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THERMODYNAMICS OF BOSON-FERMION DUALITY IN CONFINED
TWO-DIMENSIONAL MODELS

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A B S T R A C T

The partition functions of massless $(\text{QED})_2$ and $(\text{QCD})_2$ are calculated in a weak coupling limit defined by using a finite volume as infrared cutoff. A coherent state formalism is used to treat constraints from gauge invariance and show that the results are those expected from bosonization. We also present a simple derivation of the physical spectrum of $(\text{QED})_2$ confined to a bag for any coupling.

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In one space and one time dimension, fermi fields can be bosonized, i.e. they can be described in terms of bose fields^[1,2]. Recently this Coleman-Mandelstam bosonization was extended by Witten to non-Abelian quantum fields as well^[3] (for some connected works see Refs. 4-5).

In the present work we consider some aspects of the fermi-bose duality in 1+1 dimensions in terms of the thermodynamic properties of the corresponding systems. Our strategy will be to compute the various partition functions by explicitly summing over the energy eigenvalues. When doing this we must face the problem, that for fermions interacting via gauge fields, perturbation theory is not defined in 1+1 dimension even in the weak coupling limit because of infrared divergences. In fact, it is well known that there are no asymptotic states for the fermi fields. However, by introducing an infrared cut off by means of a finite volume ($=2a$), and imposing the gauge invariant (MIT bag) boundary conditions

$$\begin{aligned} (1 \pm i\gamma^1)\psi(\pm a) &= 0 \\ E(\pm a) &= F_{01}(\pm a) = 0 \end{aligned} \quad (1)$$

in obvious notation, the perturbation expansion in ga (g is the gauge coupling) is well defined^[6]. The $g = 0$ eigenfrequencies corresponding to Eq.(1) are $\omega_n^F = (2n + 1)\pi/4a$ and $\omega_n^B = n\pi/2$, where n is any positive integer, and the fermions are massless. Naively we would now compute the $g = 0$ partition function for the fermi field by treating particles and antiparticles as two non interacting fermi gases, both with energies ω_n^F . This yields

$$Z_F = \prod_{n=1}^{\infty} (1 + x^{2n-1})^2 \quad (2)$$

where we used the notation $x = \exp(-(\beta\pi/4a))$ and $\beta = 1/T$. Using properties of the Jacobi ϑ -functions (see e.g. Refs.7-8) we get

$$Z_F = \vartheta_3[0, x] Z_B \quad (3)$$

where

$$Z_B = \prod_{n=1}^{\infty} (1 - x^{2n})^{-1} \quad (4)$$

is the partition function of a bose field satisfying the (Dirichlet) boundary condition Eq.(1). On the other hand, it has been shown^[9] that, remarkably enough, the physical spectrum of the Schwinger model^[10] with the bag boundary conditions Eq.(1) is very similar to the continuum ($a \rightarrow \infty$) result. In fact, the only physical state is a massive boson with the discrete energies $\omega_n = ((\omega_n^B)^2 + g^2/\pi)^{1/2}$. The apparent contradiction is resolved by realizing that for $g \neq 0$, Eq.(1) can only be satisfied for states with no net electric charge. From the partition function Z_F , which can be written as an incoherent sum of the partition functions $Z_F^{(q)}$ evaluated in a given charge sector, we should extract the charge $q = 0$ component only. A general method for extracting a contribution to a partition function Z , due to a partial summation of physical states transforming according to an irreducible representation of a given group G , has recently been discussed in the literature in terms of quasi-coherent states^[11,12] (for some related work see Refs. 13-15). By applying the methods of Ref.12 to $U(1)$, we can express the charge q contribution to Z_F as

$$Z_F^{(q)} = \int_{-\pi+i\beta A}^{\pi+i\beta A} d\alpha / 2\pi \exp(iq\alpha) \prod_{n=0}^{\infty} (1 + \exp(i\alpha)x^{2n+1}) \prod_{m=0}^{\infty} (1 + \exp(-i\alpha)x^{2m+1}) \quad (5)$$

which, as can be seen by inspection, constitutes an integral representation of the relevant combinatorics. A is a constant background electric potential which drops out from all gauge invariant quantities. The products in Eq.(5) can again be expressed in terms of a ϑ_3 function,

$$Z_F^{(q)} = \int_{-\pi+i\beta A}^{\pi+i\beta A} d\alpha / 2\pi \exp(iq\alpha) \vartheta_3[\alpha/2, x] \prod_{n=1}^{\infty} (1 - x^{2n})^{-1} \quad (6)$$

and by using a Fourier series expansion of $\vartheta_3[\alpha/2, x]$ we finally obtain

$$Z_F^{(q)} = x^{q^2} \prod_{n=1}^{\infty} (1 - x^{2n})^{-1} \quad (7)$$

i.e.

