the future.

The author would like to thank the Theory Division of CERN, where most of this work was carried out, for its hospitality in the academic year 1983-84, and the Physics Department of the University of Florence, where some of the work was done in a brief visit in fall '83. He would also like to thank Susan Kövesi-Domokos, Gábor Domokos and Roberto Casalbuoni for helpful comments. He particularly wishes to express his appreciation to Gordon Feldman for his customary incisive comments and helpful long-distance criticisms.

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- 21) See Eqns. (3.7), (3.9), (3.10), (3.12)-(3.14), (3.20)-(3.22), (3.41)-(3.44) and (3.55)-(3.57) of ref. 18.
- 22) The solutions in ref. 18 are given as follows: for d, eqn. (3.8); for b and $\bar{b}$, eqn. (3.11); for c, σ and $\bar{\sigma}$, eqns. (3.15) and (3.16); for e and $\bar{e}$, eqns. (3.24)-(3.28); for f, $\bar{f}$, ϕ and $\bar{\phi}$, eqns. (3.45)-(3.51); for g, $\bar{g}$ and $\bar{\bar{g}}$, eqn. (3.58).
- 23) The normalization condition for $\underline{\mu}_p$ in eqn. (II.10), and for the $\underline{\lambda}$ -s, differs from the treatment in refs. 18 and 19, in order to avoid the later explicit appearance of the scale factor, s, (defined in eqns. (4.13) and (4.17) of ref. 19) in expressions for the generators.
- 24) Those for the classical groups are given in Table 1, and those for the exceptional groups in Table 2 of ref. 18.
- 25) We are using the notation of eqns. (II.28), (II.29), and (II.32) in eqns. (III.1.5), (III.1.6) and ff., but as a reminder, also write the expression out in more detail in eqn. (III.1.5). The superscripts (κ) do not as yet appear in (III.1.5) and (III.1.6).
- 26) This nonenclature is defined in ref. 14, p 346, or ref. 5, p 116. The symbol τ_1 is used for f in ref. 5.
- 27) In refs. 17 and 19, we call f "the quark", and use the symbol q for it, but since quarks can also be put into spinor irreps^{6,7}, this earlier notation is not a felicitous one in the present context.
- 28) This nonenclature is defined in ref. 14, pp. 350, 351, and in ref. 5, p. 121. The symbols σ_1 and σ_2 are used to denote the elementary spinor irreps in these references. We shall use $\sigma_{(o)}$ and $\sigma_{(e)}$ for them.

- 29) In the cases $n = 2, 3$, there are only two terminal points and $D(2) \doteq A(1) \otimes A(1)$, $D(3) \doteq A(3)$. The formalism can still be carried out, with small modifications. For $n = 2$, $N = 2$ and the result is trivial. For $n = 3$, $N = 4$ and $D(4) \supset B(3) \supset D(3)$, so that $D(3)$ is a regular non-maximal subgroup of $D(4)$. This result is also a trivial one.
- 30) The (κ) index in $N^{(\kappa)}(p-q; r+s \dots)$ and later, for $B(n)$, in $N^{(\kappa)}(p; q+r \dots)$ is suppressed, since it is not needed either to make the algebra close or to help construct irreps. It reappears in the exceptional groups, where it is needed to close the algebras.
- 31) Eqns. (B.20) and (B.21) for $E(8)$, $E(7)$, $E(6)$ and eqn. (B.24) and (B.26) for $F(4)$, in ref. 18.
- 32) See ref. 18, eqns. (3.9) and (3.10). A specific set of phase solutions is given in eqn. (3.11).
- 33) For the commutation relations in which the phases are defined, see ref. 18, eqns. (3.12)-(3.14), and for specific solutions, eqns. (3.15) and (3.16). Note error in (3.16): $\bar{\sigma}(p, q)$ should read $\bar{\sigma}(q, p)$.
- 34) See ref. 9, eqns. (2.5)-(2.7).
- 35) Note that the irrep of eqn. (III.1.56) has only a single κ superscript. This guarantees the appearance of only symmetric combinations of a^+ -s, and also that $n_{(1)} = 2$, and all other $n_{(\omega)}$ -s and $n_{(\pi)}$ -s vanish. Observe also that the only non-vanishing eigenvalues $n_{(\omega)}$ and $n_{(\pi)}$ in the irrep of eqn. (III.1.50) are $n_{(1)} = 2$ and $n_{(2)} = 1$.
- 36) These relations are spelled out in detail in ref. 19, eqns. (4.52) and (4.53), and figs. (1a) and (1b). They are applied to specific cases in Appendix A of this reference.
- 37) The l.h.s. of this relation refers to the group of higher rank, G_N , and the r.h.s. to the group of lower rank g_n ($G_N \supset g_n$, non-regular). The first row gives our version of the Dynkin-Patera indices; the second row repeats this information in terms of the symbols for the elementary irreps; the third row gives the dimension of the irrep listed above it.