$$Z_F^{(0)} = Z_B \quad (8)$$

As expected, the gauge invariant quantity $Z_F^{(0)}$ is independent of the constant background potential A . This result agrees with what one would expect using the weak coupling limit of the results of Ref.10. If we sum over all charges q in Eq.(7) we, of course, retain the naive result of Eq.(2). Note that although it follows trivially from the expressions for ω^F and ω^B that the energy levels of the bose and fermi systems are identical, the combinatorics of the degeneracy factors is both complicated and amusing. In the Table 1 we explicitly list the five lowest states in the $q = 0$ sector.

Going to the continuum limit by the usual replacement $\prod(\dots) \rightarrow \exp \int dk \ln(\dots)$ requires some care for $Z_F^{(0)}$. From Eq.(4) we get

$$Z_B = [1 + O(1/aT)] \exp(\pi a T / 3) \quad (9)$$

Making the same approximation in Eq.(5) gives a result dependent on the gauge parameter A

$$Z_F^{(0)} = \int_{-\pi+i\beta A}^{\pi+i\beta A} d\alpha / 2\pi \exp(-(2aT/\pi)\alpha^2) Z_B \quad (10)$$

If we had put $A = 0$ to start with, we would have derived

$$\ln Z_F^{(0)}(A=0) = \ln Z_B - 1/2 \ln aT + O(1/aT) \quad (11)$$

The apparent contradiction with Eq.(6) is resolved by recognizing that only the leading term is gauge invariant.

The above discussion can be extended to the weak coupling limit of $(QCD)_2$, and we can explicitly compute $Z_F^{(0)}$ for the gauge groups $U(N)$ and $SU(N)$. By applying bosonization techniques it was argued in Ref.16 that the physical spectrum for all couplings consists of N bosons, where N is the number of colours. Our calculation of the fermionic partition function including singlet states only^[16] provides an independent proof of this equivalence in the weak coupling limit.

Let us first consider the case of $(QCD)_2$ confined to a bag. Take the gauge group $G = U(N)$ and decompose $U(N) = SU(N) \otimes U(1)$. We will refer to the $U(1)$ charge as baryon number, and using similar techniques as above, we get for the colour singlet baryon number b partition function^[12]

$$Z_F^{(0,b)} = (Z_B)^N \int_G dg(\alpha_1, \dots, \alpha_N) \prod_{i=1}^N e^{ib\alpha_i} \vartheta_3(\alpha_i/2, x) \quad , \quad (12)$$

where Z_B is given by Eq.(4). Using properties of the $U(N)$ measure $dg(\alpha_1, \dots, \alpha_N)$ (where the integrals are over the eigenvalues $\{\exp(i\alpha_k)\}$, see ref.17), the integral in Eq.(12) can be performed to give

$$Z_F^{(0,b)} = (Z_B)^N \det[M_{ij}^b] = x^{Nb^2} \prod_{m=1}^N Z_B^{[m]} \quad , \quad (13)$$

where $M_{ij}^b = x^{(i-j-b)^2}$, and

$$Z_B^{[m]} = \prod_{n=m}^{\infty} (1 - x^{2n})^{-1} \quad (14)$$

For $U(N)$ singlets, i.e. $b = 0$, Eq.(13) has a simple interpretation; $Z_F^{(0,0)}$ is equal to the partition function of N non interacting bose gases, but with the $(n-1)$ lowest energy levels missing in the n :th gas. The result for $SU(N)$ singlets is obtained simply by summing Eq.(13) over all baryon numbers b , giving

$$Z_{F,SU(N)}^{(0)} = \vartheta_3(0, x^N) Z_F^{(0,0)} \quad (15)$$

Although simple, this cannot be interpreted as a collection of non interacting bosons. For $T \neq 0$, $x < 1$ so for $N \rightarrow \infty$, $\vartheta_3(0, x^N) \rightarrow 1$, which shows that $U(N)$ and $SU(N)$ are identical in the large N -limit.

The continuum limit of Eq.(12) for the $U(N)$ singlet partition function is

$$\begin{aligned} Z_F^{(0,0)} &= \int dg(\alpha_1, \dots, \alpha_N) \exp\left\{ \sum_{i=1}^N (4aT/\pi) \sum_{n=1}^{\infty} (-1)^{n-1}/n^2 \cos n\alpha_i \right\} \\ &= \exp\{N\pi aT/3\} \int_G dg(\alpha_1, \dots, \alpha_N) \exp\left\{ -(aT/\pi) \sum_{i=1}^N \alpha_i^2 \right\} \end{aligned} \quad (16)$$

For sufficiently large a the dominant contribution to the integral in Eq.(16) comes from small angles i.e.

$$Z_F^{(0,0)} = C \exp\{N\pi aT/3\} (1/aT)^{N^2/2}, \quad (17)$$

where C is an unimportant constant. The only change in Eq.(17) when going from $U(N)$ to $SU(N)$ is that N^2 gets replaced by $N^2 - 1$. Apart from the (gauge dependent) non-leading term we see that in both cases $Z_F^{(0)} = (Z_B)^N$.

In deriving Eq.(17) we assumed that the fermions transform according to the fundamental representation. It is easy to extend the analysis to any other representation. If the fermions e.g. transform according to the adjoint representation of $SU(N)$ we obtain an expression similar to Eq.(17) i.e.

$$Z_F^{(0)} = C \exp((N^2-1)\pi aT/3) (1/aT)^{(N^2-1)/2} \quad (18)$$

$Z_F^{(0)}$ is therefore equal to the partition function of N^2-1 bosons.

Here we would like to make a remark concerning the large N -limit of the partition function $Z_F^{(0)}$. In Eq.(16) we recognize the one-link partition function for two-dimensional lattice QCD with a lattice action

$$S[g] = N/\lambda \sum_{i=1}^N \alpha_i^2 \quad (19)$$

where $\lambda = \pi N/aT$. As observed by Gross and Witten^[18,19] one obtains a third order phase transition if one in Eq.(16) only keeps the $n = 1$ term (this corresponds to a low temperature, Boltzmann approximation of the partition function). In the context of two-dimensional lattice gauge theories at $T = 0$, it was shown in Ref.20 that the choice of action Eq.(19) yields no phase transition whatsoever, which otherwise would have spoiled the two-dimensional bose-fermi duality in the large N -limit.

So far we have only considered the weak coupling limit. Using the same techniques as before it is, however, also possible to give a very simple proof of the statement of Ref.9 about the physical spectrum of $(QED)_2$ in an MIT bag. The partition function can be written in the form (see e.g. Ref.21)

$$Z(\beta) = \frac{\det_F \sqrt{\det \partial^2}}{\sqrt{\det(\partial^2 + m^2)}} \quad (20)$$

where $m^2 = g^2/\pi$ enters via the axial anomaly. To derive Eq.(20) it is important to realize that the boundary terms in $\partial_\mu j_5^\mu$ cancel because of the boundary condition Eq.(1) on the electric field. In evaluating the corresponding determinants one imposes the conventional finite temperature boundary conditions for bosons and fermions in imaginary time [22]. In the continuum case it was shown in Ref.21 that $\det_F \sqrt{\det \partial^2}$ only has a trivial, zero point energy, contribution to $Z(\beta)$. This is also true in the bag case provided one evaluate $\det_F \mathcal{D}$ in the zero charge sector exactly in the same way as in proving Eq.(8). The result of Ref.9 then follows.

Recently extensions of the bose-fermi duality in two-dimensional models have been discussed in higher dimensions (see e.g. Ref.23 and the Appendix of Ref.24). By making use of a partial wave expansion, it was shown in Ref.24 that a massless Weyl spinor, in the zero fermion number sector, can be described in terms of a massless boson field. It is possible to discuss this duality in the thermodynamic framework described in the present paper [25].

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level	Fermions	Bosons
1	$\bar{1}+1$	1
2	$(1+3)^2$	$1^2, 2$
3	$(1+5)^2, \bar{3}+3$	$1^3, 1+2, 3$
4	$(1+7)^2, (3+5)^2, \bar{1}+1+\bar{3}+3$	$1^4, 1^2+2, 1+3, 2^2, 4$
5	$(1+9)^2, (3+7)^2, \bar{5}+5, (1+1+3+5)^2$	$1^5, 1^3+2, 1+2^2, 1+4, 2+3, 5$

Table: A comparison of the energy levels in the charge zero, fermion sector and in the boson sector. The notation is as follows. The integers n in the leftmost column give the total energy of the level by $\omega^{\text{tot}} = n\pi/2a$. $(n+m)^2$ in the fermion column stands for the combination $\bar{n}+m$ and $n+\bar{m}$, where a bar on an integer denotes an antifermion. The integers in the fermion column refer to ω_n^F . In the boson column, n^m means that the n :th level corresponding to ω_n^B , is populated m times.