- 38) In the language of ref. 19, the projection matrix is pseudo-orthogonal, and $s^2 = 1$, where s is the scale factor. Two roots of $B(3)$ project to each root of $G(2)$ of type (IV.1.6). The weights of the $G(2)$ roots are listed in Table 2 of ref. 18.
- 39) The symbols introduced here are $s \equiv [0]$, for the scalar irrep, and $S(2f_1)$ for the irrep constructed from the symmetric product of two f_1 -s. The notation $A(2f)$ and $(f_1 f_2)$, to appear subsequently, denotes irreps constructed from anti-symmetric products of two f -s, and from products of f_1 and f_2 , respectively.
- 40) In the language of ref. 19, the projection matrix is orthogonal. The weights of the roots are listed in Table 2 of ref. 18.
- 41) For $D(13) \supset F(4)$, $s^2 = 3$.
- 42) For type (IV.2.5), there are 2^3 possible q , r , and s , but $q+r+s$ is paired with $-q-r-s$; for type (IV.2.6), there are 2^2 possible $r+s$, but $r+s$ is paired with $-r-s$.
- 43) The factor $\frac{1}{2}$ in (IV.2.10) is traceable to the appearance of spinor weights among $F(4)$ roots; the $1/\sqrt{2}$ -s in eqns. (IV.2.12) and (IV.2.13) are traceable to the fact that, in our units, the $F(4)$ roots corresponding to (IV.2.5) and (IV.2.7) are of unit length, while those corresponding to (IV.2.6) have length $\sqrt{2}$.
[See ref. 19, eqns. (4.38) and (4.39).]
- 44) Indeed, $\psi(p,q; r,s) \equiv f(p,q; r,s)$, $\chi(p; q,r,s) \equiv \phi(p; q,r,s)$ of ref. 18.
- 45) For $D(27) \supset E(6)$ and $D(28) \supset E(7)$, $s^2=6$.
- 46) Indeed, $\psi_{(o)}(p,q; r,s,t) = e(p,q; r,s,t, \vec{L} \cdot \vec{J} 6)$ of ref. 18.
- 47) For $D(124) \supset E(8)$, $s^2=30$.
- 48) This is the sole example of all the cases we have considered, in which eqn. (4.47) of ref. 19, arising from the commutators (II.3), plays an essential role. We note that the number of $D(N)$ roots which go into $E(6)$ and $E(7)$ roots are given by s^2 (see ref. 45 and the remarks in the text below eqns. (IV.3.8) and (IV.4.7)). In contrast, for $E(8)$, $s^2=30$ and 36 $D(124)$ roots go into one of each type of $E(8)$ root. The identity $30 \equiv (\frac{1}{2})^2 \times 8 + 28$ arises from the analysis of ref. 19 in this case.
- 49) Indeed, $\psi_{(e)}(p,q; r, \dots, w) = e(p,q; r, \dots w)$ of ref. 18.